حلول بعض الاسئلة المختارة من شابتر 5

ملاحظة: الحل عبارة عن صور وليس نص مكتوب. واذا لم يوجد رقم السؤال فوق الصورة، فتكون الصورة(الحل) مكملة للصورة(للحل) السابقة.

على سبيل المثال سؤال رقم 4 من سكشن 2، له صورتان.

5.1

5.1.4

It is given that $d_m = 1 + \left(\frac{1}{2}\right)^m$ for all integers $m \geq 0$

1) Substituting $m = 0$ into the equation, we get

$$d_0 = 1 + \left(\frac{1}{2}\right)^0 = 1 + 1 = 2 \quad \left(\because \left(\frac{1}{2}\right)^0 = 1\right)$$

$$\Rightarrow d_0 = 2$$

Comment

Step 2 of 4

2) Substituting $m = 1$ into the equation, we get

$$d_1 = 1 + \left(\frac{1}{2}\right)^1 = 1 + \frac{1}{2} \quad \left(\because \left(\frac{1}{2}\right)^1 = \frac{1}{2}\right)$$

$$\Rightarrow d_1 = \frac{3}{2}$$

Comment

Step 3 of 4

3) Substituting $m = 2$ into the equation, we get

$$d_2 = 1 + \left(\frac{1}{2}\right)^2 \quad \left(\because \left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1 \times 1}{2 \times 2} = \frac{1}{4}\right)$$

$$= 1 + \frac{1}{4} = \frac{4+1}{4} = \frac{5}{4}$$

$$\Rightarrow d_2 = \frac{5}{4}$$

Comment

Step 4 of 4

4) Substituting $m = 3$ into the equation, we get

$$d_3 = 1 + \left(\frac{1}{2}\right)^3 = 1 + \frac{1}{8} \quad \left(\because \left(\frac{1}{2}\right)^3 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1 \times 1 \times 1}{2 \times 2 \times 2} = \frac{1}{8}\right)$$

$$= \frac{8+1}{8} = \frac{9}{8}$$

$\therefore$ The first four terms in the sequence $d_m = 1 + \left(\frac{1}{2}\right)^m$ for all integers $m \geq 0$ are

$2, \frac{3}{2}, \frac{5}{4}$, and $\frac{9}{8}$

5.1

5.1.13

It is given that the initial terms of the sequence are $1-\frac{1}{2},\frac{1}{2}-\frac{1}{3},\frac{1}{3}-\frac{1}{4},\frac{1}{4}-\frac{1}{5},\frac{1}{5}-\frac{1}{6},\frac{1}{6}-\frac{1}{7}$

Now, we find the formula for the sequence.

We are given

$a_1 = 1-\frac{1}{2} = \frac{1}{1}-\frac{1}{2}$

$a_2 = \frac{1}{2}-\frac{1}{3}$

$a_3 = \frac{1}{3}-\frac{1}{4}$

$a_4 = \frac{1}{4}-\frac{1}{5}$

$a_5 = \frac{1}{5}-\frac{1}{6}$

$a_6 = \frac{1}{6}-\frac{1}{7}$

$\therefore$ The formula for the sequence $1-\frac{1}{2},\frac{1}{2}-\frac{1}{3},\frac{1}{3}-\frac{1}{4},\frac{1}{4}-\frac{1}{5},\frac{1}{5}-\frac{1}{6},\frac{1}{6}-\frac{1}{7}$ is

$a_n = \frac{1}{n}-\frac{1}{n+1}$ for all integers $n \geq 1$

5.1

5.1.17

Consider that the sequence $\{a_n\}$ is defined as follows:

$a_n = \dfrac{2n+(-1)^n-1}{4}$, for all integers $n \geq 0$. ...... (1)

The objective is to find an explicit formula for a_n that includes floor notation.

Suppose n is even.

Then $n = 2k$, for some integer k.

$$\begin{aligned} a_{2k} &= \frac{2(2k)+(-1)^{2k}-1}{4} \\ &= \frac{4k+1-1}{4} \\ &= \frac{4k}{4} \\ &= k \end{aligned}$$

Since $n = 2k$, so $k = \dfrac{n}{2}$.

Thus, $a_n = \dfrac{n}{2}$, if n is even.

Comment

Step **2** of 2

Suppose n is odd.

Then $n = 2k+1$, for some integer k.

$$\begin{aligned} a_{2k+1} &= \frac{2(2k+1)+(-1)^{2k+1}-1}{4} \\ &= \frac{4k+2-1-1}{4} \\ &= \frac{4k}{4} \\ &= k \end{aligned}$$

Since $n = 2k+1$, so $k = \dfrac{n-1}{2}$.

Thus, $a_n = \dfrac{n-1}{2}$, if n is odd.

Therefore, the formula for a_n can be written as follows:

$$a_n = \begin{cases} \dfrac{n}{2}, & n \text{ is even.} \\ \dfrac{n-1}{2}, & n \text{ is odd.} \end{cases}$$

Thus, by the definition of floor function, a_n is written as, $\boxed{a_n = \left\lfloor \dfrac{n}{2} \right\rfloor}$.

5.1

5.1.18.c-e

(c)

Objective is to find the value of $\sum_{j=1}^{3} a_{2j}$.

$\sum_{j=1}^{3} a_{2j} = a_{2\times1} + a_{2\times2} + a_{2\times3}$;

$= a_2 + a_2 + a_6$;

$= -2 + 0 + (-2)$ Substitute the values.

$= -2 - 2$

$= -4$

Hence, the value of the sequence $\sum_{j=1}^{3} a_{2j} = \boxed{-4}$.

Comment

Step **4** of 5

(d)

Objective is to find the product of the sequence $\prod_{k=0}^{6} a_k$.

$\prod_{K=0}^{6} a_k = a_0 \times a_1 \times a_2 \times a_3 \times a_4 \times a_5 \times a_6$

$= 2 \times 3 \times (-2) \times 1 \times 0 \times (-1) \times -2$ Substitute the given values.

$= -24 \times 0$

$= 0$ Any number multiplied by zero is zero.

Hence, the value of the sequence $\prod_{K=0}^{6} a_k = \boxed{0}$.

Comments (1)

Step **5** of 5

(e)

Objective is to find the product of the sequence $\prod_{k=2}^{2} a_k$.

$\prod_{k=2}^{2} a_k = a_2$

$= -2$ Since $a_2 = -2$.

Hence, the value of the sequence $\prod_{k=2}^{2} a_k = \boxed{-2}$.

