حلول بعض الاسئلة المختارة من شابتر 8

ملاحظة: الحل عبارة عن صور وليس نص مكتوب. واذا لم يوجد رقم السؤال فوق الصورة، فتكون الصورة(الحل) مكملة للصورة(للحل) السابقة.

على سبيل المثال سؤال رقم 16 من سكشن 1، له صورتان

8.1

8.1.2

The object is to prove that for all integers m and n, $m-n$ is even if, and only if, both m and n are even or both m and n are odd.

Case (1):

Let *m* and *n* both be even integers, and let $m=2k$ $n=2t$ where k and t are integers.

Since even numbers are of the form 2*n*

Now, let $m-n=2k-2t$

$=2(k-t)$ Since 2 is a factor

$=2L$ Where $k-t=L$, L is an integer

$m-n$ is an even integer, since it is in the form of even difference.

$\therefore$ If *m* and *n* are even numbers, then $m-n$ is also even.

Comment

Step 2 of 2

Case (2):

Let *m* and *n* both be odd integers, then $m=2k+1$, $n=2t+1$ where k and t are integers.

Since odd numbers are in the form of 2*n* + 1

$m-n=(2k+1)-(2t+1)$

$=2k+1-2t-1$ by basic algebra

$=2k-2t$

$=2(k-t)$

$=2\text{M}$ where $M=k-t$ is an integer.

$\therefore$ $m-n$ is an even integer, since it is the form of a multiple of 2.

$\therefore$ $m-n$ is an even number, and either both *m* and *n* are even or both *m* and *n* are odd.

Hence proved.

8.1

8.1.3-D&E

(d)

Our strategy is to find five integers n such that $n\,T\,2$

$n\,T\,2 \quad \Leftrightarrow \quad 3|(n-2)$

$\Rightarrow \quad n-2=3k$ Here k is an integer

$\Rightarrow \quad n=3k+2$ Here k is an integer

To find the five integers n such that $n=3k+2$, replace k with 1, 2, 3, 4, and 5.

For $k=1$, $n=3(1)+2$

$=5$

For $k=2$, $n=3(2)+2$

$=8$

For $k=3$, $n=3(3)+2$

$=11$

For $k=4$, $n=3(4)+2$

$=14$

For $k=5$, $n=3(5)+2$

$=17$

Hence, the required integers are 5, 8, 11, 14, and 17.

(e)

All integers of the form $3k$ for some integer k are related by T to 0

Since $3k\,T\,0=3k-0=3k$, so $3\,|\,3k$

All integers of the form $3k+1,$ for some integer k are related by T to 1

Since $(3k+1)\;T\;1=3k+1-1=3k$, so $3\,|\,3k$

All integers of the form $3k+2$, for some integer k related by T to 2

Since $(3k+2)T\,2=3k+2-2=3k$, so $3\,|\,3k$

8.1

8.1.11

Step 1 of 3

Consider the two sets,

$A = \{3,4,5\}$ and $B = \{4,5,6\}$

And a relation S is 'divides' relationship between A and B i.e., $\forall (x,y) \in A \times B$,

$x\,S\,y \Leftrightarrow x \mid y$

The objective is to indicate the ordered pairs are in S and S^{-1}.

Comment

Step 2 of 3

Let $A = \{3,4,5\}$ and $B = \{4,5,6\}$ be two sets.

Let S be the "divides" relation.

That is, for all $(x,y) \in A \times B$, $x\,S\,y \Leftrightarrow x \mid y$

As $A = \{3,4,5\}$ and $B = \{4,5,6\}$, then

$A \times B = \{(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,4),(5,5),(5,6)\}$.

Comment

Step 3 of 3

Observe that, $S = \{(3,6),(4,4),(5,5)\}$.

Here 3 divides 6, 4 divides 4 and 5 divides 5. These 3 are true.

Here S is the relation $x \mid y$, so $3 \mid 6, 4 \mid 4, 5 \mid 5$.

The inverse relation $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in A \times B\}$

Also, observe that $S^{-1} = \{(6,3)(4,4)(5,5)\}$ because,

$S^{-1} = \{(y,x) \in B \times A \mid (x,y) \in S, \text{ i.e., } x \mid y\}$.

8.1

8.1.12

Step 1 of 2

(a) If $F: X \to Y$ is one-to-one, but not onto, then F^{-1} is not a function.

Because F is not onto then F^{-1} is not defined on all of Y. If $y \in Y$ such that $(y,x) \notin F^{-1}$ for any $x \in X$.

Consequently, F^{-1} does not satisfy the property of the definition of functions.

Comment

Step 2 of 2

(b) If $F: X \to Y$ is onto but not one-to-one, then F^{-1} is not a function.

Because F is not one-to-one, F^{-1} is not defined for all $x \in X$.

Since $(x_1, y) \in F$ and $(x_2, y) \in F$, but $x_1 \neq x_2$, so F^{-1} doesn't satisfy the property of the definition of the functions.

