Lecture Notes on **Sequences & Mathematical Induction**.
Birzeit University, Palestine, 2015

Sequences
& Mathematical Induction

Mustafa Jarrar

5.1 Sequences

5.2&3 Mathematical Induction

الاستقراء الرياضي

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Watch this lecture
and download the slides

Course Page: http://www.jarrar.info/courses/DMath/
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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Mustafa Jarrar: Lecture Notes on **Sequences & Mathematical Induction**.
Birzeit University, Palestine, 2015

Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

- ❑ **Part 1: What is Mathematical Induction**
- ❑ Part 2: Induction as a Method of Proof/Thinking
- ❑ Part 3: Proving *sum of integers* and *geometric sequences*
- ❑ Part 4: Proving a *Divisibility Property and Inequality*
- ❑ Part 5: Proving a *Property of a Sequence*
- ❑ Part 6: Induction Versus Deduction Thinking

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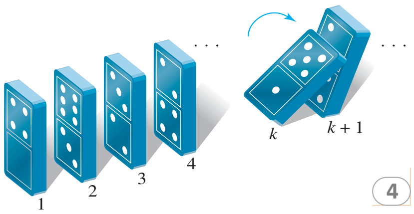
What is Mathematical Induction

Mathematical induction is one of the more **recently developed methods of proof** in mathematics.

<u>**History:**</u>
The first use of mathematical induction was by Al-kraji (1000AD) in his book Al-Fakhri to prove math sequences. In 1883 Augustus De Morgan described it carefully and named mathematical induction.

<u>**The idea:**</u>
If the k^{th} domino falls backward, it pushes the $(k+1)^{st}$ domino backward.



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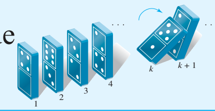
What is Mathematical Induction

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement for all integers $n \geq a$, $P(n)$ is true.



Example: how to know whether this $P(n)$ can be true?

$P(n)$: For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

Moves from specific cases to create a general rule (conjecture/حدس), this is why it is called **Principle, not a theorem**

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What is Mathematical Induction

Example

How to know whether this $P(n)$ can be true?

$P(n)$: For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

For all integers $n \geq 8$, *P(n)* is true, where *P(n)* is the sentence “n cents can be obtained using 3¢ and 5¢ coins.”

Then we need to prove that *P(n+1)* is also true

Number of Cents	How to Obtain It
8¢	3¢ + 5¢
9¢	3¢ + 3¢ + 3¢
10¢	5¢ + 5¢
11¢	3¢ + 3¢ + 5¢
12¢	3¢ + 3¢ + 3¢ + 3¢
13¢	3¢ + 5¢ + 5¢
14¢	3¢ + 3¢ + 3¢ + 5¢
15¢	5¢ + 5¢ + 5¢
16¢	3¢ + 3¢ + 5¢ + 5¢
17¢	3¢ + 3¢ + 3¢ + 3¢ + 5¢

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Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

- Part 1: *What is Mathematical Induction*
- **Part 2: Induction as a Method of Proof/Thinking**
- Part 3: **Proving** *sum of integers* and *geometric sequences*
- Part 4: **Proving** a *Divisibility Property and Inequality*
- Part 5: **Proving** a *Property of a Sequence*
- Part 6: Induction Versus Deduction Thinking

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Mathematical Induction as a Method of Proof

Proving a statement by mathematical induction is a two-step process. The first step is called the ***basis step***, and the second step is called the ***inductive step***.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true."
To prove such a statement, perform the following two steps:
Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k + 1)$ is true. To perform this step,

Suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.**]

Then
show that $P(k + 1)$ is true.

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Mathematical Induction as a Method of Proof

Example

How to know whether this statement can be true?

For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

Let the property $P(n)$ be the sentence: n¢ can be obtained using 3¢ and 5¢ coins. ← $P(n)$

Step 1 (basis step): ***Show P(8) is true:*** $P(8)$ is true as 8¢ obtained by one 3¢ and one 5¢

Step 2(inductive step): ***Show for all integers $k \geq 8$, if $P(k)$ is true then $P(k+1)$ is true:***

[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 8$. That is:]
Suppose k is any integer $k \geq 8$, k¢ obtained by 3¢ and 5¢ . ← **$P(k)$ inductive hypothesis**

[We must show that $P(k + 1)$ is true. That is:] We must show that
$(k + 1)$ ¢ can be obtained using 3¢ and 5¢ coins. ← $P(k + 1)$

Case 1 (There is a 5¢ coin among those used to make up the k¢):

replace the 5¢ coin by two 3¢ coins; the result will be $(k + 1)$¢.

