حلول بعض الاسئلة المختارة من شابتر 7

ملاحظة: الحل عبارة عن صور وليس نص مكتوب. واذا لم يوجد رقم السؤال فوق الصورة، فتكون الصورة(الحل) مكملة للصورة(للحل) السابقة.

على سبيل المثال سؤال رقم 12 من سكشن 2، له صورتان.

7.1

7.1.2

a.

The domain of g is, $\boxed{X = \{1,3,5\}}$ and the codomain of g is, $\boxed{Y = \{a,b,c,d\}}$.

Comment

Step 2 of 6

b.

From the above defined function, it follows that,

$\boxed{g(1) = g(3) = g(5) = b}$.

Comment

Step 3 of 6

c.

The range of g is, $\{b\}$.

Comment

Step 4 of 6

d.

The element a in Y does not have an inverse image. Thus, the element 3 in X is not an inverse image of a.

The inverse images of an element b are, $\{1,3,5\}$.

Hence, the element 1 in X is an inverse image of b.

Comment

Step 5 of 6

e.

The inverse images of an element b are, $\boxed{\{1,3,5\}}$ while the inverse image of c does not exist.

Comment

Step 6 of 6

f.

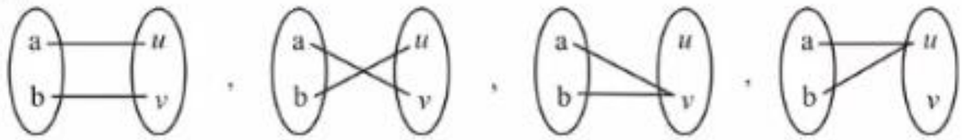
The notation of the function g as a set of ordered pairs is,

$\boxed{g = \{(1,b),(3,b),(5,b)\}.}$

7.1

7.1.4

(a)

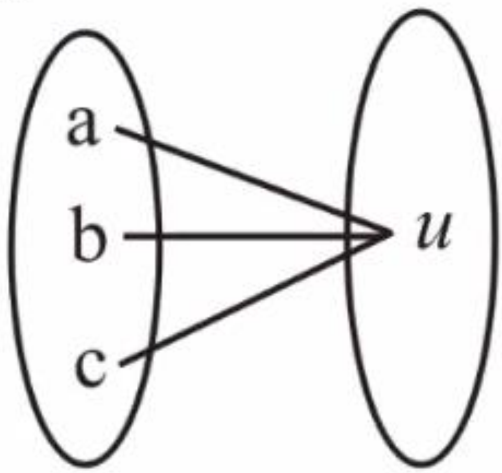


These are the functions from *X* to *Y*

Comment

Step 2 of 3

(b)

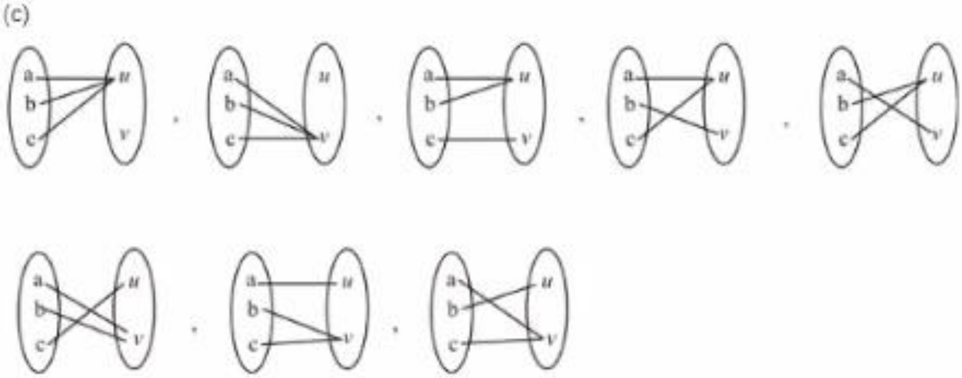


This is the only possible function from *X* to *Y*

Comment

Step 3 of 3

(c)



These are the functions from *X* to *Y*

7.1

7.1.7

(a)

The objective is to determine the value of $F(\{1,3,4\})$.

Since, $X=\{1,3,4\}$ have odd number of elements.

By the definition of F(X),

$$F(\{X\})=F(\{1,3,4\})$$
$$=\boxed{1}$$

Comments (3)

Step 2 of 4

(b)

The objective is to determine the value of $F(\phi)$.

Since, $X=\{\phi\}$ have even number of elements.

By the definition of F(X),

$$F(\{X\})=F(\phi)$$
$$=\boxed{0}$$

Since, the empty set has 0 elements (its cardinality is 0), and 0 is even number.

Comment

Step 3 of 4

(c)

The objective is to determine the value of $F(\{2,3\})$.

Since, $X=\{2,3\}$ have even number of elements.