8.1

8.1.16

Step 1 of 4

$$S = \left\{ \begin{matrix} (5,5)(5,7)(5,9)(6,6)(6,8)(6,10)(7,7)(7,9)(8,8)(7,5)(8,6)(8,10)(9,5) \\ (9,7)(9,9)(10,6)(10,8)(10,10) \end{matrix} \right\}$$

Because $x\,S\,y \Rightarrow 2\,|\,(x-y)$

Comment

Step 2 of 4

Note that 5 S 5 because 5 −5 = 0 and 2 | 0 since 0 = 2(0). Thus there is a loop from 5 to itself. Similarly there is a loop from 6 to itself, from 7 to itself, from 8 to itself, from 9 to itself and from 10 to itself.

Comment

Step 3 of 4

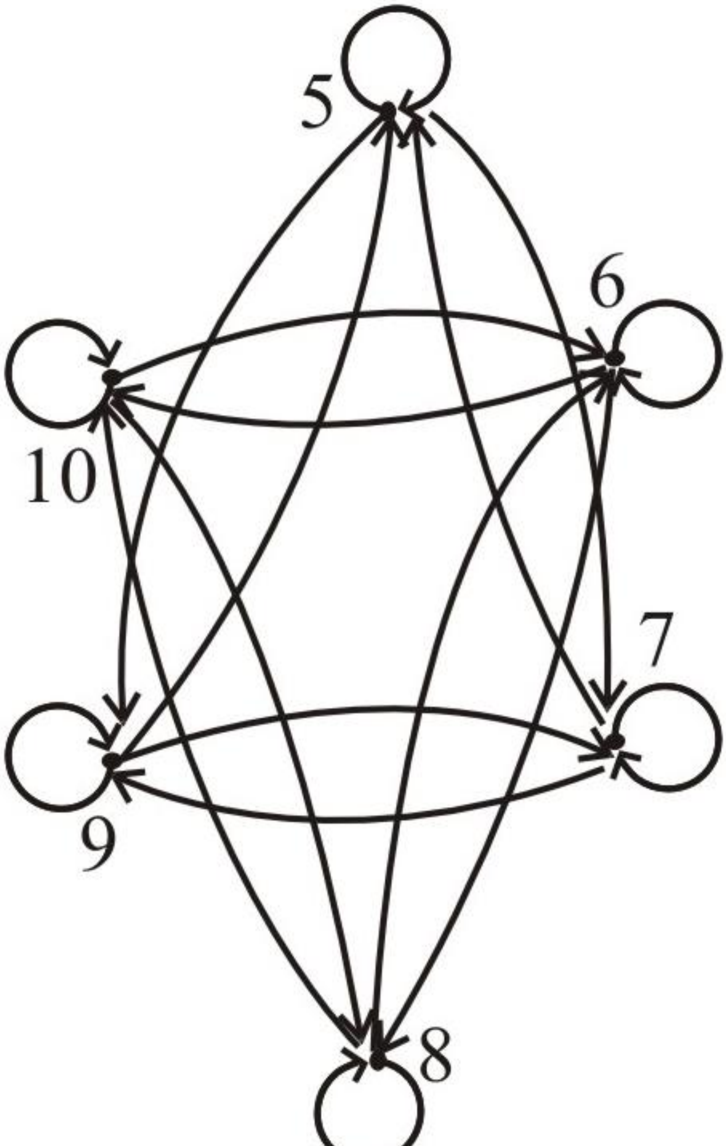
Note also that 5 S 7 because 5 – 7 = −2 = 2.(−1). And 7 S 5 because 7 – 5 = 2 = 2.1.

Hence there is an arrow from 5 to 7 and also an arrow from 7 to 5.

The other arrows in the direct graph, as shown below, are obtained by similar reasoning.

8.1

The directed graph of S is



8.1

8.1.20

Step 1 of 5

It is given that $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$

$A \times B = \{(-1,1)(-1,2)(1,1)(1,2)(2,1)(2,2)(4,1)(4,2)\}$

Because by the definition of $A \times B$, every element of *A* is mapped to all the elements of *B* as pairs.

Comment

Step 2 of 5

$R = \{(-1,1)(1,1)(2,2)\}$

Since $xRy \Rightarrow |x| = |y|$ so $|-1| = 1 \Rightarrow (-1,1) \in R$

$|1| = 1 \Rightarrow (1,1) \in R$

And $|2| = 2 \Rightarrow (2,2) \in R$

Comments (1)

Step 3 of 5

$S = \{(-1,1)(1,1)(2,2)(4,2)\}$

Because $xSy \Rightarrow x - y$ is even so $-1 - 1 = -2$ is even and $1 - 1 = 0$ is even, $2 - 2 = 0$, $4 - 2 = 2$ are even.