Case 2 (There is not a 5¢ coin among those used to make up the k ¢):

because $k \geq 8$, at least three 3¢ must have been used. So remove three 3¢ and replace them by two 5¢; the result will be $(k + 1)$¢.

Thus in either case $(k + 1)$¢ can be obtained using 3¢ and 5¢ *[as was to be shown]*.

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Sequences & Mathematical Induction

5.2&3 Mathematical Induction

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Sum of the First *n* Integers

Who can sum all numbers from 1 to 100?

1 2 3 . . . . 50 51 . . . . 98 99 100

sum is 101

sum is 101

sum is 101

sum is 101

$$1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}$$

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Proofing by Mathematical Induction

Theorem 5.2.2 Sum of the First *n* Integers

For all integers $n \geq 1$, $\quad 1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}$

Same Question: Prove that these programs prints the same results in case $n \geq 1$

```
For (i=1, i≤n; i++)
  S=S+i;
Print ("%d", S);
```

```
S=(n(n+1))/2
Print ("%d",S);
```

Proving that both programs produce the same results is like proving that:

$$1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2} \quad \leftarrow P(n)$$

Basis Step: ***Show that P(1) is true.*** $P(1)$: $1 = 1(1+1)/2$ = Thus P(1) is true

Inductive Step: ***Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is also true:***

Suppose: $1+2+3+\ldots+k = \frac{k(k+1)}{2}$ is true $\quad \leftarrow P(k)$ inductive hypothesis

$$P(k+1) = 1+2+\ldots+k + (k+1) = \boxed{\frac{(k+1)(k+2)}{2}} \quad \leftarrow P(k+1)$$

$$= P(k) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \quad = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k^2+k}{2} + \frac{2(k+1)}{2} \quad = \frac{k^2+3k+2}{2}$$

$$= \boxed{\frac{(k+1)\,(k+2)}{2}}$$

Same

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Examples of Sums

Evaluate 2 + 4 + 6 + ··· + 500.

$$\begin{aligned}2+4+6+\cdots+500 &= 2\cdot(1+2+3+\cdots+250)\\ &= 2\cdot\left(\frac{250\cdot 251}{2}\right)\\ &= 62{,}750.\end{aligned}$$

Evaluate 5 + 6 + 7 + 8 + ··· + 50.

$$\begin{aligned}5+6+7+8+\cdots+50 &= (1+2+3+\cdots+50)-(1+2+3+4)\\ &= \frac{50\cdot 51}{2}-10\\ &= 1{,}265\end{aligned}$$

For an integer $h \geq 2$, write $1 + 2 + 3 + \cdots + (h-1)$ in closed form.

$$\begin{aligned}1+2+3+\cdots+(h-1) &= \frac{(h-1)\cdot[(h-1)+1]}{2}\\ &= \frac{(h-1)\cdot h}{2}\end{aligned}$$

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Proofing by Mathematical Induction

Theorem 5.2.3 Sum of a Geometric Sequence

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} \quad \leftarrow P(0) \qquad = \frac{r-1}{r-1} = 1$$

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1} \quad \leftarrow P(k)$$

inductive hypothesis

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}. \quad \leftarrow P(k+1)$$

$$\begin{aligned} &= \sum_{i=0}^{k} r^i + r^{k+1}\\ &= \frac{r^{k+1}-1}{r-1} + r^{k+1}\\ &= \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1}\\ &= \frac{(r^{k+1}-1)+r^{k+1}(r-1)}{r-1}\\ &= \frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1}\\ &= \frac{r^{k+2}-1}{r-1}\end{aligned}$$

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Mathematics in Programming

Example : Finding the sum of a geometric series

Prove that these codes will return the same output.

```
int n, r, sum=0;
int i;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1)  {
  for(i=0 ; i<=n ; i++)  {
       sum = sum + pow(r,i);
  }
  printf("%d\n", sum);
}
```

```
int n, r, sum=0;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1) {
       sum=((pow(r,n+1))-1)/(r-1);
       printf("%d\n", sum);
 }
```

This code is proposed by a student/Zaina!

