Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Counting

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 9 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Counting

9.5 Counting Subsets of a Set: Combinations

In this lecture:

- ❑ Part 1: **Permutation versus Combinations**
- ❑ Part 2: **How to Calculate Combinations**
- ❑ Part 3: **Permutations of a Set with Repeated Elements**

Apply these rules to count elements of union and disjoint sets

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Counting Subsets of a Set
Combinations (توافيق)

Suppose we have 12 people,
How many distinct five-person teams can be selected?

كم فريق من خمسة اشخاص يمكننا ان نكون من بين اثنا عشر شخصا؟

Ordering is not important, as the result is a **set**.

- ➢ Recall that we cannot use the r-permutation rule here, because r-permutation produces ordered sets with repetition.

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Permutations التباديلvs. Combinations التوافيق

An **ordered** selection of r elements from a set of n elements is **an *r-permutation*** *P(n, r)* of the set.

➔ How many 2-permutations we can produce from {a,b,c,d}

$$= P(4,2)$$

An **unordered** selection of r elements from a set of n elements is the same as a subset of size r or an ***r-combination*** of the set.

➔ How many 2-combanations (/subsets) can produce from {a,b,c,d}

$$= \binom{4}{2}$$

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Counting Subsets of a Set: Combinations (التوافيق)

Definition

Let n and r be nonnegative integers with $r \leq n$. An ***r*-combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r}$$

which is read "n choose r," denotes the number of subsets of size r (r -combinations) that can be chosen from a set of n elements.

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Example 1

Let $S = \{Ann, Bob, Cyd, Dan\}$, each committee consisting of three of the four people in S is a 3-combination of S.

List all such 3-combinations of S.

{Bob, Cyd, Dan} leave out Ann
{Ann, Cyd, Dan} leave out Bob
{Ann, Bob, Dan} leave out Cyd
{Ann, Bob, Cyd} leave out Dan.

What is $\binom{4}{3}$ **?**

$= 4.$

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Example 2

How many unordered selections of 2 elements can be made from the set {0, 1, 2, 3}?

{0, 1}, {0, 2}, {0, 3} subsets containing 0
{1, 2}, {1, 3} subsets containing 1 but not already listed
{2, 3} subsets containing 2 but not already listed.

Thus $\binom{4}{2} = 6$

➔ How to Calculate $\binom{n}{r}$

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How to Calculate $\binom{n}{r}$

What is the relation between Permutations and Combinations?

Given the following set: {0, 1, 2, 3}

2-Combanations = $\binom{4}{2}$ = {0,1} {0,2} {0,3} {1,2} {1,3} {2,3}

Notice that
Number of ways = **2!**

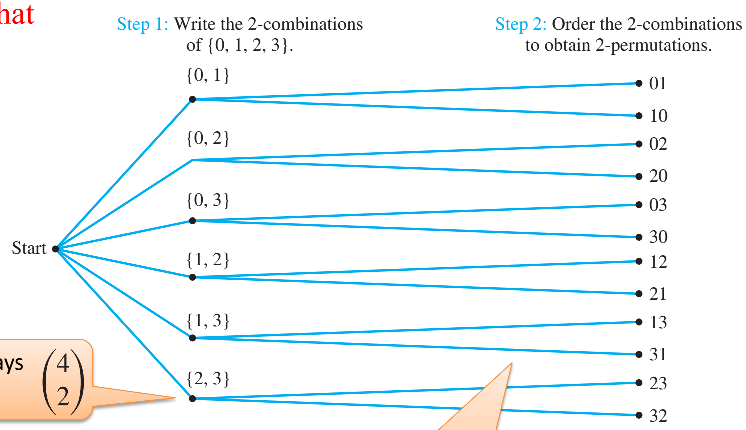
2-Permutations = P(4,2) = 01 10 02 20 03 30 12 21 31 13 23 32

$P(4,2) = \binom{4}{2} \cdot 2!$ → $\binom{4}{2} = \frac{P(4,2)}{2!}$

How to Calculate $\binom{n}{r}$

What is the relation between Permutations and Combinations?