8.1

Step 4 of 5

$R \cup S = \{(-1,1)(1,1)(2,2)(4,2)\} = S$

$R \cup S = S$

Since $R \cup S$ means the ordered pairs, which belongs to *R* or *S*

Comment

Step 5 of 5

$R \cap S = \{(-1,1)(1,1)(2,2)\} = R$

$\therefore R \cap S = R$

Since $R \cap S$ means the ordered pairs, which belongs to both *R* and *S*

8.1

8.1.22

Step 1 of 6

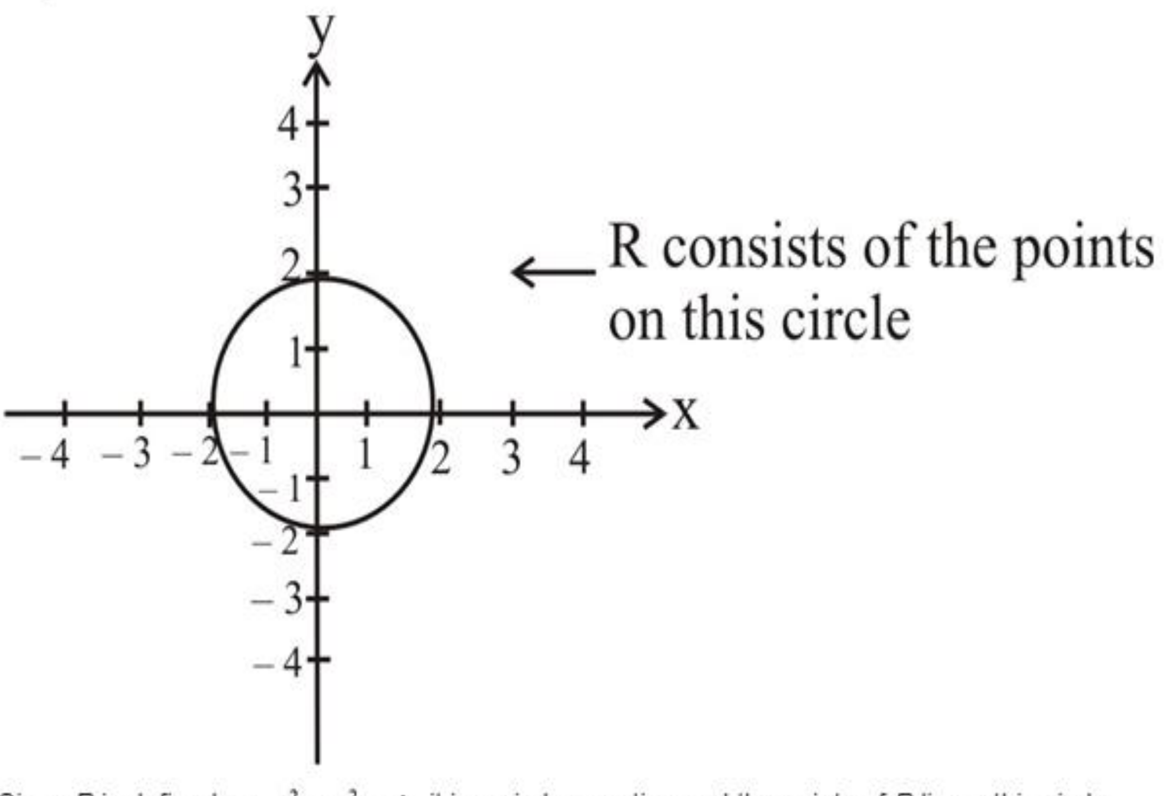
Given $R = \{(x,y) \in \square \times \square \mid x^2 + y^2 = 4\}$ and

$S = \{(x,y) \in \square \times \square \mid x = y\}$

Comment

Step 2 of 6

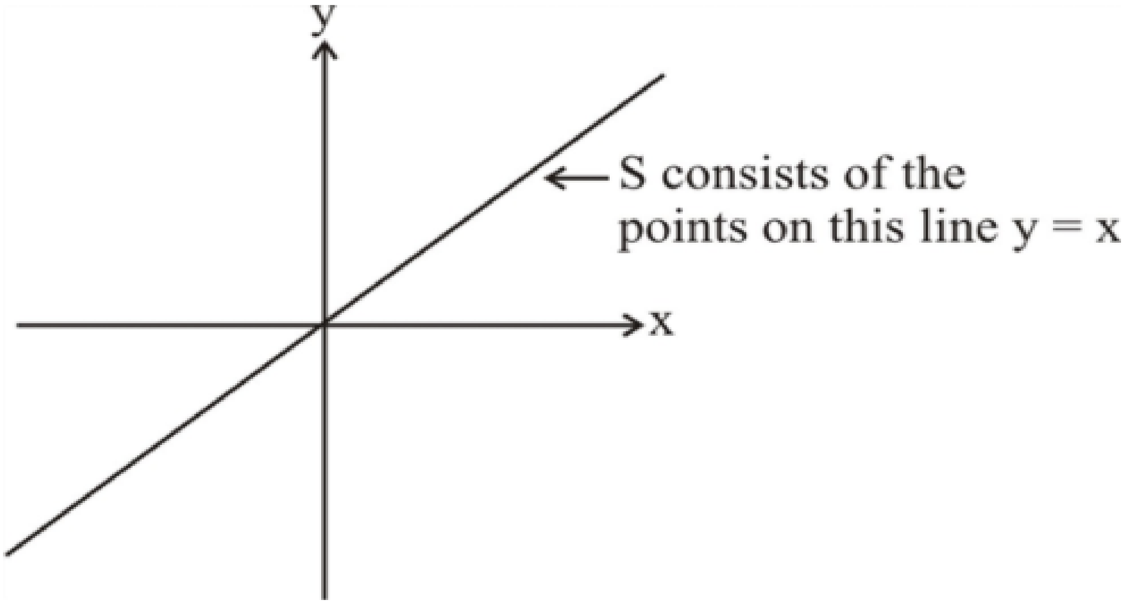
Graph of R is:



Since R is defined as $x^2 + y^2 = 4$, it is a circle equation and the points of R lie on this circle.