By the definition of F(X),

$$F(\{X\})=F(\{2,3\})$$
$$=\boxed{0}$$

Comment

Step 4 of 4

(d)

The objective is to determine the value of $F(\{2,3,4,5\})$.

Since, $X=\{2,3,4,5\}$ have even number of elements.

By the definition of F(X),

$$F(\{X\})=F(\{2,3,4,5\})$$
$$=\boxed{0}$$

7.1

7.1.10

(a) $T(1) =$ set of positive divisors of 1

$= \{1\}$ this set contain one element.

Comment

Step 3 of 7

(b) $T(15) =$ the set of positive divisors of 15

$= \{1, 3, 5, 15\}$ this set contain four elements.

Comment

Step 4 of 7

(c) $T(17) =$ the set of positive divisors of 17

$= \{1, 17\}$ this set contain two elements.

Comment

Step 5 of 7

(d) $T(5) =$ the set of positive divisors of 5

$= \{1, 5\}$ this set contain two elements.

Comment

Step 6 of 7

(e) $T(18) =$ the set of positive divisors of 18

$= \{1, 2, 3, 6, 9, 18\}$ the set contain six elements.

Comment

Step 7 of 7

(f) $T(21) =$ the set of positive divisors of 21

$= \{1, 3, 7, 21\}$

7.1

7.1.15

Consider the functions *F* and *G* are defined from real numbers to itself.

Define the product $F \cdot G : \mathbb{R} \to \mathbb{R}$ and $G \cdot F : \mathbb{R} \to \mathbb{R}$ for all $x \in \mathbb{R}$ as follows,

$$(F \cdot G)(x) = F(x) \cdot G(x)$$

$$(F \cdot G)(x) = (G \cdot F)(x)$$

Comment

Step **2** of 2

For all real number $x \in \mathbb{R}$

By the definition of $F \cdot G$,

$$(F \cdot G)(x) = F(x) \cdot G(x)$$

Now, apply the commutative law for multiplication of real numbers,

$$(F \cdot G)(x) = G(x) \cdot F(x)$$

Again by the definition of $G \cdot F$

$$(F \cdot G)(x) = (G \cdot F)(x)$$

Therefore, the functions F and G are satisfy the product $\boxed{F \cdot G = G \cdot F}$ for all real number $x \in \mathbb{R}$.

7.1

7.1.16

Consider the functions F and G from the set of all real numbers to itself.

Also, define the functions $F-G:\mathbf{R}\to\mathbf{R}$ and $G-F:\mathbf{R}\to\mathbf{R}$ as follows,

$(F-G)(x)=F(x)-G(x)$ and $(G-F)(x)=G(x)-F(x)$ for all $x\in\mathbf{R}$.

The objective is to verify whether $F-G=G-F$ is true or not and explain the reason.

Comment

Step 2 of 2

Consider the function $F-G$.

$$\begin{aligned}(F-G)(x)&=F(x)-G(x) \quad \text{(From the above definiton)}\\ &=-(G(x)-F(x))\\ &=-(G-F)(x) \quad \text{for all } x \text{ in } \mathbf{R}\end{aligned}$$

Thus, $F-G\neq G-F$ for all x in $\mathbf{R}$.

7.1

7.1.19

The objective is to prove $\log_b b = 1$ for any positive real number $b(\neq 1)$.

By the definition of the logarithm, if $a^x = y$ then,

$\log_a y = x.$

Let b be any positive real number with $b \neq 1.$

For a positive real number b,

$b^1 = b.$

This is similar to $a^x = y.$

By the definition of logarithm,

$\boxed{\log_b b = 1}.$

7.1

7.1.20

The objective is to prove $\log_b 1 = 0$ for any positive real number $b(\neq 1)$.

By the definition of the logarithm, if $a^x = y$ then,

$\log_a y = x.$

Let b be any positive real number with $b \neq 1.$

For a positive real number $b,$

$b^0 = 1.$

This is similar to $a^x = y.$

By the definition of logarithm,

$\boxed{\log_b 1 = 0}.$

Since, $\log_a y = x.$

7.1

7.1.21

Let *b* be a positive real number, and *x* be a real number with

$b \neq 1.$

Define the value of b^{-x}, as follows:

$$b^{-x} = \frac{-1}{b^x}.$$

The objective is to prove, $\log_b\left(\frac{1}{u}\right) = -\log_b(u).$

Let $v = \log_b\left(\frac{1}{u}\right).$

By the definition of a logarithm,

$$b^v = \frac{1}{u}. \quad \ldots\ldots (1)$$

Multiply both sides of equation (1), by *u*, and dividing by b^v, obtained as,

$$u = b^{-v}.$$

Thus, $-v = \log_b(u)$ [by the definition of a logarithm]

Now, multiply both sides by -1, obtained as,

$$v = -\log_b(u).$$

Therefore,

$$\boxed{\log_b\left(\frac{1}{u}\right) = -\log_b(u)}.$$

Since, $v = \log_b\left(\frac{1}{u}\right).$

7.1

7.1.22

Suppose that $\log_3 7$ is rational.

