Mustafa Jarrar: **Lecture Notes in Discrete Mathematics.**
Birzeit University, Palestine, 2015

Set Theory

6.1. Basics of Set Theory

6.2 Properties of Sets

6.3 Algebraic Proofs

6.4 Boolean Algebras

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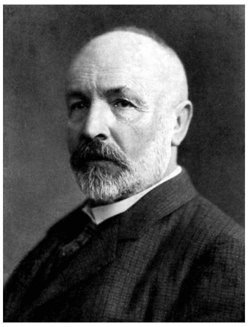
Set Theory

6.1 Basics of Sets

In this lecture:

History



Georg Cantor
1845 – 1918
Born in Saint Petersburg, Russia
Moved to Germany 1856
PhD: University of Berlin 1867
Work: University of Halle

Set theory is the branch of mathematical logic that studies sets, which informally are collections of objects.

Initiated by Georg Cantor in 1870s

Basic Concepts and Notations

Cantor suggested a set as a:

"collection into a whole $\boldsymbol{M}$ of definite and separate objects of our intuition or our thought".

$$\boldsymbol{M} = \{\text{ Ali, Adam, Sara }\}$$

Each object is called an element (or member of) of $\boldsymbol{M}$.

Ali $\in \boldsymbol{M}$ (Ali belongs to M)

Rami $\notin \boldsymbol{M}$ (Rami does not belong to M)

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Basic Concepts and Notations

The order of elements is irrelevant

{Ali, Adam, Sara} = {Adam, Sara, Ali}

Redundancy is not allowed

{Ali, Adam, Adam, Sara}

A set can be an element inside another set

$\{1, \{1\}\}$ has two elements

Notation of elements

{Ali} ≠ Ali different elements

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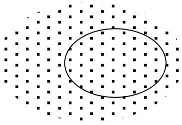
Defining Sets by a Property



$$A = \{x \in S \mid P(x)\}$$

The set of all
x is dummy

Property

Examples:

The set of all integers that are more than -2 and less than 5

$\{x \in \mathbf{Z} \mid -2 < x < 5\}$

The set of all persons who born in Palestine

$\{x \in \textbf{\textit{Person}} \mid \textit{BornIn}(x, \text{Palestine})\}$

The set of all persons who born in Palestine and love Homus

$\{x \in \textbf{\textit{Person}} \mid \textit{BornIn}(x, \text{Palestine}) \land \textit{Love}(x, \text{Homus})\}$

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Set Versus Element

In Set theory → **Set** vs. **Element** ← Mathematical Set

In JAVA → Class vs. Object

In Logic/Philosophy → Concept vs. Instance

- The extension of a set = its elements.

- In set theory: an element itself might be a set.
- In philosophy, an instance has no instances.

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6.1 Basics of Sets

In this lecture:

- ❑ Part 1: Basic Concepts and Notations
- ❑ **Part 2: Subsets, proper subsets, and Set Equalities**
- ❑ Part 3: Operations on Sets
- ❑ Part 4: Formalizing Statements in Set Theory
- ❑ Part 5: Empty Sets
- ❑ Part 6: Partitions of Sets
- ❑ Part 7: Power Sets & Cartesian Products

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Subsets

المجموعة الجزئية

$$A \subseteq B \quad \Leftrightarrow \quad \forall x,\ \text{if } x \in A \text{ then } x \in B.$$

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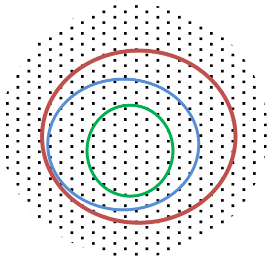
Subsets Versus JAVA SubClasses

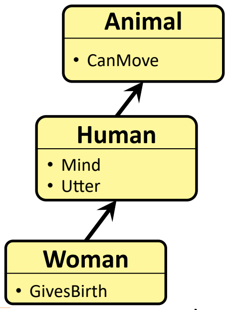


Animal = {x ∈ LivingOrganism | *CanMove*(x)}

Human = {x ∈ Animal | Has*Mind*(x) ∧ Utter (x)}

Woman = {x ∈ Human | GivesBirth(x)}

Woman ⊆ Human ⊆ Animal



Every subclass inherits the properties of its super class, thus:

- Human is a living organism that can move, has mind and utter.
- Woman is a living organism that can move, has mind and utter, and able to give birth.

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Distinction between ∈ and ⊆

Which of the following are true statements?

