Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Relations

8.1. Introduction to Relations

8.2 Properties of Relations

8.3 Equivalence Relations

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 8 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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8.2 Properties of Relations

In this lecture:

- ❑ Part 1: **Properties: Reflexivity, Symmetry, Transitivity**
- ❑ Part 2: Proving Properties of Relations
- ❑ Part 3: Transitive Closure

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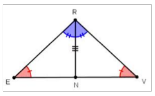

Reflexivity تناظر

R is reflexive $\Leftrightarrow$ for all x in A, $(x, x) \in R$.

R is Reflexive: Each element is related to itself.

علاقة ثنائية على مجموعة ما، وكل عنصر في المجموعة مرتبط بنفسه في إطار هذه العلاقة.

R is not reflexive: there is an element x in A such that $x \; R \; x$ [*that is, such that* $(x, x) \notin R$] .

Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?

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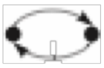
Symmetry تماثل

R is symmetric ⇔ for all *x* and *y* in *A*, ***if*** $(x, y) \in R$ then $(y, x) \in R$.



R is Symmetric: If any one element is related to any other element, then the second is related to the first.

***R* is not Symmetric:** there are elements *x* and *y* in *A* such that *x R y* but *y* ~~*R*~~ *x [that is, such that* $(x, y) \in R$ *but* $(y,x) \notin R$*]*.

Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?

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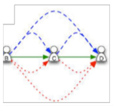
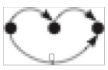


Transitivity تعدي

R is transitive ⇔ for all *x*, *y* and *z* in *A*, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.



R is Transitive: If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

***R* is not transitive:** there are elements *x*,*y* and *z* in *A* such that *xRy* and *yRz* but *x* ~~*R*~~ *z [that is, such that* $(x,y) \in R$ *and* $(y,z) \in R$ *but* $(x, z) \notin R$ *]* .

Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?

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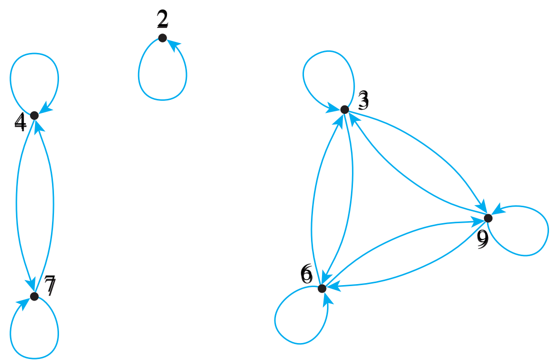
Example

Let $A = \{2,3,4,6,7,9\}$ and define a relation R on A as:

For all $x, y \in A$, $xRy \Leftrightarrow 3 | (x - y)$.



Is R Reflexive?✓ Symmetric?✓ Transitive?✓

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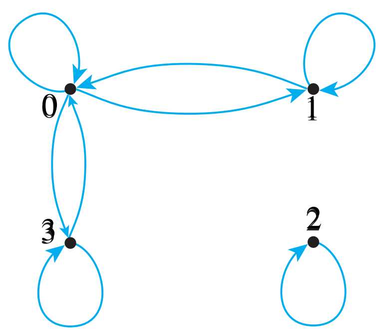
Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$

Is R Reflexive? ☑ *Symmetric?* ☑ *Transitive?* ☒



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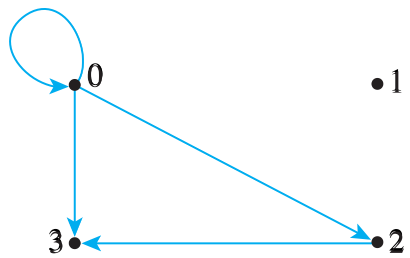
Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:
$R = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$

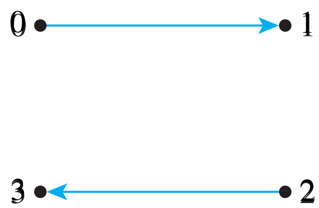
Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑



Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:
$R =\{(0,1),(2,3)\}$

Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑



Remark that the transitivity condition is vacuously true for T . To see this, observe that the transitivity condition says that
$\forall x,y,z \in A$, if $[(x,y) \in T \wedge (y,z) \in T]$ then $[(x,z) \in T]$

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Relations

8.2 Properties of Relations

In this lecture:

- ❑ Part 1: Properties: Reflexivity, Symmetry, Transitivity
- ❑ Part 2: **Proving Properties of Relations**
- ❑ Part 3: Transitive Closure

Proving Properties on Relations on Infinite Sets

Until now we discussed relation on Finite Sets

Next, we discussed relation on infinite Sets

To prove a relation is reflexive, symmetric, or transitive, first write down what is to be proved, in **First Order Logic**.