Then $\log_3 7 = \frac{p}{q}$ for some integers p and q with $q \neq 0$.

As logarithms are positive valued, p and q must be positive.

Use definition of logarithm to write,

$$3^{\frac{p}{q}} = 7$$

Apply b^{th} power to both sides,

$$\left(3^{\frac{p}{q}}\right)^q = 7^q$$

$$3^p = 7^q$$

Comment

Step 2 of 2

Let $N = 3^p = 7^q$, and take its prime factorization.

As $N = 3^p$, the prime factors are all 3 and also as $N = 7^q$, the prime factors are all 7 .

Since 3 and 7 are co-prime, no integer power of 3 is equal to an integer power of 7 .

This is a contradiction to the unique factorization theorem.

So the supposition is wrong, and therefore, $\log_3 7$ is irrational.

7.1

7.1.25

$A = \{2,3,5\}, B = \{x,y\}$

$A \times B = \{(2,x),(2,y),(3,x),(3,y),(5,x),(5,y)\}$

If $p_1(a,b) = a$ and $p_2(a,b) = b$ are the projections of $A \times B$

Comment

Step 2 of 3

a) $p_1(2,y) = 2,$

$p_1(5,x) = 5,$

The range of p_1 is *A*

Comment

Step 3 of 3

b) $p_2(2,y) = y,$

$p_2(5,x) = x$

The range of p_2 is *B*

7.1

7.1.26

We define the functions from $Z^{+} \cup \{0\} X Z^{+} \to Z$

$\text{mod}(n,d) = n \bmod d$ and

$\text{div}(n,d) = n \operatorname{div} d \ \forall \ (n,d) \in Z^{+} \cup \{0\} X Z^{+}$

Comment

Step 2 of 4

(a) $\text{mod}(67,10) = 67 \bmod 10 = 7$

$\text{div}(67,10) = 67\text{div}10 = 6$

Comment

Step 3 of 4

(b) $\text{mod}(59,8) = 59 \bmod 8 = 3$

$\text{div}(59,8) = 50\text{div}8 = 7$

Comment

Step 4 of 4

(c) $\text{mod}(30,5) = 30 \bmod 5 = 0$

$\text{div}(30,5) = 30\text{div}5 = 6$

7.1

7.1.27

$f(s) =$ number of *b* variables on the left side of the leftmost *a*

$= 0$ if there are no *a* variables in the string.

Comment

Step 2 of 3

(a) $f(aba) = 0$

The leftmost variable in the string is an *a*, and so it has no *b* variables on its left.

$f(bbab) = 2$

There is only one *a*, and so it is the leftmost *a*

To its left, there are 2 *b* variables, consequently $f(bbab) = 2$

$f(b) = 0$, since there are no *a* variables.

The range of such a function *f* is $\{0\} \cup Z^{+}$

Comment

Step 3 of 3

(b) $g(aba) = aba$

By reversing the given string

$g(bbab) = babb$

$g(b) = b$

The range of *S* is the strings *a*'s, *b*'s whose length varies from zero to infinity

7.1

7.1.34

$h\left(\frac{m}{n}\right)=\frac{m^2}{n}$ where $h:Q\to Q,\ n\neq 0,\ m$ are integers

Suppose that $\frac{m_1}{n_1}=\frac{m_2}{n_2}$

Then, $\frac{m_1^2}{n_1}\neq\frac{m_2^2}{n_2}$ for instance $\frac{3}{2}=\frac{6}{4}\not\Rightarrow\frac{3^2}{2}=\frac{6^2}{4}$,

i.e., $h\left(\frac{m_1}{n_1}\right)\neq h\left(\frac{m_2}{n_2}\right)$

$\therefore \frac{m_1}{n_1}=\frac{m_2}{n_2}\not\Rightarrow h\left(\frac{m_1}{n_1}\right)=h\left(\frac{m_2}{n_2}\right)$

$\therefore h$ is not well defined

7.1.36

Given that $J_4=\{0,1,2,3,4\}$

Student F is correct. Because $S(x)$ is not well defined if $S(x)$ is well defined then,