8.1

Graph of S is:

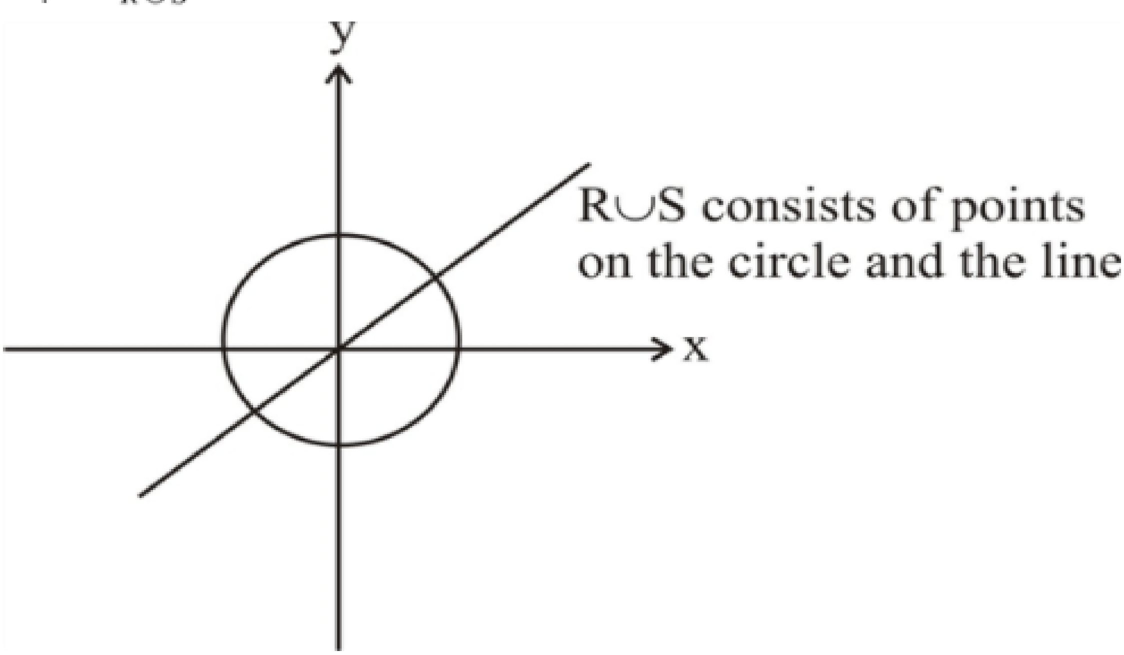


Since S is defined by the equation $y = x$, it is a line, which passes through the origin so the ordered pairs of S lie on the $y = x$ line.

Comment

Step 4 of 6

Graph of $R \cup S$ is:



Since $R \cup S$ represents the points that belong to R or S, the circle and the line represent $R \cup S$

