Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.1. Introduction to Functions

7.2 One-to-One, Onto, Inverse functions

Watch this lecture and download the slides

Course Page: http://www.jarrar.info/courses/DMath/
More Online Courses at: http://www.jarrar.info

Acknowledgement:
This lecture is based on (but not limited to) to chapter 7 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: **One-to-one Functions**
- ❑ Part 2: Onto Functions
- ❑ Part 3: one-to-one Correspondence Functions
- ❑ Part 4: Inverse Functions
- ❑ Part 5: Applications: Hash and Logarithmic Functions

One-to-One Functions

• Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

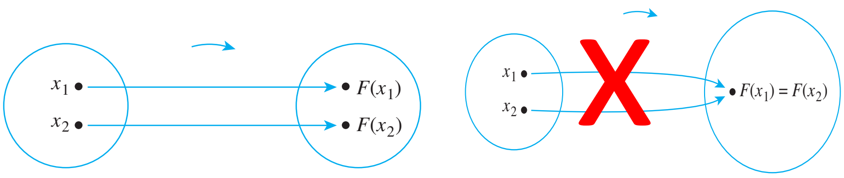
$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $\quad \text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$

Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

لا يوجد عنصرين في المجال لهما نفس الصورة في المجال المقابل



Proving/Disproving Functions are One-to-One

To prove f is one-to-one (Direct Method):

suppose x_1 and x_2 are elements of $X \mid f(x_1) = f(x_2)$, and **show** that $x_1 = x_2$.

To show that f is *not* one-to-one:

Find elements x_1 and x_2 in X so $f(x_1)= f(x_2)$ but $x_1 \neq x_2$.

5

Proving/Disproving Functions are One-to-One
Example 1

Define $f : \mathbf{R} \to \mathbf{R}$ by the rule

$$f(x) = 4x-1 \qquad \text{for all } x \in \mathbf{R}$$

Is f one-to-one? Prove or give a counterexample.

Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. *[We must show that $x_1 = x_2$]* By definition of f,

$4x_1 - 1 = 4x_2 - 1$. Adding 1 to both sides gives

$4x_1 = 4x_2$, and dividing both sides by 4 gives

$x_1 = x_2$, which is what was to be shown.

6

Proving/Disproving Functions are One-to-One
Example 2

Define $g : \mathbf{Z} \to \mathbf{Z}$ by the rule

$$g(n) = n^2 \qquad \text{for all } n \in \mathbf{Z}.$$

Is g one-to-one? Prove or give a counterexample.

Counterexample:

Let $n_1 = 2$ and $n_2 = -2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \text{ and also}$$
$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $g(n_1) = g(n_2)$ but $n_1 \neq n_2$,

and so g is not one-to-one.

,

7

Proving/Disproving Functions are One-to-One
Example 3

Define g : **MobileNumber** → **People** by the rule

$g(x) = Person$ for all $x \in$ **MobileNumber**

Is g one-to-one? Prove or give a counterexample.

Counter example:

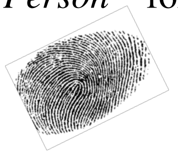
0599123456 and 0569123456 are both for Sami

,

8

Proving/Disproving Functions are One-to-One
Example 4

Define g : **Fingerprints*** → **People** by the rule
$g(x) = Person$ for all $x \in$**R** **Fingerprint**



*Small right finger

Is g one-to-one? Prove or give a counterexample.

Prove:
In biology and forensic science: "The flexibility of friction ridge skin means that no two finger or palm prints are ever exactly alike in every detail" [w].

9

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: One-to-one Functions
- ❑ Part 2: **Onto Functions**
- ❑ Part 3: one-to-one Correspondence Functions
- ❑ Part 4: Inverse Functions
- ❑ Part 5: Applications: Hash and Logarithmic Functions

10

Onto Functions

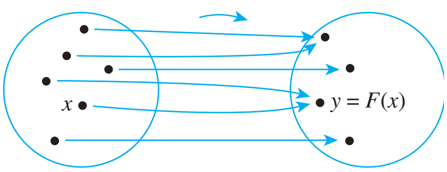
• Definition

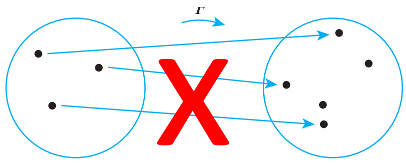
Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

لا يوجد عنصر في المجال المقابل ليس صورة لعنصر في المجال





11

Proving/Disproving Functions are Onto

To prove F is onto, (method of generalizing from the generic particular)

suppose that y is any element of Y

show that there is an element x of X with $F(x) = y$.

