حل لبعض الاسئلة المختارة من شابتر 4

ملاحظة: الحل عبارة عن صور وليس نص مكتوب. واذا لم يوجد رقم السؤال فوق الصورة، فتكون الصورة(الحل) مكملة للصورة(للحل) السابقة.

على سبيل المثال سؤال رقم 16 من سكشن 2، له صورتان.

4.1

4.1.5

Consider the statement

"There are distinct integers m and n such that $\frac{1}{m}+\frac{1}{n}$ is an integer."

The objective is to prove the statement.

Comment

Step 2 of 2

The above statement can be proved by taking example.

Let the distinct integer be $m=3, n=-3$.

And both $3,-3$ are integers.

Now substitute the value of m and n in the expression,

$$\begin{aligned}&\frac{1}{m}+\frac{1}{n}\\&=\frac{1}{3}+\frac{1}{(-3)}\\&=\frac{1}{3}-\frac{1}{3}\\&=0\end{aligned}$$

And 0 is an integer so the statement is correct.

Hence, there are distinct integers $\boxed{m \text{ and } n}$ such that $\boxed{\frac{1}{m}+\frac{1}{n}}$ is an integer.

4.1

4.1.6

Let $a=1$ and $b=0$ be real numbers

Now, we must show that $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$

$\therefore \sqrt{a+b}=\sqrt{1+0}$ [by substitution]

$=\sqrt{1}$

$=1$

$\sqrt{a}+\sqrt{b}=\sqrt{1}+\sqrt{0}$

$=1+0$

$=1$

$\therefore \sqrt{a+b}=\sqrt{a}+\sqrt{b}$

$\therefore$ There are real numbers $a=1$ and $b=0$ such that $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$

4.1

4.1.10

The objective of the question is to prove that there is an integer n such that $2n^2-5n+2$ is prime.

Comment

Step 2 of 3

Consider the statement:

"There is an integer n such that $2n^2-5n+2$ is a prime."

Consider $2n^2-5n+2$.

Constructive Proof of Existence:

When $n=3$.

Now put the value of $n=3$ in equation $2n^2-5n+2$.

Comment

Step 3 of 3

This implies;

$$\begin{aligned}2n^2-5n+2&=2(3)^2-5(3)+2\\&=2(9)-15+2\\&=18-13\\&=5\end{aligned}$$

This implies;

$$2n^2-5n+2=5$$

Here, 5 is a prime number.

So, $2n^2-5n+2$ is also prime.

Hence proved, there is an integer n such that $2n^2-5n+2$ is prime.

4.1

4.1.14

Consider the property that, $(a+b)^2 = a^2 + b^2$

Objective is to determine whether this property is true for all integers, true for no integers, or true for some integers, and false for other integers.

Comment

Step 2 of 2

For this consider that, $(a+b)^2 = a^2 + b^2 + 2ab$

Therefore,

$$(a+b)^2 = a^2 + b^2$$

Implies that,

$$a^2 + b^2 + 2ab = a^2 + b^2$$

This is possible only when, $2ab = 0$. Now since a and b are integers therefore, $2ab = 0$ implies that either $a = 0$ or $b = 0$.

Therefore, the property $(a+b)^2 = a^2 + b^2$ is true if and only if either $a = 0$ or $b = 0$.

Hence, this property is true for some integers say a is any non-zero integer and b is 0 and this property is false if a and b are all non-zero integers.

4.1

4.1.16

The average of any two numbers is found by dividing their sum by 2.

Check if the average of any two odd numbers is odd.

An integer *n* is odd if, and only if, *n* can be expressed as twice some integer plus 1. That is, if *n* can be written of the form $2k+1$ where *k* is some integer.

Comment

Step 2 of 3

Let *m* and *n* be two odd numbers. Then, *m* and *n* can be expressed as $m=2k_1+1$ and $n=2k_2+1$ for some integers k_1, k_2.

Find the average of *m* and *n*.

$$\begin{aligned}\text{Average} &= \frac{m+n}{2}\\ &= \frac{2k_1+1+2k_2+1}{2}\\ &= \frac{2(k_1+k_2)+2}{2}\\ &= k_1+k_2+1\end{aligned}$$

Comment

Step 3 of 3

Now, k_1+k_2+1 can be odd or even depending upon the value of the integers k_1 and k_2. Hence, the given property is true for some integers and false for other integers.

Consider the following examples.

Let $m=3, n=5$.

Then, their average is $\frac{3+5}{2}=\frac{8}{2}=4$ which is even.

Let $m=3, n=7$.

Then, their average is $\frac{3+7}{2}=\frac{10}{2}=5$ which is odd.

4.1

4.1.19

(a) $\forall$ Integer m and n, if m is even and n is odd, then $(m+n)$ is odd.

Comment

Step 2 of 4

$\forall$ Even integer m and odd integer n, $(m+n)$ is odd.

Comment

Step 3 of 4

If m is an even integer and n is an odd integer, then $(m+n)$ is odd.

Comment

Step 4 of 4

(b) **Theorem**: The sum of any even integer and any odd integer is odd.

Proof: Suppose m is any even integer and n is any odd integer. By definition of even, $m = 2r$ for some $Integer\ r$, and by definition of odd, $n = 2s+1$ for some integer s. By substitution and algebra,

$$m+n = 2r+(2s+1) = 2(r+s)+1.$$

Since $r\ and\ s$ are both integers, so is their sum $r+s$. Hence $m+n$ has the form twice some integer plus one, and so $(m+n)\ is\ odd$ by definition of odd.

4.1

4.1.22

(a)

The objective is to rewrite the following statement with the quantification implicit as if- then form:

For all integers m and n, if $mn = 1$ then $m = n = 1$ or $m = n = -1$.

Move the "for all" after the if-statement.

If m and n are integers with $mn = 1$, then $mn = 1$

The statement without variables is:

If the product of two integers is 1, then both integers are 1 or both integers are -1.

Comment

Step 2 of 2

(b)

The objective is to write the first sentence of a proof and the last sentence of a proof.

The **first sentence** of the proof will assume that for some values m and n for which the if-statement is true.

That is, suppose m and n are integers such that $mn = 1$.

The **last sentence** of the proof will be the conclusion which is the then-statement.

That is, $m = n = 1$ or $m = n = -1$.

4.1

4.1.25

Suppose that m is any even integer and n is any odd integer

Now, we must show that $m - n$ is odd

By the definition of even and odd integers,

$m = 2p$ and $n = 2q + 1$, for some integers p and q.