8.1

Step 5 of 6

$R \cap S$ is the common point of both *R* and *S*,

Which represents $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$

By solving $x^2 + y^2 = 4$ and $y = x$,

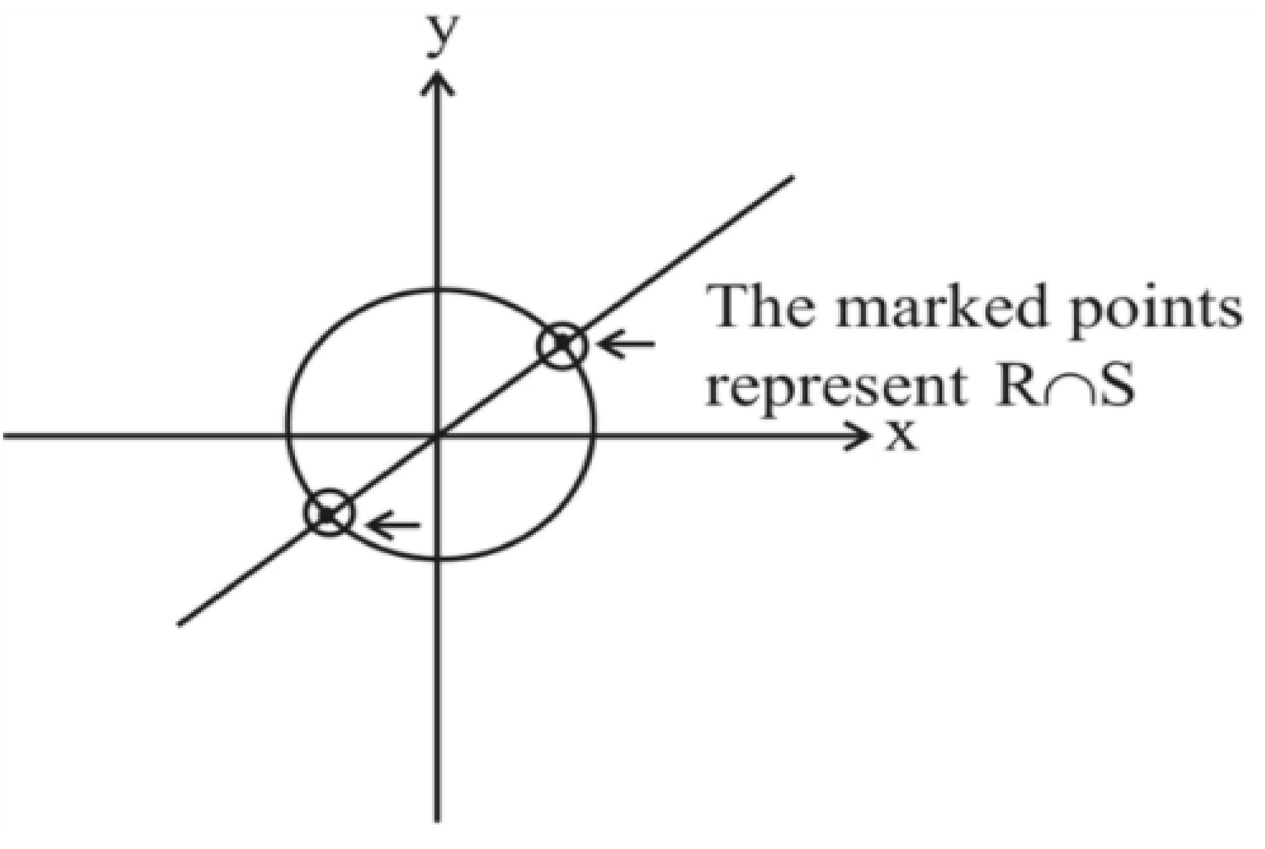
$$x^2 + y^2 = 4 \Rightarrow 2x^2 = 4$$
$$\Rightarrow x^2 = 2$$
$$\Rightarrow x = \pm\sqrt{2}$$

So the intersection points are $(\sqrt{2}, \sqrt{2})(-\sqrt{2}, -\sqrt{2})$

Comment

Step 6 of 6

Graph of $R \cap S$ is:



8.2

8.2.10

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The objective is to determine whether the following relation is reflexive, symmetric, transitive, or none of these:

The relation C defined on the set of real numbers: $x, y \in \mathbf{R}, x\,C\,y \Leftrightarrow x^2 + y^2 = 1.$

Reflexive:

For $x = 1,$ $1 \not{C} 1,$ as $1^2 + 1^2 = 2 \neq 1.$

Thus, C is not reflexive.

Comment

Step 2 of 3

Symmetric:

Suppose $x\,C\,y,$ then $x^2 + y^2 = 1.$

As sum of real numbers obeys commutative property, so $y^2 + x^2 = 1.$

So, $y\ C\ x.$

Thus, C is symmetric.

Comment

Step 3 of 3

Transitive:

For $x = 1, y = 0, z = 1,$ consider $1\,C\,0$ and $0\,C\,1.$

But $1 \not{C} 1.$

Thus, C is not transitive.

8.2

8.2.14

O is not reflexive

O is reflexive $\Leftrightarrow$ for all integers m and mOm by the definition of *O*, for all integers *m*, $m-m=0$, any number. Which is false, since *O* is an even number.

Comment

Step 2 of 3

O is symmetric

By the definition of *O*, for all integers *m* and *n*, $m-n$ is odd $\Rightarrow m-n=(2k+1)$ by the definition of odd

$\Rightarrow n-m=-(2k+1)$ number by algebra $-(2k+1)$ is also odd, which implies that $mOn \Rightarrow nOm$ for all $m,n \in z$, so *O* is symmetric.

Comment

Step 3 of 3

O is not transitive

By the definition of *O*, for all m,n and $p \in z$ $m-n$ is odd. So let $m-n=(2k+1)$ by the definition of odd where $k,l \in z$ now, $m-p=m-n+n-p$

$=(m-n)+(n-p)$

$=2k+1+2l+1$

$=2k+2l+2$

$=2(k+l+1)$

$=2t$ where $t=k+l+1$ an integer.

$m-p$ is an odd number which is false. So mOn, nOp but $m \not{O} p$

8.2

8.2.16

Step 1 of 3

A is reflexive

Since for $x \in R$, $|x| = |x| \Rightarrow$ it shows that for $x \in A,\ xAx$, which is the definition of reflexive.

Comment

Step 2 of 3

A is symmetric

Because for $x, y \in R, |x| = |y| \Rightarrow |y| = |x|$, which is true. It shows that for $x, y \in R, |y| = |x|$, which is true, it shows that for $x, y \in R, xAy \Rightarrow yAx$. So *A* follows the definition of symmetric.