$S(x)$. For each $x\in J_4-\{0\}$ would have a uniquely determined value

For,

Let $x=3$

Then $S(3)=3$ because $(3.3)\bmod 4=1$

And $S(3)=7$ because $(3.7)\bmod 4=1$

Hence $S(3)$ does not have a uniquely determined values

Therefore S is not well defined

7.1

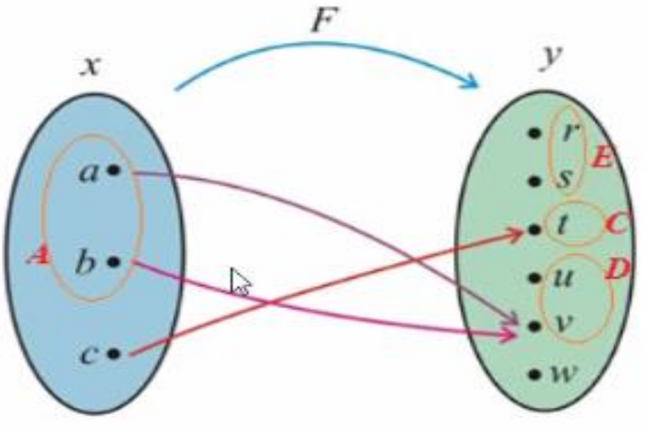
7.1.38.b

(b)

Write the sets,

$A=\{a,b\}$,$C=\{t\}$, $D=\{u,v\}$, and $\{r,s\}$

Show these in the below diagram:



As the image for a,b is v, write $f(A)=\{v\}$

As the range of the map is $\{t,v\}$, write $f(X)=\{t,v\}$

The inverse map of the set $D=\{u,v\}$ is,

$$f^{-1}(D)=f^{1}\{u,v\}$$
$$=\{a,b\}$$

The inverse map of the set $C=\{t\}$ is,

$$f^{-1}(C)=\{c\}$$

As the elements r,s are not mapped, the inverse map of the set $E=\{r,s\}$ is,

$$f^{-1}(E)=f^{-1}\{r,s\}$$
$$=\phi$$

The inverse map of the image set Y is,

$$f^{-1}(Y)=\{a,b,c\}$$
$$=X$$

7.1

7.1.46

Let F be a function from set x to set y.

And suppose $C \subseteq y$ and $D \subseteq y$ we must show that

$F^{-1}(C \cup D) = F^{-1}(C) \cup F^{-1}(D)$ in two parts

Comment

Step 2 of 3

We show $F^{-1}(C \cup D) \subseteq F^{-1}(C) \cup F^{-1}(D)$

Let $x \in F^{-1}(C \cup D)$ then we show that $x \in F^{-1}(C)$ or $x \in F^{-1}(D)$

$x \in F^{-1}(C \cup D) \Leftrightarrow F(x) \in (C \cup D)$

$\Leftrightarrow F(x) \in C \text{ or } F(x) \in D$

$\Leftrightarrow x \in F^{-1}(C) \text{ or } x \in F^{-1}(D)$

$\Leftrightarrow x \in F^{-1}(C) \cup F^{-1}(D)$

Therefore $F^{-1}(C \cup D) \subseteq F^{-1}(C) \cup F^{-1}(D)$ ...... (1)

Comment

Step 3 of 3

Now we show that

$F^{-1}(C) \cup F^{-1}(D) \subseteq F^{-1}(C \cup D)$

Let $x \in F^{-1}(C) \cup F^{-1}(D)$

$\Leftrightarrow \quad x \in F^{-1}(C) \text{ or } x \in F^{-1}(D)$

$\Leftrightarrow \quad F(x) \in C \text{ or } F(x) \in D$

$\Leftrightarrow \quad F(x) \in (C \cup D)$ by definition of union

$\Leftrightarrow \quad x \in F^{-1}(C \cup D)$

So $F^{-1}(C) \cup F^{-1}(D) \subseteq F^{-1}(C \cup D)$ ...... (2)

So by (1) and (2) we can write

$F^{-1}(C \cup D) = F^{-1}(C) \cup F^{-1}(D)$

7.1

7.1.48

Let F be a function from X to Y.

Assume that, $C \subseteq D$ and $D \subseteq y$ to prove,

$F^{-1}(C-D)=F^{-1}(C)-F^{-1}(D)$

The proof can be divided into two parts.

Part-1: $F^{-1}(C-D) \subseteq F^{-1}(C)-F^{-1}(D)$

Part-2: $F^{-1}(C-D) \supseteq F^{-1}(C)-F^{-1}(D)$

Comment

Step 2 of 3

Part -1:-

The proof of the part-1 is same as,

If $x \in F^{-1}(C-D)$, then $x \in F^{-1}(C)-F^{-1}(D)$.

By the definition of inverse image, $x \in X$ such that $f(x) \in C-D$

That means, $x \in X$ such that $f(x) \in C$ and $f(x) \notin D$.

By the definition of inverse image of the set,

$x \in F^{-1}(C)$ and $x \notin F^{-1}(D)$

By the definition of difference of two sets,

$x \in F^{-1}(C)-F^{-1}(D)$

Hence,

$F^{-1}(C-D) \subseteq F^{-1}(C)-F^{-1}(D)$ ...... (1)

Comment

Step 3 of 3

Part II: -

The proof of the part-2 is same as,

If $x \in \left(F^{-1}(C)-F^{-1}(D)\right)$, then $x \in F^{-1}(C-D)$.