Comment

Step 2 of 3

By substitution,

$m - n = 2p - (2q + 1)$

$= 2p - 2q - 1$

$= 2p - 2q - 2 + 1$

$= 2(p - q - 1) + 1$

Let $a = p - q - 1$, then a is an integer

Comment

Step 3 of 3

Thus, $m - n = 2a + 1$, where a is an integer

So by the definition of odd, $m - n$ is odd

4.1

4.1.27

The objective is to prove that the addition of two odd integers is even.

Suppose *m* and *n* are two odd integers.

Show that $m+n$ is even.

Comment

Step 2 of 2

By the definition of odd integer,

$m=2p+1 \text{ and } n=2q+1$, for some integers *p* and *q*.

Substitute the values $m=2p+1 \text{ and } n=2q+1$ in $m+n$.

$m+n=(2p+1)+(2q+1)$

$=2p+2q+2$

$=2(p+q+1)$ (by factoring out 2)

Let $K=p+q+1$.

Note that *k* is an integer, because it is the sum of integers.

Thus, $m+n=2K$, where *K* is an integer.

From the definition of an even integer, $m+n$ is even.

Hence, the sum of two odd integers is even.

4.1

4.1.32

Given Statement:

If a is any odd integer and b is any even integer, then, $2a+3b$ is even.

Let a be an odd integer and b be an even integer. Therefore, by definition of odd and even integer,

$a=2m+1$ And $b=2n$.

For some integer m and n.

Comment

Step 2 of 2

Therefore,

$2a+3b=2(2m+1)+3(2n)$ (By substitution)

$=4m+6n+2$

$=2(2m+3n+1)$ (By algebra)

Clearly being product and sum of integers $(2m+3n+1)$ is an integer and hence $2(2m+3n+1)$ is an even integer. This implies $2a+3b$ is an even integer.

4.1

4.1.35

The objective is to prove that the following statement is false:

"There exists an integer $m \geq 3$ such that $m^2 - 1$ is prime". ...... (1)

To prove that the above statement is false, consider the negation of the statement and prove that the negation statement is true.

The negation of the statement in (1) is,

"For all integers $m \geq 3$, $m^2 - 1$ is not prime".

Need to prove that the above statement is true.

Consider that m is an integer that is greater than or equal to 3.

That is, $m \geq 3$.

Comment

Step 2 of 2

Consider $m^2 - 1 = (m-1)(m+1)$ Since $x^2 - a^2 = (x-a)(x+a)$.

Since $m \geq 3$,

$$m-1 \geq 3-1 \quad , \quad m+1 \geq 3+1$$
$$m-1 \geq 2 \qquad\quad m+1 \geq 4$$

Here, $(m-1)$ and $(m+1)$ are factors of $m^2 - 1$, but neither of them is equal to one since $m-1 \geq 2$ and $m+1 \geq 4$.

Thus, $m^2 - 1$ is not a prime since for prime number the only factors are one and the number itself.

Hence, the negation statement of (1) "For all integers $m \geq 3$, $m^2 - 1$ is not prime" is true.

Therefore, the original statement "There exists an integer $m \geq 3$ such that $m^2 - 1$ is prime" must be $\boxed{\text{false}}$.

4.1

4.1.36

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The negation of the statement is "For all integers n, $6n^2+27$ is not prime.

Now proof of the negation is given below.

Suppose n is any integers.

By factoring the expression,

$$6n^2+27=3(2n^2+9)$$

The definition of the prime number is as follows.

An integer n is prime if and only is $n>1$, and for all positive integers r and s, if $n=rs$ then either r or s equal to n.

Because $(2n^2+9)$ are positive integers greater than 1, and each is smaller than $6n^2+27$.

So, $6n^2+27$ is a product of two smaller positive integers, each greater than 1.

Hence $6n^2+27$ is not prime.

4.1.40

This incorrect proof begs the question.

The word 'since' in the third sentence is completely unjustified.

The second sentence tells only what happens if k^2+2k+1 is composite.

However, at that point in the proof it has not been established that k^2+2k+1 is composite.

4.1

4.1.42

Assume $m = 2k$ and $n = 2k$

This implies that $m = n$

Then the sum of integers is given as,

$$m + n = 2k + 2k$$
$$m + n = 4k$$

This means that addition of two even numbers is divisible by 4.

Comment

Step 3 of 4

The mistake in the proof is that it is not necessary that the numbers m and n are equal numbers using the same integer k in two different quantities.

Thus, it deduces the conclusion only for this situation.

In other words, the proof does not deduce the conclusion for an arbitrarily chosen even or odd integer or sum of both.

Comment

Step 4 of 4

The correct approach to the proof is,

Assume $m = 2k_1$ and $n = 2k_2$

Then,

$$\begin{aligned} m + n &= 2k_1 + 2k_2 \\ &= 2(k_1 + k_2) \end{aligned}$$

Here, $m + n$ is divisible by 4 for the case when k_1, k_2 are both odd or both even.

Also, $m + n$ is not divisible by 4 for the case when k_1, k_2 are either odd and even or even and odd.

Hence, the mistake in the proof is that is provided for the same integer which can be different values.

4.1

4.1.48

True

Suppose that *m* and *n* are any two even integers

Then, by the definition of an even integer,

$m = 2p$ **and** $n = 2q$, where *p* and *q* are some integers.

Then the difference is

$m - n = 2p - 2q$ [by substitution]

$= 2(p - q)$ [by algebra]

Let $k = p - q$

Then *k* is an integer, because the difference of any two integers is also an integer

$\therefore m - n = 2k$

By the definition of an even integer, then $m - n$ is even

The difference of any two even integers is even

4.1.54

True

Suppose that *n* is any integer

We now claim that $4(n^2 + n + 1) - 3n^2$ is a perfect square, then

$4(n^2 + n + 1) - 3n^2 = 4n^2 + 4n + 4 - 3n^2$

$= n^2 + 4n + 4$

$= n^2 + 2(2n) + 2^2$

$= (n + 2)^2$ [by algebra]

$\therefore 4(n^2 + n + 1) - 3n^2$ is a perfect square, because $(n + 2)^2$ is a perfect square, and $n + 2$ is an integer being a sum of *n* and 2.

Thus, $4(n^2 + n + 1) - 3n^2$ is a perfect square

4.1

4.1.59

Let *a* and *b* be non-negative real numbers, i.e., $a \geq 0, b \geq 0$

We must show that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

Since *a* and *b* are non-negative real numbers, then the unique non-negative real numbers *l* and *m* respectively denote $\sqrt{a}$ and $\sqrt{b}$, such that $a = l^2$ and $b = m^2$.

Since *a* and *b* are non-negative integers, then ab is also a non-negative real number Then, there is a unique non-negative real number *k*, which denotes $\sqrt{ab}$ such that $ab = k^2$.

