Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Counting

9.1 Basics of Probability and Counting

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

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and download the slides

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 9 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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9.3 Counting Elements of Disjoint Sets: Addition Rule

In this lecture:

- ❑ Part 1: **Addition Rule**
- ❑ Part 2: **Difference Rule**
- ❑ Part 2: **Inclusion Rule**

To count elements of union and disjoint sets

Additional Rule

e.g., Number of students in this class = Number of Girls + Number of boys, in this class

Theorem 9.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets A_1, $A_2, \ldots, A_k$. Then

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

The number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

Exercise

A password consists of from 1, 2, or 3 letters chosen from {a..z} with repetitions allowed. How many different passwords are possible?

Passwords Of length 1	Passwords Of length 2	Passwords Of length 3

Number of passwords of length 1 = 26 (because there are 26 letters in the alphabet)

Number of passwords of length 2 = 26^2 (two-step process in which there are 26 ways to perform each step)

Number of passwords of length 3 = 26^3

Total = $26 + 26^2 + 26^3$ = 18,278.

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Exercise

How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

Three-Digit Integers That Are Divisible by 5

3-digits integers That ends in 0	3-digits integers That ends in 5
A_1	A_2

$A_1 \cup A_2$ = the set of all three-digit integers that are divisible by 5

$A_1 \cap A_2 = \emptyset$

[The number of 3-digit integers that are divisible by 5] $= N(A_1) + N(A_2) = 90 + 90 = 180$

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Apply these rules to count elements of union and disjoint sets

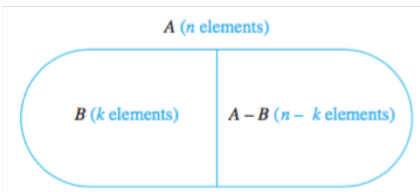
The Difference Rule

Number of students without girls =
number of all students – number of girls

Theorem 9.3.2 The Difference Rule

If A is a finite set and B is a subset of A, then

$$N(A - B) = N(A) - N(B).$$



Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

➢ **How many PINs contain repeated symbols?**

$$1{,}679{,}616 - 1{,}413{,}720 = 265{,}896$$

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

➢ **If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?**

One way:

$$\frac{265{,}896}{1{,}679{,}616} \cong 0.158 = 15.8\%.$$

Another way:

$$P(S-A) = \frac{N(S-A)}{N(S)}$$ by definition of probability in the equally likely case

$$= \frac{N(S)-N(A)}{N(S)}$$ by the difference rule

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$$ by the laws of fractions

$$= 1 - P(A)$$ by definition of probability in the equally likely case

$$\cong 1 - 0.842$$ by Example 9.2.4

$$\cong 0.158 = 15.8\%$$

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

- **If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?**

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A).$$

Another way

$$= \frac{N(S) - N(A)}{N(S)} \quad \text{by the difference rule}$$

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)} \quad \text{by the laws of fractions}$$

$$= 1 - P(A) \quad \text{by definition of probability in the equally likely case}$$

$$\cong 1 - 0.842 \quad \text{by Example 9.2.4}$$

$$\cong 0.158 = 15.8\%$$

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Apply these rules to count elements of union and disjoint sets

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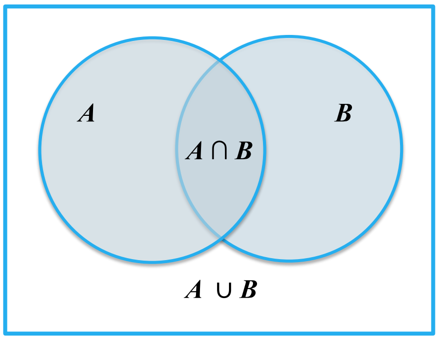
The Inclusion/Exclusion Rule

- Until now, we learned to count union of sets that they are disjoint.
- Now, we learn how to count elements in a union of sets when some of the sets **overlap** (i.e., they are not disjoint)



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Exercise

- How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

3s
1 2 3 4 5 6 . . . 996 997 998 999
3 ↕ $3 \cdot 1$, 6 ↕ $3 \cdot 2$, 996 ↕ $3 \cdot 332$, 999 ↕ $3 \cdot 333$

5s
1 2 3 4 5 6 7 8 9 10 . . . 995 996 997 998 999 1,000
5 ↕ $5 \cdot 1$, 10 ↕ $5 \cdot 2$, 995 ↕ $5 \cdot 199$, 1,000 ↕ $5 \cdot 200$

Overlap
1 2 . . . 15 . . . 30 . . . 975 . . . 990 . . . 999 1,000
15 ↕ $15 \cdot 1$, 30 ↕ $15 \cdot 2$, 975 ↕ $15 \cdot 65$, 990 ↕ $15 \cdot 66$

N (A B) = N(A) + N(B) – N(A∩B)
= 333 + 200 – 66 = 467

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Exercise

- How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

1,000 – 467 = 533

The Inclusion/Exclusion Rule

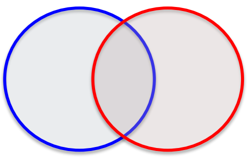
Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets

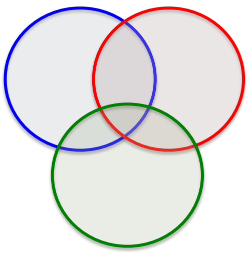
If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$





Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students did not take any of the three courses?

$$50 - 47 = 3.$$

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Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students took all three courses?

$P =$ the set of students who took precalculus
$C =$ the set of students who took calculus
$J =$ the set of students who took Java.

$$N(P \cup C \cup J) =$$
$$N(P) + N(C) + N(J) - N(P \cap C) - N(P \cap J) - N(C \cap J) + N(P \cap C \cap J)$$
$$47 = 30 + 26 + 18 - 9 - 16 - 8 + N(P \cap C \cap J).$$

$$N(P \cap C \cap J) = 6.$$

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Exercise

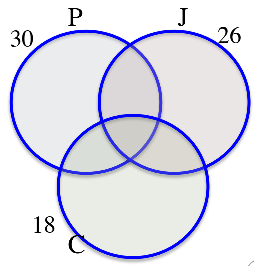
Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ How many students took precalculus and calculus but not Java?

$= (N(P \cap C)) - (N(P \cap C \cap J)) = ?$

$9 - 6 = 3$



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Exercise

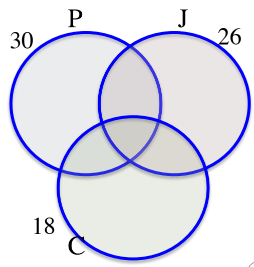
Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ How many students took precalculus but neither calculus nor Java?

$N(P) - (N(P \cap C)) - N(P \cap J)) + N(P \cap C \cap J) = ?$

$30 - 9 - 16 + 6 = 11$



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