5.1

5.1.24

It is given that $\sum_{j=0}^{0}(j+1)\cdot 2^j=(0+1)\cdot 2^0$

$=1\times 1 \qquad (\because 2^0=1 \text{ and } 0+1=1)$

$=1$

$\therefore \sum_{j=0}^{0}(j+1)\cdot 2^j=1$

5.1.27

The objective is to compute the summation of $\sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Rewrite the summation $\sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ as:

$$\sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(\frac{1}{1}-\frac{1}{1+1}\right)+\left(\frac{1}{2}-\frac{1}{2+1}\right)+\left(\frac{1}{3}-\frac{1}{3+1}\right)+\left(\frac{1}{4}-\frac{1}{4+1}\right)$$
$$+\left(\frac{1}{5}-\frac{1}{5+1}\right)+\left(\frac{1}{6}-\frac{1}{6+1}\right)+\left(\frac{1}{7}-\frac{1}{7+1}\right)+\left(\frac{1}{8}-\frac{1}{8+1}\right)$$
$$+\left(\frac{1}{9}-\frac{1}{9+1}\right)+\left(\frac{1}{10}-\frac{1}{10+1}\right)$$

Comment

Step 2 of 2

Simplify the terms as:

$$\sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}+\cancel{\frac{1}{5}}-\cancel{\frac{1}{6}}+\cancel{\frac{1}{6}}-\cancel{\frac{1}{7}}+\cancel{\frac{1}{7}}$$
$$-\cancel{\frac{1}{8}}+\cancel{\frac{1}{8}}-\cancel{\frac{1}{9}}+\cancel{\frac{1}{9}}-\cancel{\frac{1}{10}}+\cancel{\frac{1}{10}}-\frac{1}{11}$$
$$=1-\frac{1}{11}$$
$$=\frac{11-1}{11}$$
$$=\frac{10}{11}$$

Therefore, the sum of $\sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ is $\boxed{\frac{10}{11}}$.

5.1

5.1.28

Consider the following product as,

$$\prod_{i=2}^{5} \frac{i(i+2)}{(i-1)\cdot(i+1)} \text{ ...... (1)}$$

The objective is to compute the above product.

Comment

Step 2 of 2

Substitute the values of i from 2 to 5 in expression (1) and also use the definition

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

$$\prod_{i=2}^{5} \frac{i(i+2)}{(i-1)\cdot(i+1)} = \frac{2(2+2)}{(2-1)\cdot(2+1)} \times \frac{3(3+2)}{(3-1)\cdot(3+1)} \times \frac{4(4+2)}{(4-1)\cdot(4+1)} \times \frac{5(5+2)}{(5-1)\cdot(5+1)}$$

$$= \frac{2\times4}{1\times3} \times \frac{3\times5}{2\times4} \times \frac{4\times6}{3\times5} \times \frac{5\times7}{4\times6}$$

$$= \frac{2\times4\times3\times5\times4\times6\times5\times7}{1\times3\times2\times4\times3\times5\times4\times6}$$

$$= \frac{5\times7}{1\times3}$$

$$= \frac{35}{3}$$

Therefore, the required value of the product is,

$$\prod_{i=2}^{5} \frac{i(i+2)}{(i-1)(i+1)} = \boxed{\frac{35}{3}}.$$

5.1

5.1.32

Consider the summation:

$\sum_{i=1}^{k+1} i(i!)$.

Objective is to write this summation in expanded form.

Write the given sequence $\sum_{i=1}^{k+1} i(i!)$ in expanding form.

Since $\sum_{i=1}^{k+1}$ is the sum of given term define with respect to i range from 1 to $(k+1)$.

Therefore, in expanding form $\sum_{i=1}^{k+1} i(i!)$ can be written as follows:

$$\sum_{i=1}^{k+1} i(i!) = \boxed{1(1!)+2(2!)+3(3!)+4(4!)+5(5!)+\cdots+k(k!)+(k+1)(k+1)!}.$$

5.1.34

In this problem we have to evaluate the given sum

$$1(1!)+2(2!)+3(3!)+.........+m(m!);\ m=2.$$

Since for $m=2$, the given sum has only two term, i.e. first two term and hence the sum is

$$\begin{aligned} 1(1!)+2(2!) &= 1.1+2.(2.1) \qquad [2!=2.1] \\ &= 1+4 \\ &= 5 \end{aligned}$$

$$\therefore\ 1(1!)+2(2!)=5$$

5.1

5.1.36

Consider the following product of sequence:

$$\left(\frac{1\cdot 2}{3\cdot 4}\right)\left(\frac{4\cdot 5}{6\cdot 7}\right)\left(\frac{6\cdot 7}{8\cdot 9}\right)\cdots\left(\frac{m\cdot(m+1)}{(m+2)\cdot(m+3)}\right).$$

Evaluate the product for $m=1$.

The **product notation** for the given product of sequence is as follows:

$$\prod_{k=1}^{m}\left(\frac{k\cdot(k+1)}{(k+2)\cdot(k+3)}\right).$$

Comment

Step 2 of 3

For $m=1$, the given product has only one term. Substitute $m=1$ in the product notation.

$$\prod_{k=1}^{m}\left(\frac{k\cdot(k+1)}{(k+2)\cdot(k+3)}\right)=\prod_{k=1}^{1}\left(\frac{k\cdot(k+1)}{(k+2)\cdot(k+3)}\right)$$

$$=\frac{1\cdot(1+1)}{(1+2)\cdot(1+3)} \text{ Since, } \prod_{k=n}^{n}a_k=a_n$$

$$=\frac{1\cdot 2}{(3).(4)}$$

$$=\frac{2}{12}$$

$$=\boxed{\frac{1}{6}}.$$

Comment

Step 3 of 3

Or simply, when $m=1$, the value of the given product of sequence is,

$$\left(\frac{m\cdot(m+1)}{(m+2)\cdot(m+3)}\right)=\left(\frac{1\cdot(1+1)}{(1+2)\cdot(1+3)}\right)$$

$$=\frac{1\cdot 2}{3\cdot 4}$$

$$=\frac{2}{12}$$

$$=\boxed{\frac{1}{6}}.$$

5.1

5.1.39

Consider the following summation:

$$\sum_{m=1}^{n+1} m(m+1).$$

Rewrite the summation by separating off the final term.

Recall, the **Recursive definition of summation**:

Suppose m is any integer, then the following holds true:

1. $\sum_{k=m}^{m} a_k = a_m$, and,
2. $\sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n$

For all integers $n > m$.

Comment

Step 2 of 2

Assume, $a_m = m(m+1)$. Then, the given summation can be written as follows:

$$\sum_{m=1}^{n+1} a_m$$

Here, $a_m = m(m+1)$.