Now, consider $k^2 - l^2 m^2 = 0$

$\Rightarrow (k - lm)(k + lm) = 0$

We must have $k = lm = 0 \quad (\because k + lm \neq 0)$

Therefore, $\sqrt{ab} - \sqrt{a}\sqrt{b} = 0$

$\Rightarrow \sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

4.1.60

False

Counterexample

Let $a = 4$ and $b = 9$ be non-negative real numbers

Now, consider $\sqrt{a+b} = \sqrt{4+9}$ [by substitution]

$= \sqrt{13}$

$\sqrt{a} = \sqrt{4} = 2$

And

$\sqrt{b} = \sqrt{9} = 3$

$\therefore \sqrt{a} + \sqrt{b} = 2 + 3 = 5$ and $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

4.2

4.2.8

Zero product property: If a product of two real numbers is 0, then one of the numbers must be zero.

The objective is to rewrite this statement formally using quantifiers and variables in the form " $\forall$ ____, if ____ then ____".

The property is true for all real numbers. Therefore, the domain is the set of all real numbers.

Formal restatement: " $\forall$ real numbers x and y, if $xy = 0$, then $x = 0$ or $y = 0$".

Comment

Step 2 of 4

(b)

The objective is to write the contrapositive of the statement obtained in part (a).

If the statement is of the form " $\forall x \in D$, if $P(x)$ then $Q(x)$", then its contrapositive is the statement " $\forall x \in D$, if □ $Q(x)$ then □ $P(x)$".

The statement obtained in part (a) is " $\forall$ real numbers x and y, if $xy = 0$, then $x = 0$ or $y = 0$".

Comment

Step 3 of 4

In this statement, $P(x)$ is " $xy = 0$",

$Q(x)$ is " $x = 0$ or $y = 0$"

Hence, □ $P(x)$ is " $xy \neq 0$".

□ $Q(x)$ is " $x \neq 0$ and $y \neq 0$".

Hence, the contrapositive of the given statement is

" $\forall$ real numbers x and y, if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$".

Comment

Step 4 of 4

(c)

Write the informal version of the statement " $\forall$ real numbers x and y, if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$".

The statement is true for all real numbers. The hypothesis of the statement is " $x \neq 0$ and $y \neq 0$ ", which means that both the numbers are non-zero.

The conclusion of the statement is " $xy \neq 0$", which means that the product of the two numbers is non-zero.

Hence, the statement can be rewritten in the informal way as follows:

"If neither of the two real numbers is zero, then their product is also not zero".

OR

"The product of any two non-zero real numbers is non-zero".

4.2

4.2.10

The objective is to explain why $\frac{5m+12n}{4n}$ must be a rational number.

Suppose that *m* and *n* are both integers, and $n \neq 0$.

As 5 and m are integers, so their product $5m$ is also an integer.

As 12 and n are integers, so their product $12n$ is also an integer.

Also, $5m+12n$ is an integer, by addition rule of integers.

As 4 and n are integers, so their product $4n$ is also an integer.

Comment

Step 2 of 2

Note that $\frac{5m+12n}{4n}$ is the ratio of two integers.

As $4 \neq 0$ or $n \neq 0 \Rightarrow 4n \neq 0$.

Also, $5m+12n \neq 0$.

Thus, $\frac{5m+12n}{4n}$ is a quotient of two integers with a non-zero denominator and so it is rational.

Therefore, $\frac{5m+12n}{4n}$ is a rational number.

4.2.13

Given Statement:

The negative of any rational number is rational.

(a) $\forall$ *real numbers* r, if r is a rational number, then $-r$ is also a rational number.

Comment

Step 2 of 2

(b) The statement is true. Let r be a rational number, then there exist two integer p, q with $q \neq 0$ such that

$r = \frac{p}{q}$. Now

$$-r = -\frac{p}{q}$$

$$= \frac{p_1}{q}$$

As p is an integer, $p_1 = -p$ is also an integer. Therefore $-r$ is the ratio between two integers with no zero denominators. Hence $-r$ is also rational.

4.2

4.2.16

Consider the statement,

"The quotient of any two rational numbers is a rational number."

The objective is to identify whether the above statement is true or false.

And prove if it is true and give counter example if it is false.

Comment

Step 2 of 4

The above statement is false.

Counter example:

Let *x* be any rational number and $y = 0$, then *x* and *y* are both rational, but the quotient $\frac{x}{y}$ is undefined. $\left(\text{Since } \frac{x}{0} = \infty\right)$

Therefore, the quotient of any two rational numbers is not a rational number.

4.2

The correct statement is:

For all rational numbers *x* and *y*, if $y \neq 0$ then $\frac{x}{y}$ is rational.

Proof:

Let *x*, *y* be any two rational numbers and $y \neq 0$.

To show that $\frac{x}{y}$ is rational.

By definition of rational, $x = \frac{a}{b}, y = \frac{c}{d}$, for some integers

$a, b, c, \text{and } d \text{ with } b \neq 0 \text{ and } d \neq 0$.

Thus $\frac{x}{y} = \frac{\frac{a}{b}}{\frac{c}{d}}$ By substitution

$= \frac{ad}{bc}$ By basic algebra

Comment

Step 4 of 4

Let $p = ad, q = bc$, then *p* and *q* are integers because products of integers are integers and because *a*, *b*, *c*, and *d* are all integers.

Also by the zero product property $bc \neq 0$.

$\frac{x}{y} = \frac{p}{q}$, where *p* and *q* are integers and $q \neq 0$.

Therefore, $\boxed{\frac{x}{y} \text{ is rational if } y \neq 0}$.

Hence, the statement 'the quotient of any two rational numbers is a rational number' is **false**.

4.2

4.2.18

True

Suppose r and s are rational numbers, with $r < s$

By the definition of rational,

$r = \frac{m}{n}$ and $s = \frac{p}{q}$, for some integers $m, n, p,$ and q, with $n \neq 0,\ q \neq 0$

Thus, $\frac{r+s}{2} = \frac{\frac{m}{n} + \frac{p}{q}}{2}$ [by substitution]

$= \frac{mq + np}{2nq}$ [by algebra]

Comment

Step 2 of 2

Let $a = mq + np$ and $b = 2nq$

Then a and b are integers, because the products and sums are integers, and because $m, q, n,$ and p are all integers.

Also, $n \neq 0$ and $q \neq 0$ from the zero product property

Thus,

$\frac{r+s}{2} = \frac{a}{b}$, where a and b are integers and $b \neq 0$

Therefore, $\frac{r+s}{2}$ is rational by the definition of a rational number

4.2

4.2.19

Consider the below statement,

"If $a<b$ then $a<\frac{a+b}{2}<b$" is true for all $a,b\in\mathbf{R}$.