Notice that



Number of ways in step1 = $\binom{4}{2}$

$$P(4, 2) = \binom{4}{2} \cdot 2!.$$

Number of ways in step2 = **2!**

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How to Calculate $\binom{n}{r}$

$$P(4, 2) = \binom{4}{2} \cdot 2!.$$

This is an equation that relates $P(4, 2)$ and $\binom{4}{2}$.

$$\binom{4}{2} = \frac{P(4, 2)}{2!}$$

$$\binom{4}{2} = \frac{\frac{4!}{(4-2)!}}{2!} = \frac{4!}{2!(4-2)!} = 6.$$

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How to Calculate $\binom{n}{r}$

$$P(n,r) = \binom{n}{r} \cdot r!.$$

$$\binom{n}{r} = \frac{P(n,r)}{r!}.$$

$$\binom{n}{r} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}.$$

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n,r)}{r!} \quad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{second version}$$

where n and r are nonnegative integers with $r \le n$.

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How to Calculate $\binom{n}{0}$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$$

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Exercise 1

From 12 people,
How many distinct five-person teams can be selected?

كم فريق من خمسة اشخاص يمكننا ان نكون من بين اثنا عشر شخصا؟

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot 7!} = 11 \cdot 9 \cdot 8 = 792.$$

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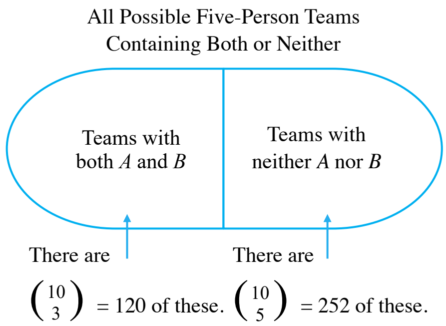
Exercise 2

From 12 people,
How many distinct five-person teams can be selected?

Suppose two members of the 12 insist on working as a pair - any team must contain either both or neither.



So the total number of teams that contain either both A and B or neither A nor B is $120 + 252 = 372$.

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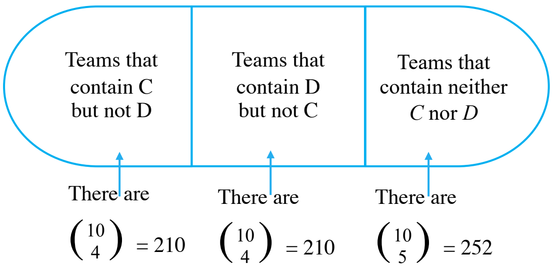
Exercise 3

From 12 people,
How many distinct five-person teams can be selected?
Suppose 2 members refuse to work together on a team.

All Possible Five-Person Teams
That Do Not Contain Both C and D



So the total number of teams that do not contain both C and D is
210 + 210 + 252 = 672.

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Exercise 4

Suppose a group of 12 consists of 5 men and 7 women.
How many 5-person teams can be chosen that consist of 3 men and 2 women?

{A, B, C, D, E, **m, n, o, p, q, s, t, r**}

↓

$\{x_1, x_2, x_3, y_1, y_2\}$

$$\left[\begin{matrix}\text{number of teams of five that}\\ \text{contain three men and two women}\end{matrix}\right] = \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!}\cdot\frac{7!}{2!5!}$$
$$= \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1}$$
$$= 210.$$

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Exercise 5

Suppose the group of 12 consists of 5 men and 7 women.
How many 5-person teams contain at **least** one man?

$$\begin{bmatrix}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of teams}\\ \text{of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{bmatrix}$$

$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5!\cdot 2!}$$

$$= 792 - \frac{7\cdot \overset{3}{\cancel{6}}\cdot \cancel{5!}}{\cancel{5!}\cdot \cancel{2}\cdot 1} = 792 - 21 = 771.$$

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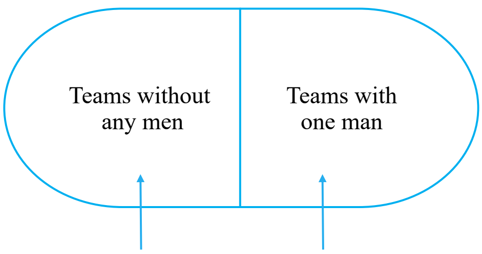
Exercise 6

Suppose the group of 12 consists of 5 men and 7 women.
How many 5-person teams contain at **most** one man?