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Examples of Sums of a Geometric Sequence

In each of (a) and (b) below, assume that m is an integer that is greater than or equal to 3. Write each of the sums in closed form.

(a) $1+3+3^2+\cdots+3^{m-2}$

$$1+3+3^2+\cdots+3^{m-2} = \frac{3^{(m-2)+1}-1}{3-1} = \frac{3^{m-1}-1}{2}.$$

(b) $3^2+3^3+3^4+\cdots+3^m$

$$3^2+3^3+3^4+\cdots+3^m = 3^2\cdot(1+3+3^2+\cdots+3^{m-2}) = 9\cdot\left(\frac{3^{m-1}-1}{2}\right)$$

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Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

- ❑ Part 1: What is Mathematical Induction
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Proofing by Mathematical Induction

Proposition 5.3.1 **Proving a Divisibility Property**

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

$3 \mid 2^{2n} - 1$ ← $P(n)$

Basis Step: ***Show that P(0) is true.***

P(0): $2^{2.0} - 1 = 2^0 - 1 = 1 - 1 = 0$ as $3 \mid 0$, thus $P(0)$ is true.

Inductive Step: ***Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is also true:***

Suppose: $2^{2k} - 1$ is divisible by 3. ← $P(k)$ inductive hypothesis

$2^{2k} - 1 = 3r$ for some integer r.

$2^{2(k+1)} - 1$ is divisible by 3. ← $P(k+1)$

$2^{2(k+1)} - 1 = 2^{2k+2} - 1$ by the laws of exponents

$= 2^{2k} \cdot 2^2 - 1 = 2^{2k} \cdot 4 - 1$
$= 2^{2k}(3+1) - 1 = 2^{2k} \cdot 3 + (2^{2k} - 1) = 2^{2k} \cdot 3 + 3r$
$= 3(2^{2k} + r)$ Which is integer

so, by definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3

Mathematics in Programming
Example : Proving Property of a Sequence

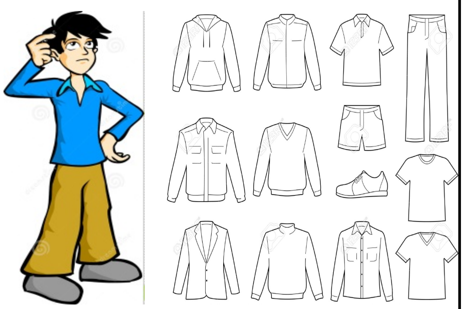
What will the output of this program be for any input n?

```
int n;
scanf("%d",&n);

if(n >= 0) {
     if( (pow(2,(2*n))  - 1) %3 == 0)          \\ does 2^2n -1 | 3?? \\
       printf("this property is true");
     else
       printf("this property isn't true");
}
```


Mathematics in Programming
Example : Proving Property of a Sequence

What will this guy choose to wear today ?
(What is the output of the program)

```
 int x, y;
scanf("%d %d", &x, &y);
if(x%2 == 0)
       x=x+1;
if(pow(x, 2)%2 != 0)
       printf("White Shirt");
else
       printf("Black Shirt");

if((pow(7, y)-1)%6==0)
       printf("Black boot");
else
       printf ("White Boot");
```



By : Ziena

Proofing by Mathematical Induction

Proposition 5.3.2 **Proving Inequality**

For all integers $n \geq 3$, $2n + 1 < 2^n$

Let P(n) be $2n+1<2^n$

Basis Step: *Show that P(3) is true.* P(3): $2.3+1 < 2^3$ which is true.