By the definition of difference of two sets and the inverse image,

$x \in F^{-1}(C)$ and $x \notin F^{-1}(D)$

$F(x) \in C$ and $F(x) \notin D$

By the definition of difference of two sets and the inverse image,

$F(x) \in (C-D)$

$x \in F^{-1}(C-D)$

Hence,

$F^{-1}(C)-F^{-1}(D) \subseteq F^{-1}(C-D)$. ...... (2)

Therefore, by equation (1) and (2), it can be conclude that,

$\boxed{F^{-1}(C-D)=F^{-1}(C)-F^{-1}(D)}$

7.2

7.2.3

We can see that every element in *A* is connected to exactly one element *b* in *B* by the function *f*.

This function *f* is not one-to-one because $f(1) = f(2)$, but $1 \neq 2$

7.2.5

(a) This is a correct statement

Comment

Step 2 of 5

(b) This is an incorrect statement. Because for an onto function, the correspondence of domain is not required.

Comment

Step 3 of 5

(c) This is a correct statement

Comment

Step 4 of 5

(d) This is an incorrect statement. The reason is given in (b).

Comment

Step 5 of 5

(e) This is a correct statement

7.2

7.2.8.a

(a)

The domain of the function *H* is $X = \{a, b, c\}$ and co-domain is $Y = \{w, x, y, z\}$.

The function *H* is defined as $H(a) = w, H(b) = y, H(c) = y$.

The objective is to find if the function *H* is one-to-one.

One-to-one function: A function $F : X \to Y$ is said to be one-to-one if, and only if, $\forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$, then $x_1 = x_2$.

For the function *H*,

$$H(b) = H(c) = y$$

But, $b \neq c$.

The function *H* is $\boxed{\text{not one-to-one}}$ as the elements *b* and *c* are both sent by *H* to the same element of *Y*, namely the element *y*.

Comment

Step 2 of 4

The objective is to find if the function *H* is onto.

Onto function: A function $F : X \to Y$ is said to be onto if, and only if, $\forall y \in Y$, $\exists x \in X$ such that $F(x) = y$.

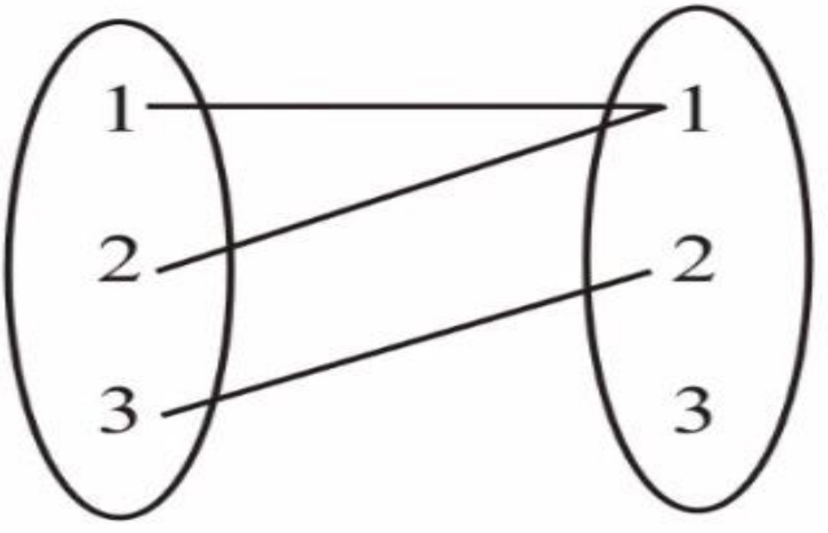
The function *H* is $\boxed{\text{not onto}}$ as for the elements $x, z \in Y$, there is no element $x_1 \in X$ such that $H(x_1) = x$ or $H(x_1) = z$.

In other words, *H* is not onto because $x, z \neq H(x_1)$ for any $x_1 \in X$.