Comment

Step 3 of 3

A is transitive

For $x, y, z \in R$ if $|x| = |y|$ and $|y| = |z| \Rightarrow |x| = |z|$

By the equality of the absolute value, which shows test for real numbers x, y, z . If $xAy, yAz,$

8.2

8.2.17

The objective is to determine whether the following relation is reflexive, symmetric and transitive or not:

For all $m, n \in \mathbf{Z}$, $m\ P\ n \Leftrightarrow \exists$ a prime number p such that $p \mid m$ and $p \mid n$.

***P* is reflexive:**

P is reflexive if and only if for all integers n, $n\ P\ n$.

By definition of P, $\exists$ a prime number p such that $p \mid n$ and $p \mid n$.

But this is false for $n = 1$.

There is no prime number that divides 1.

Therefore, P is **not reflexive.**

Comment

Step 2 of 3

***P* is symmetric:**

Suppose m and n are integers such that $m\ P\ n$.

By the definition of P, $\exists$ a prime number p such that $p \mid m$ and $p \mid n$.

This is logically equivalent to $p \mid n$ and $p \mid m$.

Hence there exists a prime number p such that $p \mid n$ and $p \mid m$.

So by the definition of P, $n\ P\ m$.

Therefore, P is **symmetric.**

Comment

Step 3 of 3

***P* is transitive:**

Suppose l, m and n are integers such that $l\ P\ m$ and $m\ P\ n$.

Then need to show that $l\ P\ n$.

For instance:

Take $l = 2, m = 6,$ and $n = 9$.

Here the prime number 2 divides both 2 and 6 and prime number 3 divides 6 and 9.

But the numbers 2 and 9 have no common prime factors.

That is $l \not P\ n$.

Therefore, P is **not transitive.**

8.2

8.2.18

Step 1 of 3

Let Q be a relation on R as for all real numbers x and y,

$xQy \Leftrightarrow x-y$ is rational.

Reflexive: $xQx \Leftrightarrow x-x=0$

and 0 is a rational no. so the relation is reflexive.

Comment

Step 2 of 3

Symmetric: $xQy \Leftrightarrow x-y$ is rational

$\Leftrightarrow -(x-y)$ is also rational

$\Leftrightarrow y-x$ is rational

$\Leftrightarrow yQx$

So the relation is symmetric.

Comment

Step 3 of 3

Transitive: $xQy \Leftrightarrow x-y$ is rational number.

$yQz \Leftrightarrow y-z$ is rational number.

$x-z=(x-y)+(y-z)$

and (x −y) is rational number.

(y −z) is rational number.

So, x −z is also rational number.

Thus the given relation is equivalence relation.

8.2

8.2.21

Let us consider a set $X = \{a,b,c\}$ and $P(X)$ be the power set of X.

Then $P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

Let a relation L is defined on a set $P(X)$ as follows:

For all $A, B \in P(X)$, ALB if and only if the number of elements in A is less than the number of elements in B.

To determine: relation is reflexive.

A relation is reflexive if and only if for all $A \in P(X)$, ALA.

But in the relation L, $(\{a\},\{a\}) \notin L$.

Thus, the relation L is not reflexive.

Comment

Step 2 of 3

To determine: relation is symmetric.

A relation is symmetric if and only if for all $A, B \in P(X)$, if ALB then BLA, i.e., if $(A,B) \in L$ then $(B,A) \in L$.

Let us assume that $\{a\}, \{a,b\} \in P(X)$, such that

$\{a\} L \{a,b\}$

But $\{a,b\}$ is not related to $\{a\}$. As the elements of $\{a,b\}$ is not less than the elements of $\{a\}$.

Thus, the relation L is not symmetric.

8.2

Step 3 of 3

To determine: relation is transitive.

A relation is transitive if and only if for all $A,B,C \in P(X)$, if ALB and BLC then ALC, i.e., if $(A,B) \in L$ and $(B,C) \in L$ then $(A,C) \in L$.

Since ALB and BLC implies that the number of elements of *A* is less than the number of elements of *B*.

hence, $ALC \Rightarrow (A,C) \in L$.

Thus, the relation *L* is transitive.

8.2

8.2.29

Let $A = \mathbf{R} \times \mathbf{R}$.

Let a relation **S** is defined on a set *A* as follows:

For all $(x_1, y_1), (x_2, y_2) \in A$, $(x_1, y_1)\mathbf{S}(x_2, y_2)$ if and only if $y_1 = y_2$.

To determine: relation is reflexive.

A relation is reflexive if and only if for all $(x, y) \in A$, $(x, y)R(x, y)$.

But the second letter is same for (x, y) and (x, y), i.e., $y = y$.

Thus, the relation **S** is reflexive.

Comment

Step 2 of 3

To determine: relation is symmetric.

A relation is symmetric if and only if for all $(x_1, y_1), (x_2, y_2) \in A$, if $(x_1, y_1)\mathbf{S}(x_2, y_2)$ then $(x_2, y_2)\mathbf{S}(x_1, y_1)$.

