Lecture Notes on **Sequences & Mathematical Induction**.
Birzeit University, Palestine, 2015

Sequences
& Mathematical Induction

Mustafa Jarrar

5.1 Sequences

5.2&3 Mathematical Induction

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Watch this lecture and download the slides

Course Page: http://www.jarrar.info/courses/DMath/
More Online Courses at: http://www.jarrar.info

Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Sequences & Mathematical Induction

5.1 Sequences

In this lecture:

- Part 1: **Why we need Sequences (Real-life examples).**
- Part 2: Sequence and Patterns
- Part 3: Summation: Notation, Expanding & Telescoping
- Part 4: Product and Factorial
- Part 5: Properties of Summations and Products
- Part 6: Sequence in Computer Loops and Dummy Variables

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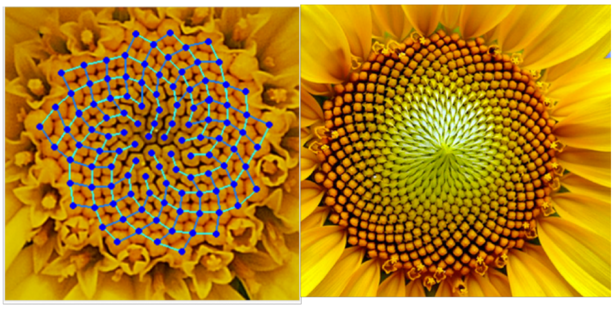
Motivation



هل يمكن النظر الى علم الرياضيات كعلم اكتشاف انماط في الحياة وتعميم هذه الانماط كنظريات وقوانين
ما هو المشترك بين الفن و علم الرياضيات

A mathematician, like a painteror poet, is a maker of patterns.
-G. H. Hardy, *A Mathematicians Apology, 1940*

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Sequences (المتتاليات)

كم عدد اسلافك حتى المستوى الاول؟ ← 2

حتى المستوى الثاني؟ ← 4

حتى المستوى الثالث؟ ← 8

حتى المستوى الرابع؟ ← 16

حتى المستوى الخامس؟ ← 32

حتى المستوى K؟ ← 2^K

Position in the row	1	2	3	4	5	6	7 . . .
Number of ancestors	2	4	8	16	32	64	128 . . .

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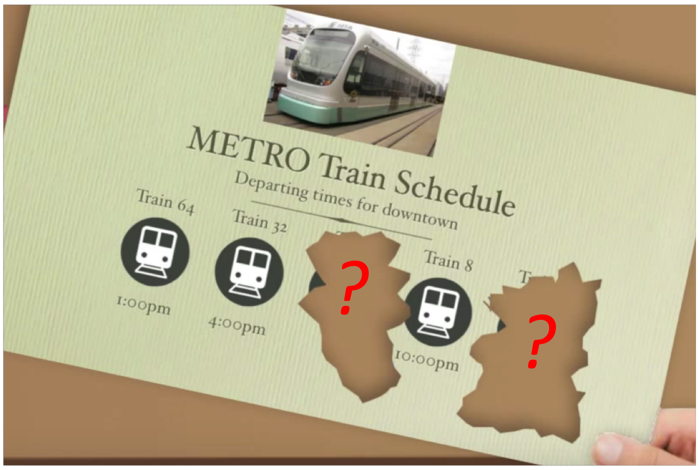
Train Schedule



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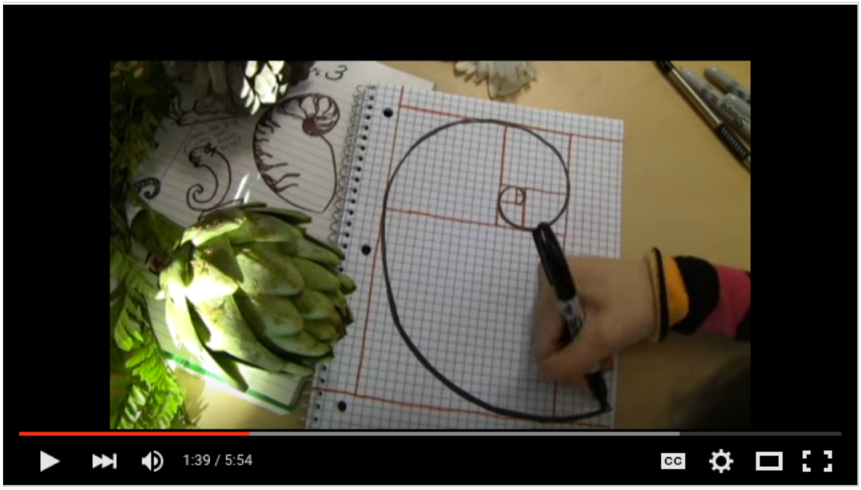
In Nature



https://www.youtube.com/watch?v=ahXIMUkSXX0

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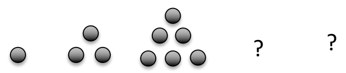
IQ Tests