Apply, the Recursive definition of summation (2).

$$\begin{aligned}\sum_{m=1}^{n+1} a_m &= \sum_{m=1}^{n} a_m + a_{n+1} \\ &= \sum_{m=1}^{n} m(m+1) + [(n+1)((n+1)+1)] \quad \text{Since, } a_m = m(m+1). \\ &= \sum_{m=1}^{n} m(m+1) + [(n+1)(n+1+1)] \\ &= \sum_{m=1}^{n} m(m+1) + [(n+1)(n+2)]\end{aligned}$$

Therefore, the summation after separating off the final term is as follows:

$$\sum_{m=1}^{n+1} m(m+1) = \boxed{\sum_{m=1}^{n} m(m+1) + [(n+1)(n+2)]}.$$

5.1

5.1.42

We have to rewrite the given sequence $\sum_{m=0}^{n}(m+1)2^m+(n+2)2^{n+1}$ as a single term summation.

Note that when $m=n+1$, $(m+1)2^m=(n+2)2^{n+1}$.

Therefore,

$$\sum_{m=0}^{n}(m+1)2^m+(n+2)2^{n+1}=\sum_{m=0}^{n+1}(m+1)2^m$$

5.1.48

Consider the following expression:

$(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)$.

Remember that, the notation for the product of a sequence of numbers analogous to the notation for their sum, $\prod$ denotes the product.

$$\prod_{k=1}^{4}a_k=a_1a_2a_3a_4$$

Comment

Step 2 of 2

To write the product notation for $(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)$.

The general term of this product can be expressed as $1-t^k$, for integers *k* from 1 to 4.

$(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)=(1-t^1)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)$ Write $t=t^1$

$$(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)=\boxed{\prod_{k=1}^{4}(1-t^k)}$$

5.1

5.1.52

Consider the sequence $n+\frac{n-1}{2!}+\frac{n-2}{3!}+\frac{n-3}{4!}+\cdots+\frac{1}{n!}$.

The objective is to write the summation notation of the given sequence.

It is given that,

$$n+\frac{n-1}{2!}+\frac{n-2}{3!}+\frac{n-3}{4!}+\cdots+\frac{1}{n!}$$

$$=\frac{n-0}{1}+\frac{n-1}{2!}+\frac{n-2}{3!}+\frac{n-3}{4!}+\cdots+\frac{n-(n-1)}{n!}$$

$$=\frac{n-0}{1!}+\frac{n-1}{2!}+\frac{n-2}{3!}+\frac{n-3}{4!}+\cdots+\frac{n-(n-1)}{n!}$$

$$=\sum_{k=0}^{n-1}\frac{n-k}{(k+1)!}$$

Hence, the summation notation of the given sequence is,

$$n+\frac{n-1}{2!}+\frac{n-2}{3!}+\frac{n-3}{4!}+\cdots+\frac{1}{n!}=\boxed{\sum_{k=0}^{n-1}\frac{n-k}{(k+1)!}}.$$

5.1.54

It is given that

$$\prod_{k=1}^{n}\frac{k}{k^2+4}$$

It is also given that $i=k+1\Rightarrow k=i-1$

Now,

If $k=1$, then $i=1+1=2\Rightarrow i=2$

If $k=n$, then $i=n+1$

And

$$\frac{k}{k^2+4}=\frac{(i-1)}{(i-1)^2+4}\quad(\because k=i-1)$$

$$\therefore\prod_{k=1}^{n}\frac{k}{k^2+4}=\prod_{i=2}^{n+1}\frac{i-1}{(i-1)^2+4}$$

5.1

5.1.57

Consider the limit $\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$.

When $i=1$, then $j=0$

When $i=n-1$, then $j=n-2$

Since $j=i-1$, then $i=j+1$

Comment

Step 2 of 2

Thus, $\frac{i}{(n-i)^2}$ can be expressed as follows:

$$\frac{i}{(n-i)^2} = \frac{j+1}{(n-(j+1))^2} \qquad (i=j+1)$$

$$= \frac{j+1}{(n-j-1)^2}$$

So, $\sum_{i=1}^{n-1} \frac{i}{(n-i)^2} = \boxed{\sum_{j=0}^{n-2} \frac{j+1}{(n-j-1)^2}}$

5.1.60

It is given that

$$2\sum_{k=1}^{n}(3k^2+4)+5\sum_{k=1}^{n}(2k^2-1)$$

$$=\sum_{k=1}^{n}2(3k^2+4)+\sum_{k=1}^{n}5(2k^2-1)$$

$$=\sum_{k=1}^{n}(6k^2+8)+\sum_{k=1}^{n}(10k^2-5)$$

$$=\sum_{k=1}^{n}(6k^2+8+10k^2-5) \quad \left(\because \sum_{k=m}^{n}a_k+\sum_{k=m}^{n}b_k=\sum_{k=m}^{n}(a_k+b_k)\right)$$

$$=\sum_{k=1}^{n}(6k^2+10k^2+8-5)$$

$$=\sum_{k=1}^{n}(16k^2+3)$$

$$\therefore 2\sum_{k=1}^{n}(3k^2+4)+5\sum_{k=1}^{n}(2k^2-1)=\sum_{k=1}^{n}(16k^2+3)$$

5.1

5.1.68

Here the expression is

$$\frac{((n+1)!)^2}{(n!)^2}$$

To compute the term follow step as below:

$$\frac{((n+1)!)^2}{(n!)^2} = \frac{[(n+1)n(n-1)(n-2)...3.2.1]^2}{[n(n-1)(n-2)...3.2.1]^2}$$

$$= \frac{(n+1)^2 [n(n-1)(n-2)...3.2.1]^2}{[n(n-1)(n-2)...3.2.1]^2}$$

$$= \boxed{(n+1)^2}$$

5.1.69

It is given that

$$\frac{n!}{(n-k)!} = \frac{n\cdot(n-1)(n-2)............(n-k+1)(n-k)(n-k-1)...........3\cdot2\cdot1}{(n-k)\cdot(n-k-1)...............3\cdot2\cdot1}$$

$$= n(n-1)(n-2)............(n-k+1)$$

$$\therefore \frac{n!}{(n-k)!} = n(n-1)(n-2)............(n-k+1)$$

5.1

5.1.70

Consider the fraction $\frac{n!}{(n-k+1)!}$.

The objective is to compute the value of the given fraction.

The definition of factorial, denoted by $n!$ for each positive integer n, states that the factorial of n is the product of all the integers from 1 to n.

$$n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$$

Comment

Step 2 of 2

It is given that,

$$\frac{n!}{(n-k+1)!} = \frac{n(n-1)\cdots(n-k+2)(n-k+1)(n-k)\cdots 3\cdot 2\cdot 1}{(n-k+1)(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1}$$

$$= \frac{n(n-1)\cdots(n-k+2)\,\cancel{(n-k+1)(n-k)\cdots 3\cdot 2\cdot 1}}{\cancel{(n-k+1)(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1}}$$

$$= n(n-1)(n-2)\cdots(n-k+2).$$

Hence, the value of the given fraction is $\frac{n!}{(n-k+1)!} = \boxed{n(n-1)(n-2)\cdots(n-k+2)}$.