Since $(x_1, y_1)\mathbf{S}(x_2, y_2)$. Which implies that

$y_1 = y_2$

$\Rightarrow y_2 = y_1$

Hence, $(x_2, y_2)\mathbf{S}(x_1, y_1)$.

Thus, the relation **S** is symmetric.

Comment

Step 3 of 3

To determine: relation is transitive.

A relation is transitive if and only if for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A$, if $(x_1, y_1)\mathbf{S}(x_2, y_2)$ and $(x_2, y_2)\mathbf{S}(x_3, y_3)$ then $(x_1, y_1)\mathbf{S}(x_3, y_3)$.

Since $(x_1, y_1)\mathbf{S}(x_2, y_2)$ and $(x_2, y_2)\mathbf{S}(x_3, y_3)$.

Which implies that

$y_1 = y_2$ and $y_2 = y_3$

$\Rightarrow y_1 = y_3$

Hence, $(x_1, y_1)\mathbf{S}(x_3, y_3)$.

Thus, the relation **S** is transitive.

8.2

8.2.33

R is not reflexive

R is reflexive $\Leftrightarrow$ for all lines *l* in *A*, lRl. By the definition of *R*, l is a line in *A*, then lRl means *l* is perpendicular to *l*, but this is false. Since every line is parallel to itself, but not perpendicular to itself, so *R* is not reflexive.

Comment

Step 2 of 3

R is symmetric

R is symmetric $\Leftrightarrow$ for all l_1, l_2 in A, if $l_1 R l_2$. Then, $l_2 R l_1$ by the definition of R, l_1, l_2 in *A* $l_1 R l_2$ means l_1 is perpendicular to l_2, which implies that l_2 is perpendicular to l_1. This is true since if $l_1 \perp l_2$, then $l_2 \perp l_1$ then $l_2 \perp l_1$, so $l_2 R l_1$.

Comment

Step 3 of 3

R is not transitive

Suppose,

R is transitive $\Leftrightarrow$ for all l_1, l_2, l_3 in *A*. If $l_1 R l_2$ and $l_2 R l_3$, then $l_1 R l_3$. By the definition of *R*, for all l_1, l_2, l_3 in *A*, $l_1 R l_2$ means l_1 is perpendicular to l_2, and $l_2 R l_3$ means l_2 is perpendicular to l_3, but l_1 is not perpendicular to l_3. Since if two lines are perpendicular to one line, then the two lines are parallel. Here, l_1 and l_3 are perpendicular to l_2 so l_1 and l_3 are parallel, so $l_1 R l_3$.

8.2

8.2.38

Step 1 of 2

Suppose two relations R and S on a set A.

The objective is to identify whether $R \cap S$ symmetric when R and S are symmetric.

Recollect the definition of symmetric property.

Let A be a set. And R is a relation on a set A.

For all $x, y \in A$, the condition if xRy then yRx then R is symmetric.

Comment

Step 2 of 2

Given: R and S are symmetric on a set A.

Claim: $R \cap S$ is symmetric, means

$$\forall x \in A, \forall y \in A, [(x,y) \in R \cap S \rightarrow (y,x) \in R \cap S]$$

Let $x, y \in A$.

Assume $(x,y) \in R \cap S$

This implies that $(x,y) \in R$ and $(x,y) \in S$. Definition of Intersection

$\Rightarrow (y,x) \in R$ and $(y,x) \in S$ Since R and S are symmetric on A.

$\Rightarrow (y,x) \in R \cap S$

Hence, If $(x,y) \in R \cap S$ then $(y,x) \in R \cap S$.

Therefore, $R \cap S$ is symmetric relation on a set A.

8.2

8.2.39

Step 1 of 1

Assume that R and S are transitive relations on a set A.

Its need to determine whether $R \cap S$ is transitive or not.

As R and S are transitive relations, $R \cap S$ also **transitive relation**.

Proof:

Given: R and S are transitive on set A

Claim: $R \cap S$ is transitive, means

$$\forall x \in A \;\; \forall y \in A \;\; \forall z \in A \;\; \left(\left[(x,y) \in R \cap S \wedge (y,z) \in R \cap S\right] \rightarrow (x,z) \in R \cap S\right)$$

Let $x, y, z \in A$. Assume $(x,y) \in R \cap S$ and $(y,z) \in R \cap S$. Then $(x,y) \in R$, $(x,y) \in S$, $(y,z) \in R$, $(y,z) \in S$

By the transitive property of R and S, $(x,z) \in R$ and $(x,z) \in S$

As $(x,z) \in R$ and $(x,z) \in S$, $(x,z) \in R \cap S$ (By the definition of intersection)

Thus, if $(x,y) \in R \cap S$ and $(y,z) \in R \cap S$, then $(x,z) \in R \cap S$

Hence, $R \cap S$ is transitive relation.