Determine the number of points in the 4th and 5th figure



1 2 3 4 5

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In programing

Any difference between these loops

1. **for** $i := 1$ **to** n
   print $a[i]$
   next i

2. **for** $j := 0$ **to** $n - 1$
   print $a[j + 1]$
   next j

3. **for** $k := 2$ **to** $n + 1$
   print $a[k - 1]$
   next k

$\sum_{k=1}^{n} a[k]$.

```
s := a[1]
for k := 2 to n
    s := s + a[k]
next k
```

```
s := 0
for k := 1 to n
    s := s + a[k]
next k
```


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Sequences
المتتاليات

$$a_m, a_{m+1}, a_{m+2}, \ . \ . \ . \ , a_n$$

a sequence is a set of elements written in a row.

Each individual element a_k is called a **term.**

The k in a_k is called a **subscript** or **index**

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Finding Terms of Sequences Given by Explicit Formulas

Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

$a_k = \dfrac{k}{k+1}$ **for some integers** $k \geq 1$

$b_i = \dfrac{i-1}{i}$ **for some integers** $i \geq 2$

Compute the first five terms of both sequences.

Solution

ترتيب الحل وترتيب الافكار مهم جداً لملاحظة الانماط

$$a_1 = \frac{1}{1+1} = \frac{1}{2} \qquad b_2 = \frac{2-1}{2} = \frac{1}{2}$$

$$a_2 = \frac{2}{2+1} = \frac{2}{3} \qquad b_3 = \frac{3-1}{3} = \frac{2}{3}$$

$$a_3 = \frac{3}{3+1} = \frac{3}{4} \qquad b_4 = \frac{4-1}{4} = \frac{3}{4}$$

$$a_4 = \frac{4}{4+1} = \frac{4}{5} \qquad b_5 = \frac{5-1}{5} = \frac{4}{5}$$

$$a_5 = \frac{5}{5+1} = \frac{5}{6} \qquad b_6 = \frac{6-1}{6} = \frac{5}{6}$$

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Finding Terms of Sequences Given by Explicit Formulas

Compute the first six terms of the sequence $c_0, c_1, c_2, \ldots$ defined as follows: $c_j = (-1)^j$ for all integers $j \geq 0$.

Solution:

$$c_0 = (-1)^0 = 1$$
$$c^1 = (-1)^1 = -1$$
$$c^2 = (-1)^2 = 1$$
$$c^3 = (-1)^3 = -1$$
$$c^4 = (-1)^4 = 1$$
$$c^5 = (-1)^5 = -1$$

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Finding an Explicit Formula to Fit Given Initial Terms

Find an explicit formula for a sequence that has the following initial terms:

$$1, \; -\frac{1}{4}, \; \frac{1}{9}, \; -\frac{1}{16}, \; \frac{1}{25}, \; -\frac{1}{36}, \; \ldots\ldots$$

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad \text{for all integers } k \geq 1.$$

OR

$$a_k = \frac{(-1)^k}{(k+1)^2} \quad \text{for all integers } k \geq 0.$$

➔How to prove such formulas of sequences?

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Summation

Upper limit

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \ldots\ a_n$$

Lower limit

index

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Example

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$.
Compute the following:

a. $\sum_{k=1}^{5} a_k$ b. $\sum_{k=2}^{2} a_k$ c. $\sum_{k=1}^{2} a_{2.k}$

Solution:

a. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$

b. $\sum_{k=2}^{2} a_k = a_2 = -1$

c. $\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 1 = 0$

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Example

When the terms of a summation are given by a formula

Compute the following summation:

$$\sum_{k=1}^{5} k^2$$

Solution:

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

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Useful Operations

- Summation to Expanded Form
- Expanded Form to Summation
- Separating Off a Final Term
- Telescoping

➔These concepts are very important to understand computer loops

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Summation to Expanded Form

Write the following summation in expanded form:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$$

Solution:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

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Expanded Form to Summation

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \ldots + \frac{n+1}{2n}$$

Solution:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \ldots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}$$

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Separating Off a Final Term and Adding On a Final Term n

Rewrite $\sum_{i=1}^{n+1} \frac{1}{i^2}$ **by separating off the final term.**

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

Write $\sum_{k=0}^{n} 2^k + 2^{n+1}$ **as a single summation.**

$$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

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Telescoping

A telescoping series is a series whose partial sums eventually only have a fixed number of terms after cancellation [wiki].

Example: $\sum_{i=1}^{n} i-(i+1)$ $=(1-2)+(2-3)+...+(n-(n+1))$

$= 1-(n+1)$

$=-n$

This is very useful in programing:

```
S=0
For (i=1;i<=n;i++)
    S= S+ i-(i+1);
```

→

```
S = -n;
```

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Telescoping

A telescoping series is a series whose partial sums eventually only have a fixed number of terms after cancellation [1].