5.1.72

Here the expression is

$$\binom{7}{4}$$

We have from definition that, the combination expression is given as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Therefore, the expression can be computed as below:

$$\binom{7}{4} = \frac{7!}{4!(7-4)!}$$

$$= \frac{7!}{4!3!}$$

$$= \frac{7\cdot 6\cdot 5\cdot \cancel{4!}}{\cancel{4!}(3\cdot 2\cdot 1)}$$

$$= \frac{(7\cdot 5)\cdot \cancel{6}}{\cancel{6}}$$

$$= \boxed{35}$$

5.1

5.1.76

The objective is to compute $\binom{n+1}{n-1}$.

For all integers n and r with $0 \le r \le n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Comment

Step 2 of 2

Apply this formula to compute $\binom{n+1}{n-1}$.

$$\begin{aligned}\binom{n+1}{n-1} &= \frac{(n+1)!}{(n-1)!((n+1)-(n-1))!} \\ &= \frac{(n+1)!}{(n-1)!(\cancel{n}+1-\cancel{n}+1)!} \\ &= \frac{(n+1)!}{(n-1)!2!} \\ &= \frac{(n+1)n(n-1)!}{(n-1)!2!}\end{aligned}$$

$$= \frac{(n+1)n}{2!} \qquad \text{cancelling the common factor } (n-1)!$$

$$= \frac{(n+1)n}{2}$$

Hence, the value of $\boxed{\binom{n+1}{n-1} = \frac{(n+1)n}{2}}$.

5.1

5.1.77

(a)

The objective is to prove that $n!+2$ is divisible by 2, for all integers $n \geq 2$.

Let n be an integer greater than 2. $n \geq 2$.

By the definition of factorial, $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$.

As $n \geq 2$, the expansion $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ contains the factor 2.

$$\begin{aligned} n!+2 &= \left[n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1\right] + 2 \\ &= 2\left[n \cdot (n-1) \cdot \ldots \cdot 4 \cdot 3 \cdot 1 + 1\right] \end{aligned}$$

Take $s = n \cdot (n-1) \cdot \ldots \cdot 4 \cdot 3 \cdot 1 + 1$, then $n!+2 = 2s$, for some integer s.

Thus, $n!+2$ is divisible by 2.

Comment

Step **2** of 3

(b)

The objective is to prove that $n!+k$ is divisible by k, for all integers $n \geq 2$ and $k = 2, 3, \ldots, n$.

Let n be an integer greater than 2. $n \geq 2$.

Let k be an integer between 2 and n.

By the definition of factorial, $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$.

As $2 \leq k \leq n$, the expansion $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ contains the factor k.

$$\begin{aligned} n!+k &= \left[n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1\right] + k \\ &= k\left[n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 + 1\right] \end{aligned}$$

Take $s = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 + 1$, then $n!+2 = 2s$, for some integer s.

Thus, $n!+k$ is divisible by k.

Comment

Step **3** of 3

(c)

The objective is to explain whether it is possible to find a sequence of $m-1$ consecutive positive integers that are not prime for $m \geq 2$.

From part **(b)**, $m!+k$ is divisible by k when $k = 2, 3, \ldots, n$.

Note that $m!+k$ are k consecutive positive integers of which none are prime.

Yes, there exists a sequence of $m-1$ consecutive positive integers that are not primes.

5.1

5.1.78

For all integer n and r with $0 \le r+1 \le n$,

$$\binom{n}{r+1} = \frac{n!}{(r+1)!(n-(r+1))!} \quad \ldots(1)$$

Comment

Step 2 of 2

Consider the right hand side of (1),

$$\begin{aligned}
&\frac{n!}{(r+1)!(n-(r+1))!} \\
&= \frac{(n-r)n!}{(r+1)(r)!(n-r)((n-r)-1)!} \quad \begin{bmatrix}\text{multiply } (n-r) \text{ on both} \\ \text{numerator and denomenator}\end{bmatrix} \\
&= \frac{n-r}{r+1}\frac{n!}{(r)!(n-r)!} \\
&= \frac{n-r}{r+1}\binom{n}{r}
\end{aligned}$$

Therefore $\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$.

5.1

5.1.79

Consider a prime number p.

An integer r that is lies between 0 and the prime number p.

i.e., $0 < r < p$.

the objective is to prove that $\binom{p}{r} = {}^{p}C_{r}$ is divisible by p.

Comment

Step 2 of 3

Consider the formula for combination $\binom{p}{r}$ is,

$$\binom{p}{r} = \frac{p!}{r!(p-r)!}$$
$$= \frac{1 \cdot 2 \cdots (p-1) \cdot p}{[1 \cdot 2 \cdots r] \cdot [1 \cdot 2 \cdots (p-r)]}$$

The numerator and the denominator have a prime factorization.

Now, observe that in the above one of the prime must be present in the numerator and all the primes in the denominator are less than p because all the numbers in the product are less than p.

Comment

Step 3 of 3

From the known factorial notation,

$$n! = n \cdot (n-1)!$$

Write the factorial $p!$ as $p \cdot (p-1)!$ in the above combination as,

$$\binom{p}{r} = \frac{p!}{r!(p-r)!}$$
$$= \frac{p \cdot (p-1)!}{r!(p-r)!}$$
$$= p \cdot \frac{(p-1)!}{r!(p-r)!}$$
$$= p \cdot \left[\frac{(p-1)!}{r!(p-r)!}\right]$$

It is divisible by p.

Observe that there is no p in the denominator.

Therefore, p cannot be cancel out.

This shows that the obtained prime factorization of the integer $\binom{p}{r}$ will contain a p.

Hence, $\binom{p}{r}$ divisible by p.

5.2

5.2.4

(a)

Consider the formula for $P(n)$ where n is positive integer with $n \geq 2$.

$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$.

Objective is to find the value of P at $n = 2$.

$P(n) = \frac{n(n-1)(n+1)}{3}$

$P(2) = \frac{2\times(2-1)\times(2+1)}{3}$ Substitute $n = 2$.

$= \frac{2\times1\times3}{3}$

$= 2$

$P(2) = 2$

Comment

Step 2 of 6

Clearly, from left hand side;

$P(2) = \sum_{i=1}^{2-1} i(i+1)$

$= \sum_{i=1}^{1} i(i+1)$

$= 1\times(1+1)$

$= 1\times2$

$= 2$

Hence, it is true that $\boxed{P(2) = 2}$.

Comment

Step 3 of 6

(b)

Objective is to write the formula for $P(k)$.

