Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Relations

8.1. Introduction to Relations

8.2 Properties of Relations

8.3 Equivalence Relations

Watch this lecture and download the slides

Course Page: http://www.jarrar.info/courses/DMath/
More Online Courses at: http://www.jarrar.info

Acknowledgement:
This lecture is based on (but not limited to) to chapter 8 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

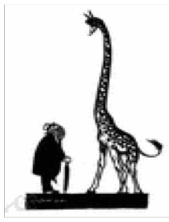
Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Relations

8.3 Equivalence Relations

In this lecture:

- ❑ Part 1: **Partitioned Sets**
- ❑ Part 2: **Equivalence Relation**
- ❑ Part 3: **Equivalence Class**

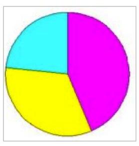


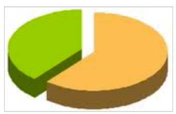
3

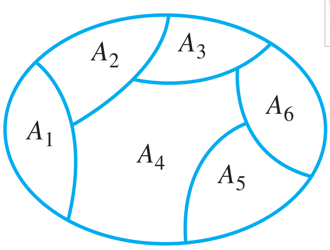
Partitioned Sets

Sets can be partitioned into disjoint sets

A **partition** of a set A is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is A.







تقسيم جامع مانع

Total (جامع)

$$A_i \cup A_2 \cup \ldots \cup A_6 = A$$

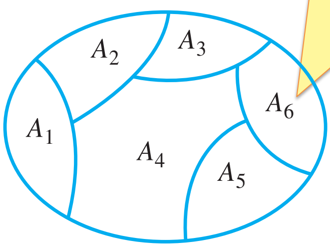
Disjoint (مانع)

$$A_i \cap A_j = \phi, \text{ whenever } i \neq j$$

4

Relations Induced by a Partition

A **relation induced by a partition**, is a relation between two element in the same partition.



تقسيم جامع مانع

Total (جامع)

$A_i \cup A_2 \cup \ldots \cup A_6 = A$

Disjoint (مانع)

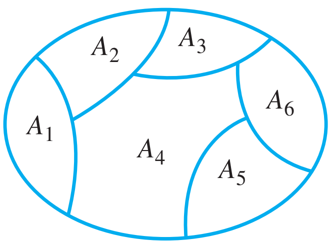
$A_i \cap A_j = \phi$, whenever $i \neq j$

Relations Induced by a Partition

- **Definition**

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$



تقسيم جامع مانع

Total (جامع)

$A_i \cup A_2 \cup \ldots \cup A_6 = A$

Disjoint (مانع)

$A_i \cap A_j = \phi$, whenever $i \neq j$

Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:
$\{0, 3, 4\}, \{1\}, \{2\}$.
Find the relation R induced by this partition.

Since {0, 3, 4} is a subset of the partition,

$0\ R\ 3$ because both 0 and 3 are in {0, 3, 4},
$3\ R\ 0$ because both 3 and 0 are in {0, 3, 4},
$0\ R\ 4$ because both 0 and 4 are in {0, 3, 4},
$4\ R\ 0$ because both 4 and 0 are in {0, 3, 4},
$3\ R\ 4$ because both 3 and 4 are in {0, 3, 4}, and
$4\ R\ 3$ because both 4 and 3 are in {0, 3, 4}.

Also, $0\ R\ 0$ because both 0 and 0 are in {0, 3, 4}
$3\ R\ 3$ because both 3 and 3 are in {0, 3, 4}, and
$4\ R\ 4$ because both 4 and 4 are in {0, 3, 4}.

7

Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:
$\{0, 3, 4\}, \{1\}, \{2\}$.
Find the relation R induced by this partition.

Since {1} is a subset of the partition,

$1\ R\ 1$ because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

$2\ R\ 2$ because both 2 and 2 are in {2}.

Hence
$R = \{(0,0),(0,3),(0,4),(1,1),(2,2),(3,0),$
$(3,3),(3,4),(4,0),(4,3),(4,4)\}$.

8

Relations Induced by a Partition

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

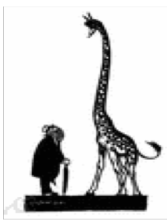
Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Relations

8.3 Equivalence Relations

In this lecture:

- ❑ Part 1: **Partitioned Sets**
- ❑ Part 2: **Equivalence Relation**
- ❑ Part 3: **Equivalence Class**



Equivalence Relation

علاقة تكافؤ

Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

➔ The relation induced by a partition is an equivalence relation

,

11

Example

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$A \; R \; B \Leftrightarrow$ the least element of A equals the least element of B.