7.2

7.2.9.c

(c) $h : X \to X$



Comment

Step 6 of 8

$h(1) = h(2)$ but $1 \neq 2$

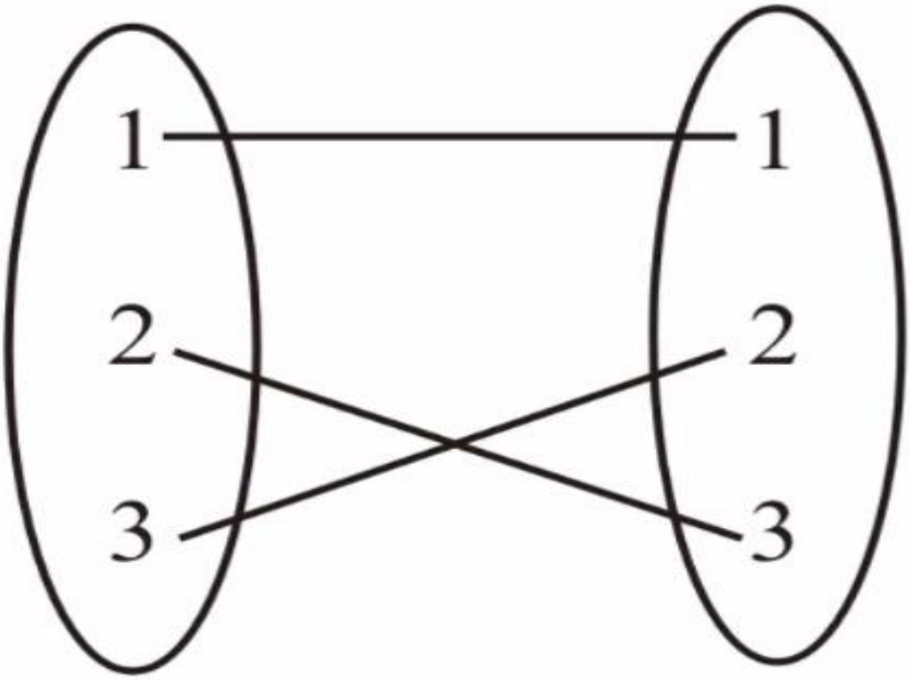
So h is not one-to-one

3 has no pre-image with respect to h. so h is not onto

7.2

7.2.9.d

(d) $K : X \to X$



Comment

Step 8 of 8

Is one-to-one while distinct elements have distinct images

Co-domain of K = range of $K = \{1,2,3\}$

$\therefore K$ is onto and clearly K is one-to-one

However, we can see that $f(2) \neq 2$ and $f(3) \neq 3$

So K is not the identity function

7.2

7.2.12

(a) $F: Z \to Z$ by the rule $F(n) = 2 - 3n$ for all integer n

(i) To prove F is one-to-one

Suppose n_1 and n_2 are two integer such that

$F(n_1) = F(n_2)$ [We must show that $n_1 = n_2$]

By definition of F

$2 - 3n_1 = 2 - 3n_2$

Subtracting 2 to both sides

$-3n_1 = -3n_2$

Dividing both side by -3 gives

$\boxed{n_1 = n_2}$

Hence F is one-to-one

Comment

Step **2** of 4

(ii) F is onto:

F is onto if given any element y in Z, it is possible to find an element in Z (domain) with the property that $y = f(x)$

Take $1 \in Z$ then we cannot find $n \in Z$ such that $f(n) = 1$

For $F(n) = 1 \Rightarrow 2 - 3n = 1 \Rightarrow n = \frac{1}{3} \notin Z$

Hence F is not onto.

7.2

b)

Given that $G:R \to R$ by the rule $G(x)=2-3x$ for all real number x

To prove Gis onto, we must prove

for all $y \in Y$**, there exists** $x \in X$ **such that** $G(x)=y$

Let $y \in R$ (co-domain)

So we can find $x \in R$ such that $G(x)=y$

$$2-3x=y$$
$$\Rightarrow x=\frac{2-y}{3} \in R$$

Comment

Step 4 of 4

Also we have

$$G(x)=G\left(\frac{2-y}{3}\right)$$
$$=2-3\left(\frac{2-y}{3}\right) \text{ (By definition of G)}$$
$$=2-(2-y)$$
$$=y$$

Therefore $\boxed{G(x)=y}$

Hence G is onto.

7.2

7.2.18

Step 1 of 1

$f(x)=\dfrac{x+1}{x-1}\ \forall x \neq 1 \in$ □

Suppose $f(x)=f(y)$ for some $x, y \in \textbf{domain}$

i.e., $\dfrac{x+1}{x-1}=\dfrac{y+1}{y-1}$

$\Rightarrow xy-x+y-1=xy-y+x-1$

$\Rightarrow x-y=y-x$

$\Rightarrow 2x=2y$

$\Rightarrow x=y$

$\therefore f(x)=f(y) \Rightarrow x=y$

$\therefore f$ is one-to-one

7.2

7.2.23

The function $F: P(\{a,b,c\}) \to \mathbb{Z}$ is defined as $F(A) = \text{the number of elements in } A$.

(a)Counterexample:

Clearly, $\{a,b\},\{b,c\} \in P\{a,b,c\}$

$\{a,b\} \neq \{b,c\}$

$F(\{a,b\}) = 2$

$F(\{b,c\}) = 2$

$\Rightarrow F(\{a,b\}) = F(\{b,c\})$

But $\{a,b\} \neq \{b,c\}$

Therefore, the function $F: P(\{a,b,c\}) \to \mathbb{Z}$ is **not** one-to-one function.

Comment

Step 2 of 2

(b)Counterexample:

Clearly, the number $-1 \in \mathbb{Z}$

Since the cardinality of the sets is always non-negative,

there is no set *S* in $P\{a,b,c\}$ satisfying $F(S) = -1$.