Substitute $n = k$ in the formula $\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$.

$P(k)$ is $\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$.

5.2

(c)

Objective is to write the formula for $P(k+1)$.

Substitute $n=k+1$ in the formula $\sum_{i=1}^{n-1} i(i+1)=\frac{n(n-1)(n+1)}{3}$.

$P(k+1)$ is $\sum_{i=1}^{(k+1)-1} i(i+1)=\frac{(k+1)((k+1)-1)((k+1)+1)}{3}$

$\sum_{i=1}^{k} i(i+1)=\frac{(k+1)(k)(k+2)}{3}$.

Comment

Step 5 of 6

(d)

Objective is to identify what must be shown in the inductive step.

Inductive step:

If the property $P(n)$ is true for $n=k$, then it is prove that $P(n)$ is also true for $n=k+1$.

i.e., for some integer $k \geq 2$

Here $k=n-1$

If $\sum_{i=1}^{k-1} i(i+1)=\frac{k(k-1)(k+1)}{3}$, then prove that

$\sum_{i=1}^{(k+1)-1} i(i+1)=\frac{(k+1)((k+1)-1)((k+1)+1)}{3}$.

$\sum_{i=1}^{k} i(i+1)=\frac{(k+1)(k)(k+2)}{3}$

Comment

Step 6 of 6

Substitute $k=n-1$ in the above formula,

$\sum_{i=1}^{n-1} i(i+1)=\frac{(n-1+1)(n-1)(n-1+2)}{3}$

$\sum_{i=1}^{n-1} i(i+1)=\frac{(n)(n-1)(n+1)}{3}$

Hence, it is true for $n=k+1$.

Therefore, the inductive step is verified.

5.2

5.2.7

Let the statement is, "for all integer $n \geq 1$,

$$1+6+11+16+\cdots+(5n-4)=\frac{n(5n-3)}{2}$$

Proof: For the given statement, the property $P(n)$ is the equation, i.e.,

$$P(n)=1+6+11+16+\cdots+(5n-4)=\frac{n(5n-3)}{2}$$

Step 1: Show that $P(1)$ is true.

To prove $P(1)$, it must be shown that when 1 is substituted into the equation in place of *n*,

The left-hand side equals the right-hand side.

Now the left-hand side of $P(1)$ is 1, and the right-hand side is $\frac{1\cdot(5(1)-3)}{2}=\frac{2}{2}=1$ also.

Thus, $P(1)$ is true.

Comment

Step 2 of 3

Step 2: Show that for all integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.

Let *k* be any integer with $k \geq 1$ and suppose $P(k)$ is true, i.e.,

$$1+6+11+16+\cdots+(5k-4)=\frac{k(5k-3)}{2} \text{ (inductive hypothesis)}$$

Now, it must be shown that $P(k+1)$ is true, i.e.,

$$1+6+11+16+\cdots+(5(k+1)-4)=\frac{(k+1)(5(k+1)-3)}{2}$$

Or, equivalently,

$$P(k+1)=1+6+11+16+\cdots+(5(k+1)-4)=\frac{(k+1)(5k+2)}{2}$$

$$=\frac{5k^2+7k+2}{2}$$

5.2

But the left-hand side of $P(k+1)$ is

$1+6+11+16+\cdots+(5(k+1)-4)$

$=1+6+11+16+\cdots+(5k-4)+(5(k+1)-4)$ (by making the next-to-last term explicit)

$=\dfrac{k(5k-3)}{2}+(5(k+1)-4)$ (by substitution from the inductive hypothesis)

$=\dfrac{k(5k-3)}{2}+(5k+1)$

$=\dfrac{5k^2-3k+10k+2}{2}$

$=\dfrac{5k^2+7k+2}{2}$

And this is right-hand side of $P(k+1)$.

Hence the property is true for $n=k+1$.

Thus, $P(n)$ is true for all the integers $n\ge 1$.

5.2

5.2.9

Consider the statement:

$$P(n):\ 4^3+4^4+4^5+\cdots+4^n=\frac{4(4^n-16)}{3}$$ For all integers $n\geq 3$.

Objective is to prove the above statement by using Mathematical induction.

Basis step: Prove that the property is true for $n=3$.

LHS:

The left-hand side is the sum of all terms from 4^3 to 4^3, i.e., just 4^3, itself.

RHS:

$$\frac{4(4^3-16)}{3}=\frac{4\times(64-16)}{3} \qquad (4^3=4^2\times4=16\times4=64)$$

$$=\frac{4\times48}{3}$$

$$=\frac{4\times3\times16}{3}$$

$$=4\times16$$

$$=4\times4^2$$

$$=4^3$$

LHS = RHS.

Hence, the statement $P(n)$ is true for $n=3$.

Comment

Step 2 of 3

Suppose that the property is true for $n=k$.

That is, the statement $P(k):4^3+4^4+4^5+\cdots+4^k=\frac{4(4^k-16)}{3}$ is true.

Now, we can show that

$$4^3+4^4+4^5+\cdots+4^k+4^{k+1}=\frac{4(4^{k+1}-16)}{3} \qquad \rightarrow(1)$$

5.2

LHS of (1):

$$\text{LHS} = 4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1}$$

$$= \left[4^3 + 4^4 + 4^5 + \cdots + 4^k\right] + 4^{k+1}$$

$$= \frac{4\left(4^k - 16\right)}{3} + 4^{k+1}$$

$$= \frac{4\left(4^k - 16\right) + 3 \times 4^{k+1}}{3} \qquad \text{Take } LCM.$$

$$= \frac{4\left(4^k - 16\right) + 3 \times 4^k \times 4}{3} \qquad \left(\text{Since } a^{m+n} = a^m \times a^n\right)$$

$$= \frac{4\left[4^k - 16 + 3 \times 4^k\right]}{3} \qquad \text{Take 4 as common}$$

$$= \frac{4\left[4^k(1+3) - 16\right]}{3}$$

$$= \frac{4\left(4^k \times 4 - 16\right)}{3}$$

$$= \frac{4\left(4^{k+1} - 16\right)}{3}$$

$$= \text{RHS of (1)}$$

$$4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1} = \frac{4\left(4^{k+1} - 16\right)}{3}.$$

Thus, the statement $P(n)$ is true for $n = k + 1$.

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for all $n \geq 3$.

5.2

5.2.15

Consider

$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$, for all integers $n \geq 1$

Let the property $P(n)$ be the equation $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$,

To show that $P(n)$ is true for all integers $n \geq 1$. Do this by using mathematical induction.