Teams with at most one Man



So the total number of teams with at most one man is **21 + 175 = 196**.

There are $\binom{5}{0}\cdot\binom{7}{5} = 21$

There are $\binom{5}{1}\binom{7}{4} = 175$

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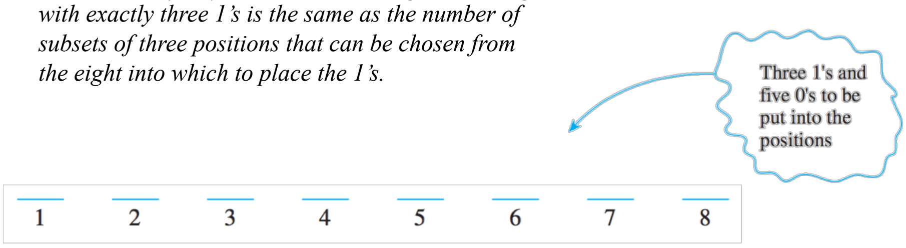
Permutations of a Set with Repeated Elements

How many eight-bit strings have exactly three 1's?

Imagine eight empty positions into which the 0's and 1's of the bit string will be placed. In step 1, choose positions for the three 1's, and in step 2, put the 0's into place.

The number of ways to construct an eight-bit string with exactly three 1's is the same as the number of subsets of three positions that can be chosen from the eight into which to place the 1's.



$$\binom{8}{3} = \frac{8!}{3!\cdot 5!} = \frac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 5!} = 56.$$

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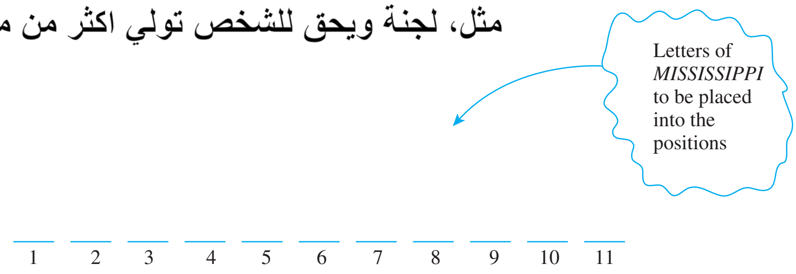
Permutations of a Set with Repeated Elements

Consider various ways of ordering the letters in the word *MISSISSIPPI*: *IIMSSPISSIP*, *ISSSPMIIPIS*, and so on.
How many distinguishable orderings are there?

مثل، لجنة ويحق للشخص تولي اكثر من منصب



Step 1: Choose a subset of 4 positions for the *S*'s.
Step 2: Choose a subset of 4 positions for the *I* 's.
Step 3: Choose a subset of 2 positions for the *P*'s.
Step 4: Choose a subset of 1 position for the *M*.

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Permutations of a Set with Repeated Elements

Consider various ways of ordering the letters in the word *MISSISSIPPI*: *IIMSSPISSIP*, *ISSSPMIIPIS*, and so on.
How many distinguishable orderings are there?

$$\begin{bmatrix}\text{number of ways to} \\ \text{Position all of the letters}\end{bmatrix} = \binom{11}{4} \cdot \binom{7}{4} \cdot \binom{3}{2} \cdot \binom{1}{1}$$

$$= \frac{11!}{4!\cancel{7!}} \cdot \frac{\cancel{7!}}{4!\cancel{3!}} \cdot \frac{\cancel{3!}}{2!\cancel{1!}} \cdot \frac{\cancel{1!}}{1!\cancel{0!}}$$

$$= \frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!} = 34{,}650.$$

Step 1: Choose a subset of 4 positions for the *S*'s.
Step 2: Choose a subset of 4 positions for the *I* 's.
Step 3: Choose a subset of 2 positions for the *P*'s.
Step 4: Choose a subset of 1 position for the *M*.

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Permutations of a Set with Repeated Elements

Theorem 9.5.2 Permutations with sets of Indistinguishable Objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other
n_2 are of type 2 and are indistinguishable from each other
$\vdots$
n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}.$$

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Double Counting and common mistakes

Read Some tips about counting from the book

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