The objective is to determine whether this is true or false.

Comment

Step 2 of 3

Suppose a and b are arbitrary real numbers such that $a<b$.

Use $b>a$, which is equivalent to $a<b$:

$$\begin{aligned}\frac{a+b}{2}&>\frac{a+a}{2}\\&=\frac{2a}{2}\\&=a\end{aligned}$$

Thus, $\frac{a+b}{2}>a$, which means that $a<\frac{a+b}{2}$ ...... (1)

Comment

Step 3 of 3

Use $a<b$, which is equivalent to $a<b$:

$$\begin{aligned}\frac{a+b}{2}&<\frac{b+b}{2}\\&=\frac{2b}{2}\\&=b\end{aligned}$$

Thus, $\frac{a+b}{2}<b$ ...... (2)

From relations (1) and (2),

$a<\frac{a+b}{2}<b$.

4.2

4.2.23

True

Suppose that k is any even integer and m is any odd integer

We must show that $(k+2)^2-(m-1)^2$ is even

Since k is even and 2 is even, then $k+2$ is even, because the known result of the sum of any two even integers is even.

We know that $k+2$ is even, then $(k+2)^2$ is also even, because the product of even integers is even.

Also, m is odd and 1 is odd, then $m-1$ is even. It follows that $(m-1)^2$ is also even because $(m-1)^2=(m-1)(m-1)$, and the product of any two even integers is even.

We know that $(k+2)^2$ is even, and $(m-1)^2$ is even

Then, $(k+2)^2-(m-1)^2$ is even, because the difference of any two even integers is even

4.2.24

The statement is "For any rational numbers r and s, $2r+3s$ is rational."

Now the objective is to derive the statement.

The derivation is shown below.

Suppose r and s are any rational numbers.

By Theorem $4.2.1$, it is said that "every integer is a rational number".

And both 2 and 3 are integers which can be expressed in quotient of two integers with a nonzero denominator.

So, both 2 and 3 are rational numbers.

Also, it is assume that r and s are rational numbers. And if and rational number is multiplied with any other rational number then the resultant number is a rational number which is proved in exercise 15.

So,

$$2\times r=\text{rational number}$$
$$3\times s=\text{rational number}$$

From the exercise 15, both $2r$ and $3s$ are rational.

And by Theorem $4.2.2$, it is said that "sum of ant two rational number is a rational number".

Hence, $2r+3s$ is rational.

4.2

4.2.28

Yes

Let $a, b, c,$ and d be integers, and $a \neq c$

Let x be a real number that satisfies the equation

$$\frac{ax+b}{cx+d}=1$$

By cross multiplication, we have

$$ax+b=cx+d$$

$$\Rightarrow ax-cx=d-b$$

$$\Rightarrow x(a-c)=d-b$$

$$\Rightarrow x=\frac{d-b}{a-c}\ ,\ a\neq c$$

Comment

Step 2 of 2

Let $p=d-b$ and $q=a-c$, then p and q are integers, because the difference of any two integers is also an integer, and $q=a-c\neq 0 \qquad (\because a\neq c)$

$\therefore x=\frac{p}{q}$, for some integer p and q with $q\neq 0$

Therefore, x is a ratio of two integers

4.2

4.2.30

Consider the quadratic equation,

$x^2 + bx + c = 0$

The objective is to prove that if one solution is rational then the other solution is also rational.

Comment

Step 2 of 4

The objective is to prove that if one solution for a quadratic equation of the form $x^2 + bx + c = 0$ is rational, then the other solution is also rational.

Here, b and c are rational numbers.

Let $x = r$ be the rational solution of $x^2 + bx + c = 0$.

Then, the quadratic equation can be written as,

$x^2 + bx + c = (x - r)(x - s)$ for some unknown number s.

Comment

Step 3 of 4

Expand $(x - r)(x - s)$ as follows:

$$\begin{aligned}(x - r)(x - s) &= x^2 - rx - sx + rs \\ &= x^2 - (r + s)x + rs\end{aligned}$$

Comment

Step 4 of 4

Compare $x^2 - (r + s)x + rs$ with $x^2 + bx + c = 0$.

$b = -(r + s)$ and $c = rs$

$s = -r - b$

Here b is rational and r is also rational.

The sum of any two rational numbers is rational.

Thus, $s = -(r + b)$ is a rational number, as b and r are rational.

Therefore, the other solution of $x^2 + bx + c = 0$ is also rational.

4.2

4.2.33.a

Given

When expressions of the form $(x-r)(x-s)$ are multiplied out, a quadratic polynomial is obtained. For instance,

$(x-2)(x-(-7))=(x-2)(x+7)=x^2+5x-14$

(a) We note following before writing answer of this problem:

(i) Sum of two odd (even) integers is even (even).

(ii) Sum of one odd and one even integer is odd.

(iii) Product of two odd (even) integers is odd (even).

(iv) Product of one odd and one even integer is even.

Comment

Step 2 of 5

Now $(x-r)(x-s)=x^2-(r+s)x+rs$.

If both r,s are odd then the coefficient are given as follows:

(i) coefficient of x^2 is equal to 1.

(ii) Coefficient of x is $-(r+s)$, i.e. sum of two odd, and hence even.

(iii) Constant is rs product of two odd, hence odd.

Comment

Step 3 of 5

If both r,s are even then the coefficient are given as follows:

(i) coefficient of x^2 is equal to 1.

(ii) Coefficient of x is $-(r+s)$, i.e. sum of two even, and hence even.

(iii) Constant is rs product of two even, hence even.

Comment

Step 4 of 5

If one of r,s is odd and other is even, then the coefficients are given as follows:

(i) coefficient of x^2 is equal to 1.

(ii) Coefficient of x is $-(r+s)$, i.e. sum of one odd and one even, and hence odd.

(iii) Constant is rs product of one odd and one even, hence even.

4.2

4.2.37

In the proof, the rational numbers *r* and *s* are both are the same,

i.e., both *r* and *s* are equal to $\frac{a}{b}$.

This incorrect proof violates the requirement that *r* and *s* are both arbitrarily chosen rational numbers.

If both *r* and *s* are equal $\frac{a}{b}$, then $r = s$

4.2.39

In the later step, it is assumed that such variables exist because of which

$$r+s=\frac{i}{j}+\frac{m}{n}=\frac{a}{b}$$

which has not been proved.

Here, there is confusion between what is known and what is still to be shown. The existence of variables *a* and *b* has been assumed when it has not been established.