In other words, there exists no set *S* satisfying $F(S) = -1$.

Therefore, the function $F: P(\{a,b,c\}) \to Z$ is **not** onto function.

7.2

7.2.27.a

Let D be the set of all finite subsets of positive integers, and define

$T : Z^{+} \rightarrow D$.

Let for all integer, $T(n) =$ the set of all of the positive divisors of n.

(a)

To prove: T is one-to-one.

Let us assume that m and n have the same image.

Now we need to prove that m and n are equal to prove T is one-to-one.

Now,

$T(m) = T(n)$

Since the two sets $T(m)$ and $T(n)$ are same.

Thus, the largest integer in these two sets will be equal.

Comment

Step 2 of 3

Let us assume that the largest integer is x.

Since the largest divisor of an integer is always the integer itself. Thus, the largest element in $T(m)$ is m.

Similarly, the largest element in $T(n)$ is n.

Now, since we assumed the largest element as x. Thus,

$m = x = n$

Since, we proved that if the images of two elements are same then the elements are also same.

Hence, T is one-to-one.

7.2

7.2.31.b

(b)

The function $G: Z^+ \times Z^+ \to Z^+$ is defined as follows:

For all $(n,m) \in Z^+ \times Z^+$, $G(n,m) = 3^n 6^m$.

The objective is to check if the function G is one-to-one.

One-to-one function: A function $F: X \to Y$ is said to be one-to-one if, and only if, $\forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$, then $x_1 = x_2$.

Comment

Step 5 of 6

Let $(n_1, m_1), (n_2, m_2) \in Z^+ \times Z^+$.

Let $G(n_1, m_1) = G(n_2, m_2)$. Check if $(n_1, m_1) = (n_2, m_2)$.

$G(n_1, m_1) = G(n_2, m_2)$

$\Rightarrow 3^{n_1} 6^{m_1} = 3^{n_2} 6^{m_2}$

$\Rightarrow 3^{n_1} (2 \cdot 3)^{m_1} = 3^{n_2} (2 \cdot 3)^{m_2}$

$\Rightarrow 2^{m_1} 3^{n_1 + m_1} = 2^{m_2} 3^{n_2 + m_2}$ (as $a^m a^n = a^{m+n}$)

According to the unique prime factorization theorem, every integer greater than 1 is either a prime number or can be **uniquely** represented as the product of primes.

Comment

Step 6 of 6

The term $2^{m_1} 3^{n_1 + m_1}$ is expressed as product of prime factors 2 and 3. Also, the term $2^{m_2} 3^{n_2 + m_2}$ is expressed as product of prime factors 2 and 3.

Therefore, $2^{m_1} 3^{n_1 + m_1} = 2^{m_2} 3^{n_2 + m_2}$ is possible only if the powers of 2 and 3 are equal. That is, $m_1 = m_2$ and $n_1 + m_1 = n_2 + m_2$.

$n_1 + m_1 = n_2 + m_2$

$\Rightarrow n_1 + m_1 = n_2 + m_1$ (as $m_1 = m_2$)

$\Rightarrow n_1 = n_2$ (subtracting m_1 from both the sides)

Hence, if $G(n_1, m_1) = G(n_2, m_2)$, then $(n_1, m_1) = (n_2, m_2)$.

Therefore, the function G is one-to-one.

7.2

7.2.35

Our strategy is to prove that, for all real numbers $a, b,$ and x with b and x are positive and $b \neq 1$,

$$\log_b (x)^a = a \log_b x$$

Proof:

Let a, b, and x are positive real numbers, and $b \neq 1$

Suppose that, $\log_b (x)^a = m$

Recall the fact that, for each positive real numbers s and real number t,

$$\log_b s = t \Leftrightarrow b^t = s \quad \text{...... (1)}$$

As $\log_b (x)^a = m$, get $b^m = x^a$ [By the fact (1)]

$$(b^m)^{\frac{1}{a}} = (x^a)^{\frac{1}{a}}$$

$$b^{\frac{m}{a}} = x^{\frac{a}{a}}$$

$$b^{\frac{m}{a}} = x^1$$

$$b^{\frac{m}{a}} = x$$

$\Rightarrow \log_b x = \dfrac{m}{a}$ [By the fact (1)]

$$a \log_b x = m$$

Replace m with $\log_b (x)^a$ in the above one

$$a \log_b x = \log_b (x)^a$$

Thus, $\boxed{\log_b (x)^a = a \log_b x}$.

7.2

7.2.40

The objective is to prove that for all subsets $A \subseteq X, F^{-1}(F(A)) = A$ if $F: X \to Y$ is one-to-one.

Need to show that $F^{-1}(F(A)) \subseteq A$ and $A \subseteq F^{-1}(F(A))$.

Suppose that $x \in F^{-1}(F(A))$, then $F(x) \in F(A)$.