For instance, for symmetry

$\forall x, y \in A$, if $x \, R \, y$ then $y \, R \, x$.

Then use **direct methods** of proving

Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows:
For all real numbers x and y. $x \, R \, y \Leftrightarrow x = y$.

Is R Reflexive? *Symmetric?* *Transitive?*

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Properties of Less Than

Define a relation R on **R** (the set of all real numbers) as follows:
For all $x, y \in R$, $x \, R \, y \Leftrightarrow x < y$.

Is R Reflexive? *Symmetric?* *Transitive?*

Solution

***R is not reflexive*:** R is reflexive if, and only if, $\forall x \in \mathbf{R}$, $x \, R \, x$. By definition of R, this means that $\forall x \in \mathbf{R}, x < x$. But this is false: $\exists x \in \mathbf{R}$ such that $x \not< x$. As a counterexample, let $x = 0$ and note that $0 \not< 0$. Hence R is not reflexive.

***R is not symmetric*:** R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x \, R \, y$ then $y \, R \, x$. By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$. But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$. As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$. Hence R is not symmetric.

***R is transitive*:** R is transitive if, and only if, for all $x, y, z \in \mathbf{R}$, if $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$. By definition of R, this means that for all $x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers (Appendix A, T18). Hence R is transitive.

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Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows: For all integers m and n, $m\ T\ n \Leftrightarrow 3|(m-n)$.

Is R Reflexive? *Symmetric?* *Transitive?*

For all $m \in \mathbf{Z}$, $3|(m-m)$.

Suppose m is a particular but arbitrarily chosen integer.
[We must show that m T m.]
Now, $m-m = 0$.
But $3 \mid 0$ since $0 = 3 \cdot 0$.
Hence $3|(m-m)$.
Thus, by definition of T, $mT\ m$
[as was to be shown].

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Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows: For all integers m and n, $m\ T\ n \Leftrightarrow 3|(m-n)$.

Is R Reflexive? *Symmetric?* *Transitive?*

For all m, $n \in$ **Z**, **if** $3|(m-n)$ then $3|(n-m)$.

Suppose m and n are particular but arbitrarily chosen integers that satisfy the condition $m\ T\ n$.

[We must show that n T m.]
By definition of T, since $m\ T\ n$ then $3 \mid (m-n)$. By definition of "divides," this means that $m - n = 3k$, for some integer k. Multiplying both sides by -1 gives $n - m = 3(-k)$. Since $-k$ is an integer, this equation shows that $3 \mid (n - m)$. Hence, by definition of T, $n\ T\ m$
[as was to be shown].

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Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows: For all integers m and n, $m\ T\ n \Leftrightarrow 3 | (m-n)$.

Is R Reflexive? *Symmetric?* *Transitive?*

For all $m, n \in \mathbf{Z}$, **if** $3|(m-n)$ and $3|(n-p)$ then $3|(m-p)$.

Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m\ T\ n$ and $n\ T\ p$. *[We must show that $m\ T\ p$.]* By definition of T, since m T n *and* $n\ T\ p$, *then* $3|(m-n)$ *and* $3|(n-p)$. By definition of "divides," this means that $m - n = 3$r and $n - p = 3$s, for some integers r and s. Adding the two equations gives (m−n)+(n−p)=3r+3s, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3|(m - p)$. Hence, by definition of T, $m\ T\ p$ *[as was to be shown]*.

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- ❑ Part 3: **Transitive Closure**

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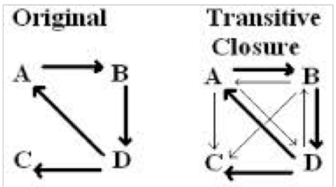
The Transitive Closure of a Relation

The **smallest** transitive relation that contains the relation.

• **Definition**

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.



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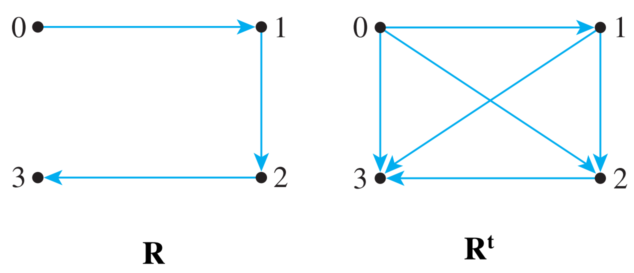
Exercise

Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as:

$R = \{(0, 1), (1, 2), (2, 3)\}$.

Find the transitive closure of R.

$R^t = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$.



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