To prove F is *not* onto, you will usually

find an element y of Y | $y \neq F(x)$ for *any* x in X.

12

Proving/Disproving Functions are Onto
Example 1

Define f : $\mathbf{R} \to \mathbf{R}$
$f(x) = 4x - 1$ for all $x \in \mathbf{R}$

Is f onto? Prove or give a counterexample.

Let $y \in \mathbf{R}$. *[We must show that $\exists x$ in $\mathbf{R}$ such that $f(x) = y$.]* Let $x = (y+1)/4$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$f(x) = f\left(\frac{y+1}{4}\right) \quad \text{by substitution}$$
$$= 4 \cdot \left(\frac{y+1}{4}\right) - 1 \quad \text{by definition of } f$$
$$= (y+1) - 1 = y \quad \text{by basic algebra.}$$

[This is what was to be shown.]

13

Proving/Disproving Functions are Onto
Example 2

Define h : $\mathbf{Z} \to \mathbf{Z}$ by the rules
$h(n) = 4n - 1$ for all $n \in \mathbf{Z}$.

Is h onto? Prove or give a counterexample.

Counterexample:

The co-domain of h is $\mathbf{Z}$ and $0 \in \mathbf{Z}$. But $h(n) \neq 0$ for any integer n. For if $h(n) = 0$, then

$$4n - 1 = 0 \quad \text{by definition of } h$$

which implies that

$$4n = 1 \quad \text{by adding 1 to both sides}$$

and so

$$n = \frac{1}{4} \quad \text{by dividing both sides by 4.}$$

But $1/4$ is not an integer. Hence there is no integer n for which $f(n) = 0$, and thus f is not onto.

14

Proving/Disproving Functions are Onto
Example 3

Define g : **MobileNumber** $\rightarrow$ **People** by the rule
$g(x) = Person$ for all $x \in$ **MobileNumber**

Is g onto? Prove or give a counterexample.

Counter example:
Sami does not have a mobile number

15

Proving/Disproving Functions are Onto
Example 4

Define g : **Fingerprints** $\rightarrow$ **People** by the rule
$g(x) = Person$ for all $x \in$ **Fingerprint**

Is g onto? Prove or give a counterexample.

Prove:
In biology and forensic science: there is no person without fingerprint

16

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.2 Properties of Functions

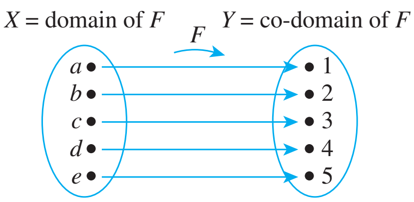
In this lecture:

One-to-One Correspondences

• Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.

لا يوجد عنصر في المجال المقابل ليس صورة لعنصر في المجال، او صورة لعنصرين في المجال



String-Reversing Function

Let T be the set of all finite strings of x's and y's. Define
$g : T \to T$ by the rule: For all strings $s \in T$,
$g(s)$ = the string obtained by writing the characters of s in reverse order. E.g., g("Ali") = "ilA"

Is g a one-to-one correspondence from T to itself?

19

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: One-to-one Functions
- ❑ Part 2: Onto Functions
- ❑ Part 3: one-to-one Correspondence Functions
- ❑ Part 4: **Inverse Functions**
- ❑ Part 5: Applications: Hash and Logarithmic Functions

20

Inverse Functions

Theorem 7.2.2

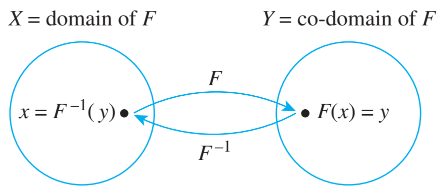
Suppose $F: X \rightarrow Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \rightarrow X}$ that is defined as follows:

Given any element y in Y,

$\boldsymbol{F^{-1}(y)}$ = that unique element x in X such that $F(x)$ equals y.

In other words,

$$F^{-1}(y)=x \quad \Leftrightarrow \quad y=F(x).$$



➔ Is it always that the inverse of a function is a function?

21

Finding Inverse Functions

The function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by the formula

$f(x) = 4x-1$ for all real numbers x

(was shown one-to-one and onto)

Find its inverse function?