Example: $\sum_{k=1}^{n} \frac{1}{k(k+1)}$

```
S=0;
For (k=1;k<=n;k++)
    S=S+ 1/k*(k+1);
```

$$=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

```
S = 1- (1/(n+1);
```

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Product Notation

• Definition

If m and n are integers and $m \le n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

$$\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120 \qquad \prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$$

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Factorial Notation

• Definition

For each positive integer n, the quantity ***n* factorial** denoted ***n*!,** is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted 0!, is defined to be 1:

$$0! = 1.$$

0! =1
1! =1
2! = 2·1 = 2
3! =3·2·1=6
4! = 4·3·2·1 = 24
5! = 5·4·3·2·1 = 120
6! = 6·5·4·3·2·1 = 720
7! = 7·6·5·4·3·2·1 = 5,040
8! = 8·7·6·5·4·3·2·1
= 40,320
9! = 9·8·7·6·5·4·3·2·1
= 362,880

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Factorial Notation

A recursive definition for factorial

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \,.\, (n\text{-}1)! & \text{if } n \geq 1 \end{cases}$$

0! =1
1! =1
2! = 2·1 = 2
3! =3·2·1=6
4! = 4·3·2·1 = 24
5! = 5·4·3·2·1 = 120
6! = 6·5·4·3·2·1 = 720
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8! = 8·7·6·5·4·3·2·1 = 40,320
9! = 9·8·7·6·5·4·3·2·1 = 362,880

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Computing with Factorials

$$\frac{8!}{7!} = \frac{8 \cdot \cancel{7!}}{\cancel{7!}} = 8$$

$$\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

$$\frac{(n+1)!}{n!} = \frac{(n+1) \cdot \cancel{n!}}{\cancel{n!}} = n + 1$$

$$\frac{n!}{(n-3)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \cancel{(n-3)!}}{\cancel{(n-3)!}} = n \cdot (n-1) \cdot (n-2)$$
$$= n^3 - 3n^2 + 2n$$

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Properties of Summations and Products

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequence[s of real numbers and c]
is any real number, then the following equations hold for any [integer $n \geq m$:]

Which is better in programing?

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

2. $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law

3. $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$.

➔ Remember to apply these in programing Loops

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Example

Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expressions as a single summation or product:

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k \qquad \prod_{k=m}^{n} a_k \cdot \prod_{k=m}^{n} b_k$$

$$\begin{aligned} \sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k &= \sum_{k=m}^{n} (k+1) + 2 \cdot \sum_{k=m}^{n} (k-1) \\ &= \sum_{k=m}^{n} (k+1) + \sum_{k=m}^{n} 2 \cdot (k-1) \\ &= \sum_{k=m}^{n} ((k+1) + 2 \cdot (k-1)) \\ &= \sum_{k=m}^{n} (3k-1) \end{aligned}$$

$$\begin{aligned} \left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) &= \left(\prod_{k=m}^{n} (k+1)\right) \cdot \left(\prod_{k=m}^{n} (k-1)\right) \\ &= \prod_{k=m}^{n} (k+1) \cdot (k-1) \\ &= \prod_{k=m}^{n} (k^2 - 1) \end{aligned}$$

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Change of Variable

Observe: $\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$ $\qquad$ $\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2.$

Hence: $\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2.$

Also Observe: $\sum_{j=2}^{4} (j-1)^2 = (2-1)^2 + (3-1)^2 + (4-1)^2$

$$= 1^2 + 2^2 + 3^2$$

$$= \sum_{k=1}^{3} k^2.$$

Replaced Index by any other symbol (called a **dummy variable**).

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Programing Loops

Any difference between these loops

1. **for** $i := 1$ **to** n
   print $a[i]$
   next i

2. **for** $j := 0$ **to** $n - 1$
   print $a[j + 1]$
   next j

3. **for** $k := 2$ **to** $n + 1$
   print $a[k - 1]$
   next k

$\sum_{k=1}^{n} a[k]$.

$s := a[1]$
for $k := 2$ **to** n
  $s := s + a[k]$
next k

$s := 0$
for $k := 1$ **to** n
  $s := s + a[k]$
next k

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Change Variables

Transform the following summation by making the specified change of variable.

$$\sum_{k=0}^{6} \frac{1}{k+1}$$ Change variable $j = k+1$

For (k=0; k≤6; k++)
Sum = Sum + 1/(k+1)

$$\sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}.$$

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{k=1}^{7} \frac{1}{k}$$

For (k=1; k≤7; k++)
Sum = Sum + 1/(k)

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Change Variables

Transform the following summation by making the specified change of variable.

$$\sum_{k=1}^{n+1} \frac{k}{n+k}$$

For (k=1; k<=n+1; k++)
Sum = Sum + k/(n+k)

Change of variable: j = k - 1

$$\sum_{j=0}^{n} \frac{j+1}{n+(j+1)} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

For (k=0; k<=n; k++)
Sum = Sum + (k+1)/(n+k+1)

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Programing Loops

All questions in the exams will be loops

Thus, I suggest:
Convert all previous examples into loops and play with them

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