Instead, the proof should have statement such as "show that there exists integers *a* and *b* such that $r+s=\frac{a}{b}$.

4.3

4.3.5

Consider the expression $6m(2m+10)$, where *m* is an integer.

The objective is to find that provided expression is divided by 4 or not.

Comment

Step 2 of 3

Simplify provided expression as follows,

$$\begin{aligned} 6m(2m+10) &= 6m[2(m+5)] \\ &= 12m(m+5) \\ &= 12(m^2+5m) \end{aligned}$$

Comment

Step 3 of 3

As *m* is an integer. So, m^2+5m be also an integer.

As m^2+5m is multiple of 12 and 12 is divided by 4. Therefore, $12(m^2+5m)$ that is, $6m(2m+10)$ is divided by 4.

Hence, $6m(2m+10)$ is divided by 4.

4.3

4.3.12

Yes

Let $n = 4k + 1$

We must show that $n^2 - 1$ is divisible by 8

Now, $n^2 - 1 = (4k+1)^2 - 1$ [by substitution]

$= 16k^2 + 8k + 1 - 1$ [by algebra]

$= 16k^2 + 8k$

$= 8(2k^2 + 1)$

From the definition of divisibility,

$16k^2 + 8k = 8(2k^2 + k)$

And $2k^2 + k$ is an integer, because k is an integer, and sums and products of integers are integers.

4.3

4.3.15

The objective is to prove that $a|(b+c)$ if $a|b$ and $a|c$.

From the definition of divisibility,

$a|b \Rightarrow b = ak$, for some integer k

$a|c \Rightarrow c = as$, for some integer s

Substitute the values of b and c in $b+c$.

$$\begin{aligned} b+c &= (ak+as) \\ &= a(k+s) \end{aligned}$$

As k and s are integers, follows that $p = k+s$ is also an integer.

Thus, $b+c = ap$ for some integer p.

Therefore, by the definition of divisibility, a is divisible by $b+c$.

That is, $a|(b+c)$.

4.3..16

The objective is to prove that $a|(b-c)$ if $a|b$ and $a|c$.

From the definition of divisibility,

$a|b \Rightarrow b = ak$, for some integer k

$a|c \Rightarrow c = as$, for some integer s

Substitute the values of b and c in $b-c$.

$$\begin{aligned} b-c &= (ak-as) \\ &= a(k-s) \end{aligned}$$

As k and s are integers, follows that $p = k-s$ is also an integer.

Thus, $b-c = ap$ for some integer p.

Therefore, by the definition of divisibility, a is divisible by $b-c$.

That is, $a|(b-c)$.

4.3

4.3.21

True

Suppose that $2l$ and $2m$ are any two even integers

Then the product of the even integers is $2l \cdot 2m = 4lm$

$\therefore$ By the definition of divisibility,

$(2l)(2m)$ is a multiple of 4.

Therefore, the product of any two even integers is a multiple of 4

4.3.22

Consider the statement, "A necessary condition for an integer to be divisible by 6 is that it be divisible by 2".

The objective is to prove this statement is true or false.

Comment

Step 2 of 3

Let n be an integer divisible by 6.

By definition, for some integer k,

$$\frac{n}{6} = k$$
$$n = 6k$$

Thus,

$$\begin{aligned} n &= 2 \cdot 3 \cdot k \\ &= 2 \cdot (3k) \end{aligned}$$

Comment

Step 3 of 3

As $3k$ is an integer since it is the product of integers. Therefore,

$$n = 2 \cdot (\text{Integer})$$

Thus, by definition of divides $2 \mid n$.

Hence, proved that provided statement is true.

4.3

4.3.23

This statement is true

Suppose n is any integer that is divisible by 16

From the definition of divisibility,

$n = 16p$, for some integer p

However, $16p = 8(2p)$

Let $k = 2p$

Then, k is an integer because p is an integer

Thus, $n = 8k$, for some integer k,

Since from the definition of divisibility, n is divisible by 8

Therefore, an integer that is divisible by 16 is also divisible by 8

The given statement can be rewritten formally as

For all integers n, if n is divisible by 16, then n is divisible by 8

4.3.25

The statement is false

Counterexample

Let $a = 3$, $b = 4$, and $c = 15$

Then a is a factor of c

i.e., $3 | 15$

But $ab \nmid c$

$ab = 3 \cdot 4 = 12$, and 15 is not divisible by 12

Therefore, ab is a not factor of c

4.3

4.3.26

The statement is: "for all integers a,b,c if $ab|c$ then $a|c$ and $b|c$."

Assume that $ab|c$ then by the definition of divisible there exists an integer p such that $c=(ab)p.$

Rewrite c as:

$c=a(bx)$ using associative property

$c=(ab)x=(ba)x=b(ax)$ using commutative and associative property

As b and x are integers, so bx is also integer.

Similarly, a and x are integers implies that ax is also integer.

Comment

Step 2 of 2

This can be written as:

$c=ak$ and $c=bl$ Taking $bx=k$ and $ax=l.$

Thus, c is divisible by a and divisible by $b.$

Hence, if $ab|c$ then $a|c$ and $b|c.$

4.3.27

The statement is false.

Write the Counter example:

Let $a=5,$ $b=6$, and $c=4$

Then $a|(b+c)$

That is, $5|(6+4)$

$5|10$

But,

$5\nmid 6$ and $5\nmid 4$

So,

$a\nmid b$ and $a\nmid c$

Hence, the above statement is false.

4.3

4.3.31

Consider the statement "For all integers *a* and *b*, if $a|10b$ then $a|10$ or $a|b$."

The objective is to determine whether the above statement is true or false.

The given statement is false.

The counterexample is as follows:

Let $a=20$ and $b=2$.

Substitute $a=20$ and $b=2$ in $a|10b$.

$$\begin{aligned} a|10b &= 20|10(2) \\ &= 20|20 \end{aligned}$$

Here, 20 divides 20 since $\frac{20}{20}$ is an integer.

Comment

Step 2 of 2

Substitute $a=20$ and $b=2$ in $a|10$.

$$a|10 = 20|10$$

Here, 20 doesn't divides 10 since $\frac{10}{20}$ is not an integer.

Substitute $a=20$ and $b=2$ in $a|b$.

$$a|b = 20|2.$$

Here, 20 doesn't divides 2 since $\frac{2}{20}$ is not an integer.

Therefore, the given statement is $\boxed{\text{false}}$.

4.3

4.3.33

Suppose that number of nickels, dimes and quarters are x, y, z respectively.

Values of coins as follows:

1 nickel $= \$0.05$

1dime $= \$0.10$

1 quarter $= \$0.25$

Total $= \$(0.05x + 0.10y + 0.25z)$

$= 5x + 10y + 25z$ Cents

Comment

Step 2 of 2

Since 5, 10, and 25 are divisible by 5. Therefore, $5x + 10y + 25z$ is divisible by 5.