If x is not in A, then there is some element $y \in A$ such that $x \neq y$ and $F(x) = F(y)$ but this contradicts the fact that F is one-to-one.

So, $x \in A$.

Thus, $x \in F^{-1}(F(A))$ implies that $x \in A$.

Therefore, $F^{-1}(F(A)) \subseteq A$.

Comment

Step 2 of 3

Suppose that $x \in A$, then $F(x) \in F(A)$ since F is one-t-one.

So $x \in F^{-1}(F(A))$.

Thus, $x \in A$ implies that $x \in F^{-1}(F(A))$.

Therefore, $A \subseteq F^{-1}(F(A))$.

Hence, $\boxed{F^{-1}(F(A)) = A}$.

Comment

Step 3 of 3

b.

The objective is to prove that for all subsets A_1 and A_2 in X, $F(A_1 \quad A_2) = F(A_1) \quad F(A_2)$.

As F is one-to-one,

$$
\begin{aligned}
& F(x) \in F(A_1 \quad A_2) \\
& \Leftrightarrow x \in A_1 \quad A_2 \\
& \Leftrightarrow x \in A_1 \text{ and } x \in A_2 \\
& \Leftrightarrow F(x) \in F(A_1) \text{ and } F(x) \in F(A_2) \\
& \Leftrightarrow F(x) \in F(A_1) \quad F(A_2)
\end{aligned}
$$

Therefore, $\boxed{F(A_1 \quad A_2) = F(A_1) \quad F(A_2)}$.

7.2

7.2.49

Let $G: \mathbf{R} \to \mathbf{R}$ be any function defined by the formula

$G(x) = 2 - 3x$ for all real numbers x.

To prove: G is one-to-one.

Let us assume that $G(x_1) = G(x_2)$ for some x_1 and x_2 in $\mathbf{R}$.

Now, we need to prove that $x_1 = x_2$ to prove G is one-to-one.

By the definition of G,

$$2 - 3x_1 = 2 - 3x_2$$

Now subtract 2 to both sides, we get

$$2 - 3x_1 - 2 = 2 - 3x_2 - 2$$

$$\Rightarrow -3x_1 = -3x_2$$

Again, divide both sides by -3, we get

$$\frac{-3}{-3}x_1 = \frac{-3}{-3}x_2$$

$$\Rightarrow x_1 = x_2$$

Which implies that G is one-to-one.

Hence proved the desired result.

Comment

Step 2 of 2

To find: inverse function.

Let us claim that $G^{-1}(y) = \frac{2-y}{3}$.

By the inverse function, it is true if and only if

$$G\left(\frac{2-y}{3}\right) = y$$

Now, by the definition of by the definition of G,

$$G\left(\frac{2-y}{3}\right) = 2 - 3\left(\frac{2-y}{3}\right)$$

$$= 2 - 2 + y$$

$$= y$$

Hence, $G^{-1}(y) = \frac{2-y}{3}$.

7.2

7.2.53

Given that $f: R \to R$, $f(x) = \frac{x}{x^2+1}$

To show f is one to one correspondence, we need to show that f is one one and onto

To show f is one to one, let

$$f(x_1) = f(x_2)$$
$$\frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$
$$x_1(x_2^2+1) = x_2(x_1^2+1)$$

$x_1x_2(x_1 - x_2) - 1(x_1 - x_2) = 0$

$(x_1 - x_2)(x_1x_2 - 1) = 0$

$x_1 = x_2$ Or $x_1x_2 = 1$

$f(2) = f\left(\frac{1}{2}\right)$, but $2 \neq \frac{1}{2}$

Hence, f is not one to one

Comment

Step 2 of 3

$x^2 + 1 > x$ For $x \notin [0,1]$

For $x \in [0,1]$

$0 \le x \le 1$

$x^2 \le 1$

$x^2 + 1 \ge 1 \ge x$

$x \le x^2 + 1$ For all x

Comment

Step 3 of 3

$f(x) = \frac{x}{x^2+1} \le 1$ For all x

Hence, for all $y \in (1, \infty)$, there exist no $x \in R$ such that $f(x) = y$

Therefore, f is not onto.

Hence f is not one to one correspondence.

7.2

7.2.57

Suppose $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$

Then $f : X \to Y$ is one to one if and only if

$f(x_i) = f(x_j)$ Such that $x_i \neq x_j$

Domain of f: one dimensional array $a[1], a[2], \ldots, a[n]$

Introduce a new variable: *answer*

Comment

Step 2 of 2

Algorithm body:

While ($i \leq n-1$ and *answer* = "one to one")

$j := i+1$

While ($j \leq n$ and *answer* = "one to one")

if $(f(a[i]) = f(a[j])$ **and** $a[i] \neq a[j])$

then *answer* = "not one to one"

$j := j+1$

end while

$i := i+1$

end while

Output: answer