Remember that $n! = 1 \times 2 \times 3 \times .... \times (n-1) \times n$

Comment

Step 2 of 4

Show that $P(1)$ is true:

That is to show that $1(1!) = (1+1)! - 1$ $P(1)$

The left hand side of the equation is $1(1!) = 1$ and right-hand side is

$$(1+1)! - 1 = 2! - 1$$
$$= 2 - 1$$
$$= 1$$

It follows that $1 = 1$

Hence $P(1)$ is true.

5.2

Show that for all integers $n \geq 1$, $P(k)$ is true then $P(k+1)$ is also true:

Suppose $P(k)$ is true.

Then the inductive hypothesis is

$$\sum_{i=1}^{k} i(i!) = (k+1)! - 1, \quad (k \geq 1)$$

Now show that $P(k+1)$ is true.

That is to show that

$$\sum_{i=1}^{k+1} i(i!) = ((k+1)+1)! - 1,$$

Or, equivalently that

$$\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1,$$

The left-hand side of $P(k+1)$ is

$\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)((k+1)!)$ Write into two terms

$= (k+1)! - 1 + (k+1)((k+1)!)$ By $P(k)$

$= (k+1)!\left[1 + (k+1)\right] - 1$ Taking common term $(k+1)!$

$= (k+1)!(k+2) - 1$ Simplify

$= (k+2)! - 1$ $n! = n(n-1)!$

which is right hand side of $P(k+1)$

Comment

Step 4 of 4

Hence from the principle of mathematical induction,

$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$, is true, for all integers $n \geq 1$

5.2

5.2.17

Consider the statement:

For all integers $n \geq 0$, $\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!}$.

Objective is to prove this statement by using mathematical induction.

$$\prod_{i=0}^{n}\left\{\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right\}=\frac{1}{(2n+2)!} \qquad \forall n \geq 0$$

Suppose that the given statement is $P(n)$.

Substitute $n=0$ in the statement.

$$\begin{aligned} P(0) &= \frac{1}{2\cdot 0+1}\cdot\frac{1}{2\cdot 0+2} \\ &= \frac{1}{1}\cdot\frac{1}{2} \\ &= \frac{1}{2} \\ &= \text{LHS} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{1}{(2\cdot 0+2)!} \\ &= \frac{1}{2!} \\ &= \frac{1}{2} \end{aligned}$$

$$\text{LHS} = \text{RHS}$$

Hence, the statement is true for $P(0)$. $\cdots\cdots(1)$

5.2

Assume that the statement is true for $n = m$.

i.e., $P(m) = \prod_{i=0}^{m}\left\{\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right\} = \frac{1}{(2m+2)!}$ ......(2)

When $n = m+1$, consider the product of the $(m+1)$ terms.

i.e., multiply the $m+1^{th}$ term on both sides of equation (2).

$$\prod_{i=0}^{m}\left\{\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right\}\cdot\frac{1}{2(m+1)+1}\cdot\frac{1}{2(m+1)+2} = \frac{1}{(2m+2)!}\cdot\frac{1}{2(m+1)+1}\cdot\frac{1}{2(m+1)+2}$$

$$\prod_{i=0}^{m+1}\left\{\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right\} = \frac{1}{(2m+2)!}\cdot\left\{\frac{1}{2(m+1)+1}\cdot\frac{1}{2(m+1)+2}\right\}$$

$$\prod_{i=0}^{m+1}\left\{\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right\} = \frac{1}{(2m+2)!}\cdot\left\{\frac{1}{2m+2+1}\cdot\frac{1}{2m+2+2}\right\}$$

$$= \frac{1}{(2m+2)!}\cdot\frac{1}{(2m+3)(2m+4)}$$

$$= \frac{1}{(2m+4)!}$$

$$= \frac{1}{(2(m+1)+2)!}$$

This expression is in the required form.

So, the statement is true when $n = m+1$. ......(3)

From equations (1), (2), and (3) satisfy the hypotheses of mathematical induction.

So, by the result of mathematical induction, the given statement is true for all integers $n \geq 0$.

i.e., $\prod_{i=0}^{n}\left\{\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right\} = \frac{1}{(2n+2)!}\ \forall n \geq 0.$

5.2

5.2.21

Consider a sequence $5+10+15+...+300$.

The objective is use the formula for the sum of the first n integers and to write the sequence in closed form.

Comment

Step 2 of 2

Rewrite the sequence $5+10+15+...+300$ as,

$5(1+2+3+...+60)$.

The sum of the sequence of the first n integers $1+2+3+...+n$ in closed form is, $\frac{n(n+1)}{2}$.

Apply the formula for the sum of the first n integers over $5(1+2+3+...+60)$ with $n=60$.

Therefore, the sum is,

$$\begin{aligned} 5(1+2+3+...+60) &= 5\left(\frac{60(60+1)}{2}\right) \\ &= 5\left(\frac{60(61)}{2}\right) \\ &= 5\left(\frac{3660}{2}\right) \\ &= 5(1830) \end{aligned}$$

$=9150$

Hence, the sum of the sequence $5+10+15+...+300$ in closed form is $\boxed{9150}$.

5.2

5.2.23

Consider the series $7+8+9+\cdots+600$.

The objective is to find the sum of the first n integers.

The sum of the first n integers is $1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

Rewrite the series as follows:

$$7+8+9+\cdots+600=(1+2+3+4+\cdots+600)-(1+2+3+4+5+6)$$

The sum of the first 600 integers is,

$$\begin{aligned}1+2+3+4+\cdots+600&=\frac{600(600+1)}{2}\\&=\frac{600(601)}{2}\\&=300(601)\end{aligned}$$

Comment

Step 2 of 2

Thus, the sum of the series is:

$$\begin{aligned}7+8+9+\cdots+600&=(1+2+3+4+\cdots+600)-(1+2+3+4+5+6)\\&=\frac{600(601)}{2}-21\\&=300(601)-21\\&=180{,}279\end{aligned}$$

Hence, the sum of the series is $\boxed{180{,}279}$.

5.2

5.2.27

Consider the series $5^3+5^4+5^5+\cdots+5^k$.

Here, k is an integer with $k\geq 3$.

The objective is to find the sum of the above series.

The series $5^3+5^4+5^5+\cdots+5^k$ is a geometric series.

Rewrite the above series as follows:

$$5^3+5^4+5^5+\cdots+5^k=5^3\left(1+5+5^2+5^3+\cdots+5^{k-3}\right).$$

Comment

Step 2 of 3

Consider the geometric series as,

$$1+r+r^2+r^3+\cdots+r^n.$$

Here, r is any real number except 1 and the integer $n\geq 1$.

The formula for the sum of a geometric series is,

$$\sum_{i=0}^{n} r^i=\frac{r^{n+1}-1}{r-1}.$$

Comment

Step 3 of 3

Now, consider the geometric series as,

$$1+5+5^2+5^3+\cdots+5^{k-3}$$

Apply the formula for the sum of a geometric series to the above geometric series with $r=5$.