$2 \in \{1, 2, 3\}$

✗ $\{2\} \in \{1, 2, 3\}$

✗ $2 \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{1, 2, 3\}$

✗ $\{2\} \subseteq \{\{1\}, \{2\}\}$

$\{2\} \in \{\{1\}, \{2\}\}$

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Subsets Notations

Not Subset:

$$A \nsubseteq B \Leftrightarrow \exists x \,.\, x \in A \text{ and } x \notin B$$

Notations:

$A = B$		A equals B
$A \subset B$	$B \supset A$	A is subset of B
$A \subseteq B$	$B \supseteq A$	A is subset or equal of B
$A \not\subset B$	$B \not\supset A$	A is not a subset of B
$A \nsubseteq B$	$B \nsupseteq A$	A is not a subset but not equal of B
$A \subsetneq B$	$B \supsetneq A$	A is a subset but not equal of B

Examples: Person $\supset$ Man, $Z \supset Z+$, $R \supset Z$

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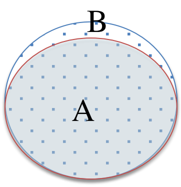
Proper Subsets

Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.



Examples of proper subsets:

Man $\subsetneq$ Person

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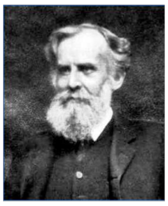
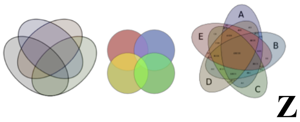
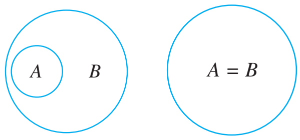
Venn Diagrams



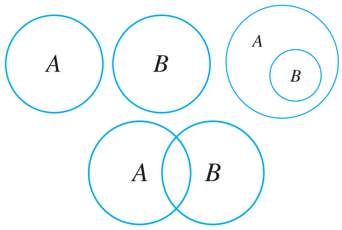
John Venn, British
(1834-1923)
Represented sets as diagrams in1881.
used to teach elementary set theory,



A ⊆ B



A ⊄ B



Z: integers numbers (اعداد صحيحة)
Q: rational numbers (اعداد نسبية)
R: real numbers (اعداد حقيقة)

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Proving and Disproving Subset Relations

Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$

$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}$.

Prove that $A \subseteq B$.

Suppose x is a particular but arbitrarily chosen element of A.

Show that $x \in B$, means show that $x = 3\cdot(\text{integer})$.

$$x = 6r + 12$$
$$= 3\cdot(2r + 4).$$

Let $s = 2r + 4$.

Also, $3s = 3(2r + 4)$
$= 6r + 12$
$= x$

Therefore, x is an element of B.

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Set Equality

Definition

Given sets A and B, A **equals** B, written $\boldsymbol{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically: $A=B \Leftrightarrow A \subseteq B$ **and** $B \subseteq A$.

<u>Example</u>: Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$

$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$

Is $A = B$?

Yes. To prove this, both subset relations $A \subseteq B$ and $B \subseteq A$ must be proved.

***Part 1, Proof That $A \subseteq B$*:**

…..

***Part 2, Proof That $B \subseteq A$*:**

…..

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6.1 Basics of Sets

In this lecture:

- ❑ Part 1: Basic Concepts and Notations
- ❑ Part 2: Subsets, proper subsets, and Set Equalities
- ❑ Part 3: **Set Operations (Union, Intersection, Difference, Complement)**
- ❑ Part 4: Formalizing Statements in Set Theory
- ❑ Part 5: Empty Sets
- ❑ Part 6: Partitions of Sets
- ❑ Part 7: Power Sets & Cartesian Products

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Operations on Sets

• Definition

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$**,** is the set of all elements that are in at least one of A or B.

2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$**,** is the set of all elements that are common to both A and B.

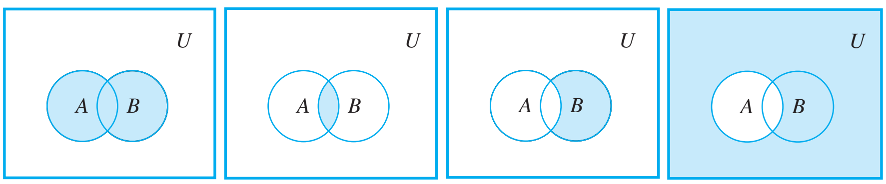
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$**,** is the set of all elements that are in B and not A.

4. The **complement** of A, denoted $\boldsymbol{A^c}$**,** is the set of all elements in U that are not in A.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Operations on Sets



Shaded region represents $A \cup B$.

Shaded region represents $A \cap B$.

Shaded region represents $B - A$.

Shaded region represents A^c.