Prove that R is an equivalence relation on X.

by definition of *R:*

R *is reflexive:* A *R* A

R *is Symmetric :* B *R* A

R *is transitive :* A *R* C

,

12

Example

Let S be the set of all digital circuits with a fixed number n of inputs. Define a relation E on S as follows: For all circuits $C1$ and $C2$ in S,

$$C_1 \mathbf{E} C_2 \Leftrightarrow C1 \text{ has the same input/output table as } C2.$$

E *is reflexive*: Suppose C is a digital logic circuit in S. *[We must show that C **E** C.]* Certainly C has the same input/output table as itself. Thus, by definition of **E**, C **E** C

E *is symmetric*: Suppose C_1 and C_2 are digital logic circuits in S such that C_1 **E** C_2. By definition of **E**, since C_1 **E** C_2, then C_1 has the same input/output table as C_2. It follows that C_2 has the same input/output table as C_1. Hence, by definition of **E**, C_2 **E** C_1

E *is transitive*: Suppose C_1, C_2, and C_3 are digital logic circuits in S such that C_1 **E** C_2 and C_2 **E** C_3. By definition of **E**, since C_1 **E** C_2 and C_2 **E** C_3, then C_1 has the same input/output table as C_2 and C_2 has the same input/output table as C_3. It follows that C_1 has the same input/output table as C_3. Hence, by definition of **E**, C_1 **E** C_3

13

Example

Let L be the set of all allowable identifiers in a certain computer language, and define a relation R on L as follows: For all strings s and t in L,

$s\ R\ t \Leftrightarrow$ the first eight characters of s equal the first eight characters of t.

***R is reflexive*:** Let $s \in L$. Clearly s has the same first eight characters as itself. Thus, by definition of R, $s\ R\ s$.

***R is symmetric*:** Let s and t be in L and suppose that $s\ R\ t$. By definition of R, since $s\ R\ t$, the first eight characters of s equal the first eight characters of t. But then the first eight characters of t equal the first eight characters of s. And so, by definition of R, $t\ R\ s$

***R is transitive*:** Let s, t, and u be in L and suppose that $s\ R\ t$ and $t\ R\ u$. By definition of R, since $s\ R\ t$ and $t\ R\ u$, the first eight characters of s equal the first eight characters of t, and the first eight characters of t equal the first eight characters of u. Hence the first eight characters of s equal the first eight characters of u. Thus, by definition of R, $s\ R\ u$

14

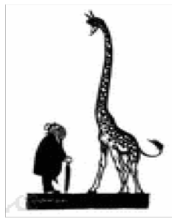
Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Relations

8.3 Equivalence Relations

In this lecture:

- ❑ Part 1: **Partitioned Sets**
- ❑ Part 2: **Equivalence Relation**
- ❑ Part 3: **Equivalence Class**



15

Equivalence Class

• Definition

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a,** denoted **$[a]$** and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$$[a] = \{x \in A \mid x \; R \; a\}$$

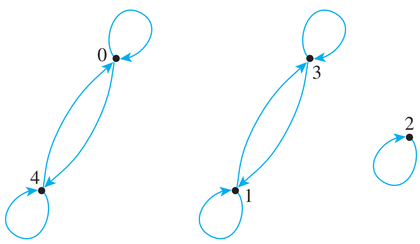
for all $x \in A$, $\quad x \in [a] \quad \Leftrightarrow \quad x \; R \; a$.

16

Example

Let $A = \{0,1,2,3,4\}$ and define a relation R on A as :
$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$.

Find the distinct equivalence classes of R.



$[0] = \{x \in A \mid x\ R\ 0\} = \{0, 4\}$
$[1] = \{x \in A \mid x\ R\ 1\} = \{1, 3\}$
$[2] = \{x \in A \mid x\ R\ 2\} = \{2\}$
$[3] = \{x \in A \mid x\ R\ 3\} = \{1, 3\}$
$[4] = \{x \in A \mid x\ R\ 4\} = \{0, 4\}$

$[0] = [4]$ and $[1] = [3]$.

, 17

Equivalence Class

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a\ R\ b$, then $[a] = [b]$.

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

• Definition

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

, 18

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set $\mathbf{Z}$ of all integers. That is, for all integers m and n,

$$m\ R\ n \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n \pmod{3}.$$

Describe the distinct equivalence classes of R.

For each integer a,

$$\begin{aligned}[a] &= \{x \in \mathbf{Z} \mid x\ R\ a\} \\ &= \{x \in \mathbf{Z} \mid 3 \mid (x-a)\} \\ &= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.\end{aligned}$$

Therefore

$$[a] = \{x \in \mathbf{Z} \mid x = 3k + a, \text{ for some integer } k\}.$$

,

19

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set $\mathbf{Z}$ of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m\equiv n \pmod{3}.$$

Describe the distinct equivalence classes of R.

$$\begin{aligned}[0] &= \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\} \\ &= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\} \\ &= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\}, \\ [1] &= \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\} \\ &= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\}, \\ [2] &= \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\} \\ &= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.\end{aligned}$$

,

20

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n (\text{mod}3).$$

Describe the distinct equivalence classes of R.

The distinct equivalence classes:

$$\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\},$$
$$\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\},$$
$$\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}.$$

,

21

Exercise

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$$A = \mathbf{Z} \times (\mathbf{Z} - \{0\}).$$

Define a relation R on A as follows: For all $(a, b), (c, d) \in A$,

$$(a,b)R(c,d) \Leftrightarrow ad = bc.$$

Describe the distinct equivalence classes of R

For example, the class (1,2):

$$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth.

,

22