Thus, the sum of the series $5^3+5^4+5^5+\cdots+5^k$ is,

$$\begin{aligned}5^3+5^4+5^5+\cdots+5^k &= 5^3\left(1+5+5^2+5^3+\cdots+5^{k-3}\right)\\ &= 5^3\cdot\frac{5^{(k-3)+1}-1}{5-1} && \text{Substituite } r=5\\ &= 5^3\cdot\frac{5^{k-2}-1}{4} && \text{Simplify}\\ &= \frac{125}{4}\left(5^{k-2}-1\right)\end{aligned}$$

Thus, the required sum of the series is, $\boxed{\frac{125}{4}\left(5^{k-2}-1\right)}$.

5.2

5.2.31

Consider the formula for m and n are integers with $n \geq 0$ and a and r are real numbers.

$$ar^m + ar^{m+1} + ar^{m+2} + \cdots + ar^{m+n}$$

Objective is to find a formula for the sum.

$$ar^m + ar^{m+1} + ar^{m+2} + \cdots + ar^{m+n} = a\left(r^m + r^{m+1} + r^{m+2} + \cdots + r^{m+n}\right)$$

Take a as common

$$= a\left(r^m + r^m r^1 + r^m r^2 + \cdots + r^m r^n\right)$$

$$\left[\text{Use } x^{a+b} = x^a x^b\right]$$

$$= ar^m\left(1 + r + r^2 + \cdots + r^n\right)$$

Take r^m as common.

$$= ar^m\left(\frac{r^{n+1} - 1}{r - 1}\right)$$

Therefore, the formula for sum is $\boxed{ar^m\left(\frac{r^{n+1} - 1}{r - 1}\right)}$.

5.3

5.3.2

General formula

$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=n+1$ for all integers $n \geq 1$

Proof (by mathematical induction)

The property is the equation

$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=n+1$ for all integers $n \geq 1$

Show that the property is true for $n=1$

LHS

$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=\left(1+\frac{1}{1}\right)=1+1=2$

RHS

When $n=1,\ n+1=1+1=2$

$\therefore$ LHS = RHS

$\therefore$ The property is true for $n=1$

Comment

Step **2** of 5

Show that for all integers $k \geq 1$, if the property is true for $n=k$, then it is also true for $n=k+1$

Assume that $\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)=(k+1)$ for some integer $k \geq 1$ (the inductive hypothesis)

We must show that

$$\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)=(k+1)+1$$

$$=k+2 \qquad \rightarrow (1)$$

Comment

Step **3** of 5

LHS of (1)

$$\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right)=\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)\left(1+\frac{1}{k+1}\right)$$

$=(k+1)\cdot\left(1+\frac{1}{k+1}\right)$ ($\therefore$ by substituting the inductive hypothesis)

5.3

$$=(k+1)\left(\frac{(k+1)+1}{k+1}\right)$$

$$=(k+1)\left(\frac{k+2}{k+1}\right)$$

$$=k+2$$

= RHS of (1)

Thus, the property is true for $n=k+1$

Comment

Step 5 of 5

$\therefore$ The property is true for all integers $n \geq 1$

i.e., $\prod_{i=1}^{n}\left(1+\frac{1}{i}\right)=n+1$ for all integers $n \geq 1$

5.3

5.3.7

Consider that $P(n)$ is the property $2^n < (n+1)!$, where *n* is a positive integer.

a)

The objective is to verify whether $P(2)$ is true or not.

Let $n = 2$.

Then, substitute $n = 2$ in $P(n)$.

Therefore, $P(2)$ is $2^2 < (2+1)!$

$4 < (3)!$

$4 < 6$

Clearly, $2^2 < (2+1)!$.

Thus, $P(2)$ is true.

Comment

Step 2 of 4

b)

The objective is to write the expression for $P(k)$.

Let $n = k$.

Then, substitute $n = k$ in $P(n)$.

Therefore, the expression for $P(k)$ is $\boxed{2^k < (k+1)!}$.

5.3

c)

The objective is to write the expression for $P(k+1)$.

Let $n = k+1$.

Then, substitute $n = k+1$ in $P(n)$.

Therefore, the expression for $P(k+1)$ is $\boxed{2^{k+1} < ((k+1)+1)!}$.

Comment

Step 4 of 4

d)

To show that the result is true for all integers $n \geq 2$, first assume that the result is true for $k \geq 2$.

That is $2^k < (k+1)!$, for $k \geq 2$.

Prove that that result is true for $n = k+1$ as,

Consider

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2 \\ &< ((k+1)!) \cdot 2 && \text{Since by assumption } 2^k < (k+1)! \\ &< (k+2)! && \text{Since } ((k+2)!) \cdot 2 < (k+2)!,\ \forall k \geq 2 \\ &< ((k+1)+1)! \end{aligned}$$

Therefore, the value $2^{k+1} < ((k+1)+1)!$ for all $k \geq 2$

That is, $2^{k+1} < ((k+1)+1)!$ is true, for $n = k+1$.

Therefore, the inductive step is $2^{k+1} < ((k+1)+1)!$, for $n = k+1$.

Hence, the statement $P(n)$ is true for all $n \geq 2$

5.3

5.3.9

Consider the statement as,

" $P(n): 7^n - 1$ is divisible by 6," for each integer $n \geq 0$.

The objective is to prove the above statement by mathematical induction.

Basis step:

For $n = 0$,

$P(0): 7^0 - 1 = 1 - 1 = 0$

Thus, 0 is divisible by 6 since zero is divisible by all integers.

Hence, $P(0)$ is true.

Inductive step:

Let $P(k)$ be true for some k for an arbitrary integer $k \geq 0$.

That is, $P(k): 7^k - 1$ is divisible by 6.

Thus, by the definition of divisibility, there exists an integer p, such that $7^k - 1 = 6p$.

Comment

Step 2 of 2

Now, need to prove that $P(k+1)$ is true.

Consider $7^{k+1} - 1$.

$$
\begin{aligned}
7^{k+1} - 1 &= 7^k \times 7 - 1 \\
&= 7 \times (7^k) - 1 \\
&= 7 \times (7^k - 1 + 1) - 1
\end{aligned}
$$

$$
\begin{aligned}
&= 7 \times (6p + 1) - 1 \quad (\text{Since } 7^k - 1 = 6p) \\
&= 42p + 7 - 1 \\
&= 42p + 6 \\
&= 6(7p + 1)
\end{aligned}
$$

The value of $(7p+1)$ is an integer since product or sum of integers is also an integer.

Hence, by the definition of divisibility, $7^{k+1} - 1$ is divisible by 6 since there is an integer $q = 7p + 1$ Such that $7^{k+1} - 1 = 6q$.