Distinction between ∩ and ∧

Between sets → $\cap$ and $\wedge$ ← Between predicate and propositions

$\cup$ and $\vee$

Indexed Collection of Sets

• Definition

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0,\ A_1,\ A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

Finding Unions and Intersections of More than Two Sets

For each positive integer i, let A_i = $\left\{x \in \mathbf{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$

A_1: set of all real numbers between -1 and 1
A_2: set of all real numbers between -1/2 and 1/2
A_3: set of all real numbers between - 1/3 and 1/3

Find $A_1 \cup A_2 \cup A_3$ $= (-1,1)$, because $\left(-\frac{1}{2}, \frac{1}{2}\right)\left(-\frac{1}{3}, \frac{1}{3}\right)$ included

Find $A_1 \cap A_2 \cap A_3$ $= \left(-\frac{1}{3}, \frac{1}{3}\right)$, because (-1,1)$\left(-\frac{1}{2}, \frac{1}{2}\right)$ are included

Find $\bigcup_{i=1}^{\infty} A_i = (-1,1)$

Find $\bigcap_{i=1}^{\infty} A_i = \{0\}$

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- ❑ Part 6: Partitions of Sets
- ❑ Part 7: Power Sets & Cartesian Products

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Formalizing Statements in Set Theory

All smart students

Smart ∩ Student

Students who are not Smart

Student ∩ Smart^c / Student - Smart

There are no smart students from Palestine

Smart ∩ Student ∩ Palestinian = Ø

Formalizing Statements in Set Theory

There are no smart students from Palestine among the winners

Smart ∩ Student ∩ Winner ∩ Palestinian = Ø

All Palestinian Americans except Women

(Palestinian ∩ American) – Women / Palestinian ∩ American ∩ women^c

All Students except Ali

Students – {Ali}

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6.1 Basics of Sets

In this lecture:

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The Empty Set

The empty set is not the same thing as nothing; rather, it is a set with nothing inside it and a set is always something. This issue can be overcome by **viewing a set as a bag—an empty bag undoubtedly still exists.**

Example: the set $D = \{x \in \mathbf{R} \mid 3 < x < 2\}$.

Axioms about the empty set:

$\forall A . \emptyset \subseteq A$

$\forall A . A \cup \emptyset \subseteq A$

$\forall A . A \cap \emptyset \subseteq \emptyset$

$\forall A . A \times \emptyset = \emptyset$

$\forall A . A \times \emptyset \Rightarrow A = \emptyset$

While the empty set is a standard and widely accepted mathematical concept, it remains an ontological curiosity, **whose meaning and usefulness are debated by philosophers and logicians**.

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Disjoint Sets

• Definition

Two sets are called **disjoint** if, and only if, they have no elements in common. Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Man ∩ Woman = ∅

• Definition

Sets $A_1, A_2, A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

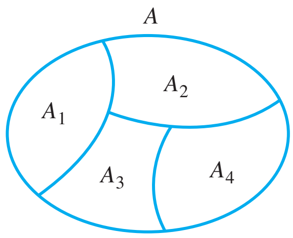
Partitions of Sets

تقسيم جامع مانع

• Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.



Man ∩ Woman = ∅

Person = Man ∪ Woman

Example

Let **Z** be the set of all integers and let

$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \quad \text{for some integer } k\}$,

$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{for some integer } k\}$,

$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{for some integer } k\}$.

Is $\{T_0, T_1, T_2\}$ a partition of Z?

Yes. By the quotient-remainder theorem, every integer n can be represented in exactly one of the three forms

$$n=3k \quad \text{or} \quad n=3k+1 \quad \text{or} \quad n=3k+2$$

It also implies that every integer is in one of the sets T_0, T_1, or T_2.
So $\mathbf{Z} = T_0 \cup T_1 \cup T_2$.

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- ❑ Part 7: **Power Sets & Cartesian Products**

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Power Sets

• Definition

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Find the power set of the set $\{x, y\}$. That is, find $\mathscr{P}(\{x, y\})$

$$= \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

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n-tuples

• Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Order *n*-tuples:

Is (1,2) =(2,1)?

Is $(3, (-2)^2, 1/3) = (\sqrt{9}, 4, \frac{3}{9})$?

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Cartesian Products

• Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\mathbf{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example: Let $A_1 = \{x, y\}$, $A_2 = \{1,2,3\}$, and $A_3 = \{a,b\}$.

$A_1 \times A_2 =$

$= \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3)\}$

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Example

Let A = {Ali, Ahmad},
B = {AI, Dmath, DB},
C = {Pass, Fail}

Find (A × B) × C =

Find A × B × C =

,

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