Inductive Step: *Show that for all integers $k \geq 3$, if P(k) is true then P(k + 1) is also true*:

Suppose: $2k+1<2^k$ is true ← $P(k)$ inductive hypothesis

$2(k+1)+1 < 2^{k+1}$ ← $P(k+1)$

$2k+3 = (2k+1)+2$ by algebra

$< 2^k + 2^k$ as $2k - 1 < 2^k$ by the hypothesis and because $2 < 2^k$ $(k \geq 2)$

$\therefore 2k+3 < 2 \cdot 2^k = 2^{k+1}$

[This is what we needed to show.]

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Mathematics in Programming
Example : Proving Property of a Sequence

What will the output of this program be for any input n>=3 ?

```
scanf("%d", &n);

If (n>=3)
{
    If (2n+1 < pow (2,n))      // is 2n+1 < 2^n ?? //
         printf ("true");
    Else
         printf ("false");
}
```

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Sequences & Mathematical Induction

5.2&3 Mathematical Induction

In this lecture:

Proving a Property of a Sequence
Example

Define a sequence a_1, a_2, a_3 . . . as follows:

$$a_1 = 2$$
$$a_k = 5a_{k-1} \qquad \text{for all integers } k \geq 2.$$

Write the first four terms of the sequence.

$\boldsymbol{a_1 = 2}$
$\boldsymbol{a_2} = 5a_{2-1} = 5a_1 = 5 \cdot 2 = \mathbf{10}$
$\boldsymbol{a_3} = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$
$a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = \mathbf{250}$

➔ **The terms of the sequence satisfy the equation $a_n = 2 \cdot 5^{n-1}$**

Proving a Property of a Sequence
Example

Prove this property:

$$a_n = 2 \cdot 5^{n-1} \text{ for all integers } n \geq 1$$

Basis Step: *Show that P(1) is true.* $a_1 = 2 \cdot 5^{1-1} = 2 \cdot 5^0 = 2$

Inductive Step: *Show that for all integers $k \geq 1$, if P(k) is true then P(k + 1) is also true:*

Suppose: $a_k = 2 \cdot 5^{k-1}$ ← $P(k)$ inductive hypothesis

$a_{k+1} = 2 . 5^k$ ← $P(k+1)$

$= 5a_{(k+1)-1}$ by definition of $a_1, a_2, a_3 \ldots$

$= 5a_k$

$= 5 . (2 . 5^{k-1})$ by the hypothesis

$= 2 . (5 . 5^{k-1})$

$= 2 . 5^k$

[This is what we needed to show.]

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Induction Versus Deduction Reasoning

Deduction Reasoning

If every man is a person and Sami is Man, then Sami is a Person

If my highest mark this semester is 82%, then my average will not be more than 82%

Induction Reasoning

For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

We had a quiz each lecture in the past months, so we will have a quiz next lecture

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Induction Versus Deduction Reasoning

Deduction Reasoning

Based on facts, definitions, , theorems, laws

Moves from general observation to specific results

Provides proofs

Induction Reasoning

Based on observation, past experience, patterns

Moves from specific cases to create a general rule

Provides conjecture/ حدس

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More slides

More Examples of Patterns

$$1-\frac{1}{2}=\frac{1}{2}$$

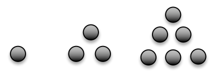
$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=\frac{1}{3}$$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)=\frac{1}{4}$$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k},$$

More Examples of Patterns

Determine the number of points in the 4th and 5th figure



? ?

1 2 3 4 5

More Examples of Patterns

Determine the next 2 terms of the sequences

4, 8, 16, 32, 64

Induce the formulate that could be used to determine any term in the sequence

More slides from Students

Student: Ehab, 2016

Not Reviewed or verified

Example[1]

prove the following property:
for all integers $n \geq 1$, $1\times2 + 2\times3 + 3\times4 + \ldots + (n)(n+1) = \frac{(n)(n+1)(n+2)}{3}$

basis step : show p(1) is true.