Thus, $P(k+1)$ is true.

Therefore, by the principle of mathematical induction, the statement

" $P(n): 7^n - 1$ is divisible by 6," is true for each integer $n \geq 0$.

5.3

5.3.10

Proof (by mathematical induction)

For the given statement, the property is the sentence " n^3-7n+3 is divisible by 3 for each integer $n \geq 0$ "

Show that the property is true for $n=0$

When $n=0$, the property is the sentence

$0^3-7\times 0+3$

This is divisible by 3

However, $0^3-7\times 0+3 = 0-0+3$

$=3$

And 3 is divisible by 3 because $3=3\times 1$

Thus, the property is true for $n=0$

Comment

Step 2 of 3

Show that for any integer $k \geq 0$, if the property is true for $n=k$, then it is also true for $n=k+1$

Let *k* be any integer where $k \geq 0$

Assume that the property is true for $n=k$

i.e., assume that " k^3-7k+3 is divisible by 3" is true (the inductive hypothesis)

We must show that the property is true for $n=k+1$

i.e., we must show that " $(k+1)^3-7(k+1)+3$ is divisible by 3"

Now,

$(k+1)^3-7(k+1)+3$

$=k^3+3k^2+3k+1-7k-7+3 \quad \left(\therefore (a+b)^3=a^3+3a^2b+3ab^2+b^3\right)$

$=\left(k^3-7k+3\right)+\left(3k^2+3k+1-7\right)$

$=\left(k^3-7k+3\right)+\left(3k^2+3k-6\right)$

5.3

5.3.17

The objective is to prove $1+3n \le 4^n$, for every integer $n \ge 0$ using mathematical induction.

Let $P(n)$ be the inequality $1+3n \le 4^n$ for every integer $n \ge 0$.

Basis step:

Show that the statement is true for $n=0$.

$$1+3(0) \overset{?}{\le} 4^0$$
$$1 \le 1$$

Thus, the inequality is true for $n=0$.

Comment

Step 2 of 2

Inductive Step:

Assume that the inequality is true for $n=k$.

That is, $1+3k \le 4^k$.

Need to show that the result is true for $n=k+1$.

Consider,

$$\begin{aligned} 1+3(k+1) &= 1+3k+3 \\ &\le 4^k+3 && \text{By assumption} \\ &\le 4^k+3\cdot 4^k && \text{since } 3<3\cdot 4^k \\ &= 4^k(1+3) \\ &= 4^{k+1} \end{aligned}$$

Thus, the result is true for $n=k+1$.

Therefore, $1+3n \le 4^n$, for every integer $n \ge 0$.

5.3

5.3.18

Consider the statement " $5^n + 9 < 6^n$, for all integers $n \geq 2$". The objective of the problem is to prove the given statement by using mathematical induction.

Let $P(n)$ be the statement $5^n + 9 < 6^n$. Take, $n = 2$, then

$$5^2 + 9 < 6^2$$
$$34 < 36$$

Thus, $P(2)$ is true, since $34 < 36$.

Comment

Step 2 of 3

Now, suppose that $P(k)$ is true for an arbitrary integer $k \geq 2$. That is,

$$5^k + 9 < 6^k$$

Then, prove that $P(k+1)$ and this establishes the proof of the statement for all $n \geq 2$ by induction. That is, to prove that $5^{k+1} + 9 < 6^{k+1}$. Now,

$$5^{k+1} + 9 = 5 \cdot 5^k + 9$$

By using the inductive hypothesis in the alternative form, $5^k < 6^k - 9$. Then,

$$5^{k+1} + 9 < 5 \cdot (6^k - 9) + 9$$
$$= 5 \cdot 6^k - 45 + 9 \quad \text{Using the distributive property}$$
$$= 5 \cdot 6^k - 36 \quad \text{Simplify}$$

Comment

Step 3 of 3

But, $5 \cdot 6^k < 6 \cdot 6^k$ or 6^{k+1} and $-36 < 0$, so that this quantity is less than 6^{k+1}. Therefore, $5^{k+1} + 9 < 6^{k+1}$, as desired.

Thus, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all nonnegative integers $n \geq 2$

5.3

5.3.23

a)

For the given statement, the property is the inequality " $n^3 > 2n+1$ for all integers $n \geq 2$"

Show that the property is true for $n = 2$

When $n = 2$, the property is the inequality, " $2^3 > 2\times2+1$"

But $2^3 = 8$ & $2\times2+1 = 4+1 = 5$

Then, $8 > 5$

$\Rightarrow 2^3 > 2\times2+1$

Thus, the property is true for $n = 2$

Comment

Step 2 of 8

Show that for all integers $k \geq 2$, if the property is true for $n = k$, then it is also true for $n = k+1$

Let k be any integer where $k \geq 2$

Assume that the property is true for $n = k$

i.e., suppose that $k^3 > 2k+1$

We must show that the property is true for $n = k+1$

i.e., we must show that $(k+1)^3 > 2(k+1)+1$

Comment

Step 3 of 8

Now,

$(k+1)^3 = k^3 + 3k^2 + 3k + 1$

Since $k^3 > 2k+1$ (by the inductive hypothesis)

And $3k^2 + 3k + 1 > 2$

Clearly,

$\Rightarrow k^3 + 3k^2 + 3k + 1 > 2k + 1 + 2$

$\Rightarrow (k+1)^3 > 2(k+1)+1$

Thus, the property is true for $n = k+1$

Thus, the property is true for all integers $n \geq 2$

Thus, proved.

Comment

Step 5 of 8

b)

For the given statement, the property is the inequality " $n! > n^2$ for all integers $n \geq 4$"

Show that the property is true for $n = 4$

When $n = 4$, the property is the inequality " $4! > 4^2$"

But $4! = 4 \times 3 \times 2 \times 1 = 24$

$4^2 = 4 \times 4 = 16$

Clearly, $24 > 16$

$\Rightarrow 4! > 4^2$

Thus, the property is true for $n = 4$

Comment

Step 6 of 8

Show that for all integers $k \geq 4$, if the property is true for $n = k$, then it is also true for $n = k + 1$

Let k be any integer with $k \geq 4$

Assume that the property is true for $n = k$

i.e., suppose that $k! > k^2$

We must show that the property is true for $n = k + 1$

i.e., we must show that $(k+1)! > (k+1)^2$

Comment

Step 7 of 8

We know that

$k! > k + 1$ for all $k > 2$

$\Rightarrow (k+1) \times k! > (k+1)(k+1)$

$\Rightarrow (k+1)! > (k+1)^2$

Thus, the property is true for $n = k + 1$

Thus, the property is true for all integers $n \geq 4$