But the total amount $\$4.72 = 472$ Cents can't divisible by 5.

Hence it is **not possible** to have a combination of nickels, dimes and quarters that add up to $\$4.72$.

4.3

4.3.36

(a) We are given that an integer is 637,425,403,705,125,

The sum of the digits is 54

From the division property, it is divisible by 9

Thus, it is also divisible by 3 (by transitivity of divisibility)

Also, the number is not divisible by 4 because the two rightmost digits are 25, which is not divisible by 4.

Then, 637,425,403,705,125 is not divisible by 4, because the rightmost digit is 5.

Then 637 425 403 705 125 is divisible by 5

Comment

Step 2 of 5

(b) We are given that an integer is 12,858,306,120,312

The sum of the digits is 42

From the division property, 42 is not divisible by 9,

Therefore, 12,858,306,120,312 is not divisible by 9.

Also, 42 is divisible by 3

Then the number 12,858,306,120,312 is divisible by 3

Also, the number is divisible by 4, because the two rightmost digits are 12, which is divisible by 4.

Also, the number is not divisible by 5 because the rightmost digit is neither 0 nor 5.

Comment

Step 3 of 5

(c) We are given that an integer is 517,924,440,926,512

The sum of the digits is 61

From the division property, 61 is not divisible by 9, and so is not divisible by 3 Also, the number is divisible by 4 because the two rightmost digits are 12 , which is divisible by 4.

The number is not divisible by 5 because the rightmost digit is neither 0 nor 5

4.3

Step 4 of 5

(d) We are given that an integer is 14,328,083,360,232

The sum of the digits is 45

From the division property, 45 is divisible by 9

Thus, it is also divisible by 3 (by transitivity of divisibility)

Comment

Step 5 of 5

Also, the number is divisible by 4 because the two rightmost digits are 32, which is divisible by 4.

Also, the number is not divisible by 5 because the rightmost digit is neither 0 nor 5.

4.3

4.3.39

(a)

The objective is to find the standard factored form for a^3.

The standard factored form for $a^3 = \left(p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}\right)^3$

$= \left(p_1^{e_1}\right)^3 \cdot \left(p_2^{e_2}\right)^3 \cdots \left(p_k^{e_k}\right)^3$

$= p_1^{3e_1} \cdot p_2^{3e_2} \cdots p_k^{3e_k}$

Hence, the standard factored form for a^3 is $a^3 = \boxed{p_1^{3e_1} \cdot p_2^{3e_2} \cdots p_k^{3e_k}}$.

Comment

Step 3 of 4

(b)

Consider the perfect cube, $2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k$

The objective is to find the least positive integer k.

Now, find the lowest positive integer k, such that $2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k$ is a perfect cube.

Perfect cube is equals an integer to the third power.

Take the least positive integer $k = 2^2 \cdot 3^1 \cdot 7^2 \cdot 11$

$2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k = 2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot 2^2 \cdot 3^1 \cdot 7^2 \cdot 11$

$= 2^6 \cdot 3^6 \cdot 11^3 \cdot 7^3$

$= \left(2^2\right)^3 \cdot \left(3^2\right)^3 \cdot 11^3 \cdot 7^3$

$= \left(2^2 \cdot 3^2 \cdot 11 \cdot 7\right)^3$

$= (2772)^3$

Comment

Step 4 of 4

By the definition of a perfect cube,

$k = 2^2 \cdot 3^1 \cdot 7^2 \cdot 11$

$= 4 \cdot 3 \cdot 49 \cdot 11$

$k = 6468$

Hence, the least positive integer is $k = \boxed{6468}$.

4.3

4.3.48

The objective is to prove that for any nonnegative integer n, if the sum of the digits of n is divisible by 3, then n is divisible by 3.

Suppose n is any positive integer with $k+1$ digit.

So, $n = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + .. + a_k \cdot 10^k$.

Let s be the sum of the digits:

$$s = a_0 + a_1 + a_2 + .. + a_k$$

Now,

$$n - s = (a_0 - a_0) + (a_1 \cdot 10 - a_1) + (a_2 \cdot 10^2 - a_2) + .. + (a_k \cdot 10^k - a_k)$$
$$n - s = a_1(10-1) + a_2(10^2 - 1) + ... + a_k(10^k - 1)$$

Comment

Step 2 of 2

If $b_x = 10^x - 1$, then

$$n - s = a_1 \cdot b_1 + a_2 \cdot b_2 + ... + a_n \cdot b_n$$

So,

$$b_x = 10^x - 1$$
$$= 99...$$

It is seen that $b_x = 9...9(9 \text{ occurs } x \text{ times})$.

Implies that, all the numbers b_x are divisible by 3 because it is divisible by 9.

Hence, all the numbers $a_x b_x$ are divisible by 3.

So, their sum $n - s$ is divisible by 3.

Therefore, n is divisible by 3 then so is sum s and vice versa.

4.4

4.4.4

Consider the values $n = 3$ and $d = 11$.

The objective is to find the integers *q* and *r* such that $n = qd + r$ and $0 \le r < d$.

Comment

Step 2 of 3

Find the integers *q* and *r* as,

Theorem:

Given any integer *n* and positive integer *d*, there exist unique integer *q* and *r* such that

$n = qd + r$ and $0 \le r < d$

Comment

Step 3 of 3

By the quotient-remainder theorem,

For $n = 3$ and $d = 11$

$3 = 0(11) + 3$ ...... (1)

Compare equation (1) with $n = qd + r$, obtain $q = 0$ and $r = 3$ and $0 \le 3 < 11$.

Therefore, the integers $q = 0$ and $r = 3$.

4.4

4.4.8

(a) 50 div 7

$$\begin{array}{r|l} & \mathbf{7} \leftarrow 50\ div\ 7 \\ 7 & 50 \\ & 49 \\ \hline & 1 \end{array}$$

i.e., $50 = 7(7) + 1$

$\Rightarrow 50 \text{ div } 7 =$ the quotient when 50 is divided by 7

= 7

Comment

Step 2 of 2

(b) 50 mod 7

$$\begin{array}{r|l} & 7 \\ 7 & 50 \\ & 49 \\ \hline & \mathbf{1} \leftarrow 50 \bmod 7 \end{array}$$

i.e., $50 = 7(7) + 1$

$\Rightarrow 50 \bmod 7 =$ the remainder when 50 is divided by 7

$= 1$

4.4

4.4.10

(a) 30 div 2

$$\begin{array}{r|l} & 15 \leftarrow 30 \text{ div } 2 \\ \hline 2 & 30 \\ & 30 \\ \hline & 0 \end{array}$$

i.e., $30 = 2(15) + 0$

$\Rightarrow 30 \text{ div } 2 =$ quotient when 30 is divided by 2

= 15

Comment

Step 2 of 2

(b) 30 mod 2

$$\begin{array}{r|l} & 15 \\ \hline 2 & 30 \\ & 30 \\ \hline & 0 \quad \leftarrow 30 \bmod 2 \end{array}$$

i.e., $30 = 2(15) + 0$

$\Rightarrow 30 \bmod 2 = \text{remainder when 30 is divided by 2}$

$= 0$

4.4

4.4.17

The objective is to prove that the product of any two consecutive integers is even.