P(1): $1\times2 = \frac{(1)(2)(3)}{3}$

left-hand side is $1\times2 = 2$
right-hand side is $\frac{(1)(2)(3)}{3} = 2$

thus p(1) is true

inductive step : Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is also true:
suppose that p(k) is true

$p(k) = 1\times2 + 2\times3 + 3\times4 + \ldots + (k)(k+1) = \frac{(k)(k+1)(k+2)}{3}$ ← P(k) inductive hypothesis

$p(k+1) = 1\times2 + 2\times3 + 3\times4 + \ldots + (k)(k+1) + (k+1)((k+1)+1)$
$= [1\times2 + 2\times3 + 3\times4 + \ldots + (k)(k+1)] + (k+1)((k+1)+1)$
$= \frac{(k)(k+1)(k+2)}{3} + (k+1)(k+2)$
$= \frac{(k)(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$
$= \frac{(k+1)(k+2)(k+3)}{3}$ = right side [This is what we needed to show.]

Then p(k) works for all $n \geq 1$.

[1] CALCULUS with Analytic Geometry, Earl W.Swokwski

Example[1]

Show that For any integer $n \geq 5$, $4n < 2^n$.

basis step : show P(n = 5) is true.
$4n = 4 \times 5 = 20$, and $2^n = 2^5 = 32$.
Since 20 < 32, thus p(n=5) is true

inductive step : Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:
suppose p(k) is true for $k \geq 5$ ← P(k) inductive hypothesis

p(k+1): $4(k + 1) = 4k + 4$, and, by assumption $[4k] + 4 < [2^k] + 4$
Since $k \geq 5$, then $4 < 32 \leq 2^k$. Then we get
$2^k + 4 < 2^k + 2^{k=}$
$= 2 \times 2^k$
$= 2^1 \times 2^k$
$= 2^{k+1}$
Then $4(k+1) < 2^{k+1}$,hence p(k+1) is true *[This is what we needed to show.]*

[1] question taken from this book: CALCULUS with Analytic Geometry, Earl W.Swokwski

Example[1]

show that For all $n \geq 1$, $8^n - 3^n$ is divisible by 5.

basis step : show that p(1) is true
$8^1 - 3^1 =$
$= 8 - 3$
$= 5$ which is clearly divisible by 5.
inductive step : Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:
Suppose p(k) is true ($8^k - 3^k$ is divisible by 5) ← P(k) inductive hypothesis

$8^{k+1} - 3^{k+1} =$
$= 8^{k+1} - 3 \times 8^k + 3 \times 8^k - 3^{k+1}$
$= 8^k(8 - 3) + 3(8^k - 3^k)$
$= 8^k(5) + 3(8^k - 3^k)$
The first term in $8^k(5) + 3(8^k - 3^k)$ has 5 as a factor (explicitly), and the second term is divisible by 5 (by assumption). Since we can factor a 5 out of both terms, then the entire expression, $8^k(5) + 3(8^k - 3^k) = 8^{k+1} - 3^{k+1}$, must be divisible by 5.

[This is what we needed to show.]

[1] question taken from this book: CALCULUS with Analytic Geometry, Earl W.Swokwski

Example[1]

$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2 (n+1)^2}{4}$ .show that this equation is true for all integers n $\geq$1.

Basis step: show that p (1) is true.

Left Side = $1^3 = 1$

Right Side = $\frac{1^2 (1+1)^2}{4} = 1$

hence p (1) is true.

Inductive step: Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:

suppose that p (k) is true ← P(k) inductive hypothesis

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$

$= \frac{k^2 (k+1)^2}{4} + (k+1)^3$

$= \frac{k^2 (k+1)^2}{4} + \frac{4(k+1)^3}{4}$

$= \frac{(k+1)^2 [k^2 + 4k + 4]}{4}$

$= \frac{(k+1)^2 [(k+2)^2]}{4}$

= right side *[This is what we needed to show.]*

[1] question taken from this book: CALCULUS with Analytic Geometry, Earl W.Swokwski

Example[1]

Prove that for any integer number n$\geq$1 , $n^3 + 2n$ is divisible by 3

Basis Step: show that p (1) is true.

Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

3 is divisible by 3 ,hence p (1) is true.

Inductive Step: Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:

suppose that p (k) is true ← P(k) inductive hypothesis

$(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3[k^3 + 2k] + 3[k^2 + k + 1]$

$= 3[[k^3 + 2k] + k^2 + k + 1]$

Hence $(k+1)^3 + 2(k+1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

[This is what we needed to show.]

[1] question taken from this book: CALCULUS with Analytic Geometry, Earl W.Swokwski