Solution For any *[particular but arbitrarily chosen]* y in $\mathbf{R}$, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But

$$\begin{aligned} & f(x) = y \\ \Leftrightarrow \quad & 4x - 1 = y && \text{by definition of } f \\ \Leftrightarrow \quad & x = \frac{y+1}{4} && \text{by algebra.} \end{aligned}$$

Hence $f^{-1}(y) = \dfrac{y+1}{4}$.

22

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.2 Properties of Functions

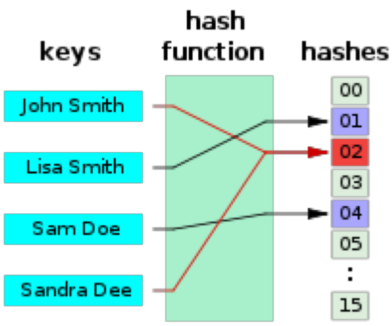
In this lecture:

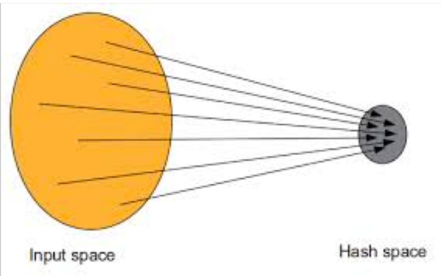
- ❑Part 1: One-to-one Functions
- ❑Part 2: Onto Functions
- ❑Part 3: one-to-one Correspondence Functions
- ❑Part 4: Inverse Functions
- ❑Part 5: **Applications: Hash and Logarithmic Functions**

23

Hash Functions

- Maps data of arbitrary length to data of a fixed length.
- Very much used in databases and security





24

Hash Functions

How to store long (ID numbers) for a small set of people

For example: ***n*** is an ID number, and ***m*** is number of people we have

Hash(n) = n mod m

0	356-63-3102
1	
2	513-40-8716
3	223-79-9061
4	
5	328-34-3419
6	

collision?

,

Exponential and Logarithmic Functions

$$\mathrm{Log}_b\, x = y \quad \Leftrightarrow \quad b^y = x$$

,

Relations between Exponential and Logarithmic Functions

Laws of Exponents

If b and c are any positive real numbers and u and v are any real numbers, the following laws of exponents hold true:

$$b^u b^v = b^{u+v} \quad 7.2.1$$

$$(b^u)^v = b^{uv} \quad 7.2.2$$

$$\frac{b^u}{b^v} = b^{u-v} \quad 7.2.3$$

$$(bc)^u = b^u c^u \quad 7.2.4$$

The exponential and logarithmic functions are one-to-one and onto. Thus the following properties hold:

For any positive real number b with $b \neq 1$,

$$\text{if } b^u = b^v \text{ then } u = v \quad \text{for all real numbers } u \text{ and } v, \quad 7.2.5$$

and

$$\text{if } \log_b u = \log_b v \text{ then } u = v \quad \text{for all positive real numbers } u \text{ and } v. \quad 7.2.6$$

Relations between Exponential and Logarithmic Functions

We can derive additional facts about exponents and logarithms, e.g.:

Theorem 7.2.1 Properties of Logarithms

For any positive real numbers b, c and x with $b \neq 1$ and $c \neq 1$:

a. $\log_b(xy) = \log_b x + \log_b y$

b. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b(x^a) = a \log_b x$

d. $\log_c x = \frac{\log_b x}{\log_b c}$

How to prove this?

Using the One-to-Oneness of the Exponential Function

Prove that:

$$\log_c x = \frac{\log_b x}{\log_b c}.$$

Solution Suppose positive real numbers b, c, and x are given. Let

$$(1)\ \ u = \log_b c \qquad (2)\ \ v = \log_c x \qquad (3)\ \ w = \log_b x.$$

Then, by definition of logarithm,

$$(1')\ \ c = b^u \qquad (2')\ \ x = c^v \qquad (3')\ \ x = b^w.$$

Substituting (1′) into (2′) and using one of the laws of exponents gives

$$x = c^v = (b^u)^v = b^{uv} \qquad \text{by 7.2.2}$$

But by (3), $x = b^w$ also. Hence

$$b^{uv} = b^w,$$

and so by the one-to-oneness of the exponential function (property 7.2.5),

$$uv = w.$$

Substituting from (1), (2), and (3) gives that

$$(\log_b c)(\log_c x) = \log_b x.$$

And dividing both sides by $\log_b c$ (which is nonzero because $c \neq 1$) results in

$$\log_c x = \frac{\log_b x}{\log_b c}.$$

29