Let n and $n+1$ are two arbitrarily consecutive integers.

There exist two cases for choosing n, either n is even or n is odd.

Case 1:

If n is even, then by definition $n = 2k$, for some integer k.

Substitute $n = 2k$ in $n(n+1)$.

$$\begin{aligned} n(n+1) &= 2k(2k+1) \\ &= 2(2k^2+k) \end{aligned}$$

As k is an integer, $2k^2+k$ is also an integer.

Take $p = 2k^2+k$ then $n(n+1) = 2p$.

Therefore, $n(n+1)$ is an even integer.

Comment

Step 2 of 2

Case 2:

If n is odd, then by definition $n = 2k+1$, for some integer k.

Substitute $n = 2k+1$ in $n(n+1)$.

$$\begin{aligned} n(n+1) &= (2k+1)(2k+2) \\ &= 2(2k+1)(k+1) \end{aligned}$$

As k is an integer, $(2k+1)(k+1)$ is also an integer.

Take $p = (2k+1)(k+1)$ then $n(n+1) = 2p$.

Thus, $n(n+1)$ is an even integer.

Therefore, the product of any two consecutive integers is even.

4.4

4.4.19

Consider the expression,

$n^2 - n + 3$ For all integers n.

The objective is to prove the given expression $n^2 - n + 3$ is odd for all integers n.

Comment

Step 2 of 5

It can be prove in two cases.

Let n be any integer.

Case1: $(n \text{ is even})$

Then $n = 2k$, for some integer k (since n is even).

$n^2 - n + 3 = (2k)^2 - 2k + 3$ Replace n with $2k$

$= 4k^2 - 2k + 2 + 1 \qquad (\because 3 = 2 + 1)$

$= 2(2k^2 - k + 1) + 1$

$n^2 - n + 3 = 2(2k^2 - k + 1) + 1$

Comment

Step 3 of 5

Let $t = 2k^2 - k + 1$

Then t is an integer because products, differences, and sums of integers are integers.

Thus, the expression can be written as,

$n^2 - n + 3 = 2t + 1$, Here t is an integer.

By the definition of odd, $n^2 - n + 3$ is odd.

4.4

Case 2 (n is odd):

Then $n = 2k+1$, for some integer k (since n is odd)

$n^2 - n + 3 = (2k+1)^2 - (2k+1) + 3$ Replace n with $2k+1$

$= 4k^2 + 1 + 4k - 2k - 1 + 3$

$= 4k^2 + 2k + 3$

$= 4k^2 + 2k + 2 + 1$

$= 2(2k^2 + k + 1) + 1$

Comment

Step 5 of 5

Let $t = 2k^2 + k + 1$

Then t is an integer because products, differences, and sums of integers are integers.

Thus, the expression can be written as,

$n^2 - n + 3 = 2t + 1$, Here t is an integer.

By the definition of odd, $n^2 - n + 3$ is odd.

Therefore, from the two cases,

The expression $n^2 - n + 3$ is odd for all integers n.

4.4

4.4.21

Consider b is an integer and $b \bmod 12 = 5,$

That is division of b by 12 gives a remainder 5.

Therefore, there exist an integer q such that,

$$b = 12q + 5$$

Multiply both sides by 8.

$$\begin{aligned} 8b &= 8(12q+5) \\ &= 8q(12)+40 \\ &= 8q(12)+36+4 \\ &= 12(8q+3)+4 \end{aligned}$$

As q is an integer, then $8q+3$ is also an integer.

Thus, if a division of $8b$ by 12 give a remainder of 4, then $8b \bmod 12 = 4$.

Hence, $8b \bmod 12 = \boxed{4}$.

4.4.22

The objective is to determine the value of $10c \bmod 15,$ when $c \bmod 15 = 3,$ and c is any constant.

When c is divided by 15, the remainder is 3.By definition of modular, there exists an integer k such that $c = 15k + 3.$

Multiply both sides by 10.

$$\begin{aligned} 10c &= 10(15k+3) \\ &= 10 \cdot 15k + 30 \\ &= 15(10k+2) \\ &= 15(10k+2)+0 \end{aligned}$$

Comment

Step 2 of 2

As $10k+2$ is an integer, so $10c$ is divisible by 15.

That is, $10c \bmod 15 = 0.$

Therefore, the value of $10c \bmod 15$ is $\boxed{0}$.

4.4

4.4.25

Let a be an integer such that $a \bmod 7 = 5$ i.e. division of a by 7 gives a remainder of 5. Therefore, there exists an integer r such that

$$a = 7r + 5.$$

Comment

Step 3 of 4

Let b be an integer such that $b \bmod 7 = 6$ i.e. division of b by 7 gives a remainder of 6. Therefore, there exists an integer q such that

$$b = 7q + 6.$$

Comment

Step 4 of 4

This implies

$$ab = (7r+5)(7q+6)$$

$$= 49q\,r + 42\,r + 35q + 30$$

$$= 7(7qr + 6r + 5q + 4) + 2$$

As q, r both are integers, $(7qr + 6r + 5q + 4)$ is also an integer. If we divide ab by 7, the reminder will be 2. Hence

$$\boxed{ab \bmod 7 = 2}$$

4.4

4.4.26

We know that for any integer n and positive integer $d > 0$,

$n \bmod d = r$

$\Leftrightarrow n = dq + r$

Where q, r are integers and $0 \le r < d$

Suppose $n \bmod d = 0$.

Then there exist an integer q such that $n = dq + 0$

$n = dq$

d is a factor of n

Therefore, n is divisible by d

Comment

Step 2 of 2

Conversely, suppose that n is divisible by d. Then there exist an integer q such that $n = dq$

Therefore

$n = dq + 0$

$n \bmod d = 0$

Therefore, a necessary and sufficient condition for a nonnegative integer n to be divisible by a positive integer d is that $n \bmod d = 0$

4.4

4.4.28.a

(a) We know by the quotient-remainder theorem that when three consecutive integers

n, $n + 1$, and $n + 2$ are divided by 3, the remainders 0, 1, and 2 are left in the same order starting from one of 0 or 1 or 2.

So these integers can, without loss of generality, be written in one of the three forms

as $3q$, $3q+1$, or $3q + 2$.

Now, consider the product of these consecutive integers

Comment

Step 2 of 6

Case (1)

$n = 3q$, for some integer q

Now, $n(n+1)(n+2) = 3q(3q+1)(3q+2)$

$= 3m$, where $m = q(3q+1)(3q+2)$

So from the definition of divisibility, $n(n+1)(n+2)$ is divisible by 3

Comment

Step 3 of 6

Case (2)

$n = 3q+1$, for some integer q

Now, $n(n+1)(n+2) = (3q+1)(3q+2)(3q+3)$

$= 3(3q+1)(3q+2)(q+1)$

$= 3m$, where $m = (3q+1)(3q+2)(q+1)$

So from the definition of divisibility, $n(n+1)(n+2)$ is divisible by 3

Comment

Step 4 of 6

Case (3)

$n = 3q+2$, for some integer q

Now, $n(n+1)(n+2) = (3q+2)(3q+3)(3q+4)$

$= 3(3q+2)(q+1)(3q+4)$

$= 3m$, where $m = (3q+2)(3q+4)(q+1)$

So by definition of divisibility $n(n+1)(n+2)$ is divisible by 3

Therefore, the product of any three consecutive integers is divisible by 2

4.4

4.4.31

(b)

Let m and n are positive integers.

$m^2 - n^2 = 56$

Since, $m^2 - n^2 = (m-n)(m+n)$. Thus, 56 is the product of $m+n$ and $m-n$.

And, by the part (a), $m+n$ and $m-n$ are either both odd or both even.

Now the factorization of 56 is $2^3.7$.

Since both terms should be even or odd. Thus, the possible factors are $(2,28),(4,14),$ and $(8,7)$.

But for the factor $(8,7)$, $m+n$ and $m-n$ are not both odd or not both even. Thus, it will not be considered here.

Comment

Step 6 of 9

When $m+n=28$ and $m-n=2$, then

$2m = 30$

$\Rightarrow m = 15$

And, for $m=15$

$n = 28 - m$

$= 28 - 15$

$= 13$

When $m+n=14$ and $m-n=4$, then

$2m = 18$

$\Rightarrow m = 9$

And, for $m=9$

$n = 14 - m$

$= 14 - 9$

$= 5$

Hence, the solutions are $m=15$ and $n=13$, or $m=9$ and $n=5$.

4.4

4.4.32

The objective is to find the parity of $2a-(b+c)$, with $a-b$ and $b-c$ are even integers for integers $a,b,$ and c.

Rewrite $2a-(b+c)$ as follows:

$$\begin{aligned}2a-(b+c)&=(a+a)-(b+c)\\&=(a-b)+(a-c)\end{aligned}$$

As $a-b$ and $b-c$ are even, and sum of two even integers is even, and then $(a-b)+(b-c)=a-c$ is also even.

Thus, by applying the rule of sum of even integers is even, conclude that $(a-b)+(a-c)=2a-(b+c)$ is even.

4.4.39

Let $n,\ n+1,\ n+2$, and $n+3$ be four consecutive integers

Now,

$n+(n+1)+(n+2)+(n+3)=n+n+n+n+1+2+3$

$=4n+6$

$=4n+4+2$

$=4(n+1)+2$

$=4k+2$.

Where $n+1=k$, for some integer k

$\therefore n+(n+1)+(n+2)+(n+3)=4k+2$

Thus, the sum of any four consecutive integers is of the form $4k+2$, for some integer k

4.4

4.4.44

The objective is to show that $|x|\cdot|y|=|xy|$ for all real numbers x and y.

Case 1:

Suppose $x<0, y<0$, then $|x|=-x, |y|=-y$.

So,

$$\begin{aligned}|x||y| &= (-x)(-y)\\ &= xy\end{aligned}$$

On the other hand, $x<0, y<0 \Rightarrow xy>0$.

So, $|xy|=xy$

Therefore, $|x|\cdot|y|=|xy|$.

Comment

Step **2** of 3

Case 2:

Suppose $x<0, y>0$, then $|x|=-x, |y|=y$.

So,

$$\begin{aligned}|x||y| &= (-x)(y)\\ &= -xy\end{aligned}$$

On the other hand, $x<0, y>0 \Rightarrow xy<0$.

So, $|xy|=-xy$

Therefore, $|x|\cdot|y|=|xy|$.

Comment

Step **3** of 3

Case 3:

Suppose $x>0, y>0$, then $|x|=x, |y|=y$.

So,

$$\begin{aligned}|x||y| &= (x)(y)\\ &= xy\end{aligned}$$

On the other hand, $x>0, y>0 \Rightarrow xy>0$.

So, $|xy|=xy$

Therefore, $|x|\cdot|y|=|xy|$.

From the above cases, conclude that $|x|\cdot|y|=|xy|$.

4.4

4.4.49

It is given that m and n are integers, d is a positive integer such that $m \bmod d = n \bmod d$

Suppose $m \bmod d = k = n \bmod d$

So from the quotient-remainder theorem, we have

$m = d\,p + k$ and $n = dq + k$, for some integers p and q

If $p \neq q$, then $dp \neq dq$

$\Rightarrow dp + k \neq dq + k$

$\Rightarrow m \neq n$

So from this, we can say that $m \bmod d = n \bmod d$ need not imply $m = n$

Comment

Step 2 of 3

On the other hand, $m - n = (dp + k) - (dq + k)$

$= dp - dq$

$= d\,(p- q)$

Since $p \neq q$, $p - q \neq 0$

So $m - n = d\,(p - q)$

$\Rightarrow m - n$ is an integer multiple of d

In other words, d divides $m - n$

Comment

Step 3 of 3

For instance, 13 mod7 = 20 mod7

However, $13 \neq 20$

4.4

4.4.50

We are given $d \mid (m - n)$

So from the quotient-remainder theorem, we have

$(m - n) = ds + 0.$ ---------- (1)

Suppose $m \bmod d = p$, $n \bmod d = q$, for some integers p and q

i.e., $m = da + p$, $n = db + q$, for some integers a and b

$\Rightarrow m - n = d(a - b) + (p - q)$ ---------- (2)

Now, from the uniqueness of the quotient-remainder theorem, equations (1) and (2) give

$p - q = 0$

$\Rightarrow p = q$

$\Rightarrow m \bmod d = n \bmod d$