8.2

8.2.50

Step 1 of 4

Let us consider the set, $A = \{0,1,2,3\}$

Suppose that $R_8 = \{(0,0),(1,1)\}$ be a relation defined on A

Determine whether the relation R_8 is, Irreflexive, asymmetric, intransitive, or none of these

Recall the definition that "A relation on a set A is defined to be

Irreflexive if, and only if, for all $x \in A$, $x \not R x$

Asymmetric if, and only if, for all $x, y \in A$, if $x \, R \, y$ then $y \not R x$

Intransitive if, and only if, for all $x, y, z \in A$, if $x \, R \, y$ and $y \, R \, z$ then $x \not R z$"

Comment

Step 2 of 4

To determine the irreflexivity of the relation R_8 on A :

Observe that, 0 and 1 are the elements of the set $A = \{0,1,2,3\}$, $(0,0) \in R_8$ and $(1,1) \in R_8$

Thus, property of irreflexive does not satisfied.

Therefore, the relation R_8 is **not irreflexive**

8.2

Step 3 of 4

To determine the asymmetricity of the relation R_8 on A :

Observe that, $\forall 0,1 \in A$, 0 is related to 0 and 1 is related to 1 only.

Thus, 0 and 1 are not mapped to the different elements.

So, property of asymmetric satisfied by default.

Therefore, the relation R_8 is **asymmetric**

Comments (1)

Step 4 of 4

To determine the intransitivity of the relation R_8 on A :

Observe that, $\forall 0,1 \in A$, 0 is related to 0 and 1 is related to 1 only.

Thus, 0 and 1 are not mapped to the different elements.

Thus, property of intransitive does satisfy by default.

Therefore, the relation R_8 is **intransitive**.

8.2

8.2.53

Step 1 of 3

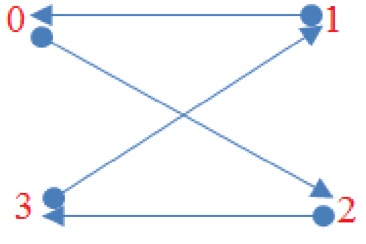
Let $T = \{(0,2),(1,0),(2,3),(3,1)\}$ be a relation defined on $A = \{0,1,2,3\}$.

The objective is to find the transitive closure of T.

Every ordered pair in T is in T', so

$\{(0,2),(1,0),(2,3),(3,1)\} \subseteq T'$.

Thus, the directed graph of T contains the arrows:



Comment

Step 2 of 3

From the figure, observe that the arrows are going from 0 to 2 and 2 to 3, T' must have an arrow going from 0 to 3.

Hence, $(0,3) \in T'$.

As $(0,3) \in T'$ and $(3,1) \in T'$, and T' is transitive, $(0,1) \in T'$.

The arrows going from 2 to 3 and 3 to 1 then T' must have an arrow going from 2 to 1.

Hence, $(2,1) \in T'$.

As $(2,1) \in T'$ and $(1,0) \in T'$, and T' is transitive, $(2,0) \in T'$.

The arrows going from 3 to 1 and 1 to 0 then T' must have an arrow going from 3 to 0.

Hence, $(3,0) \in T'$.

As $(3,0) \in T'$ and $(0,3) \in T'$, and T' is transitive, $(3,3) \in T'$.

8.2

Step 3 of 3

The arrows going from 1 to 0 and 0 to 2 then T' must have an arrow going from 1 to 2.

Hence, $(1,2) \in T'$.

As $(1,2) \in T'$ and $(2,1) \in T'$, and T' is transitive, $(2,2) \in T'$.

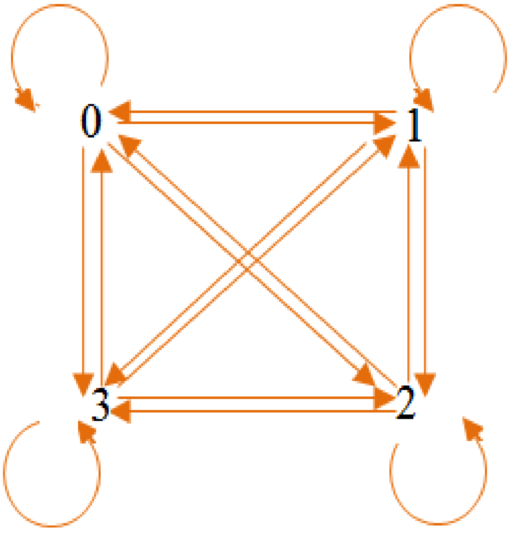
Also, $(1,0),(0,1) \in T'$ then $(1,1) \in T'$ and $(0,2),(2,0) \in T'$ then $(0,0) \in T'$.

Similarly, for $(3,1),(1,2) \in T'$ then $(3,2) \in T'$ and for $(1,2),(2,3) \in T'$ then $(1,3) \in T'$.

Thus, T' contains at least the following ordered pairs:

$$\boxed{\left\{\begin{matrix}(0,2),(1,0),(2,3),(3,1),(0,3),(0,1),(2,1),(2,0), \\ (3,0),(1,2),(2,2),(1,1),(0,0),(3,3),(3,2),(1,3)\end{matrix}\right\}}.$$

The directed graph of T' is:



8.3

8.3.2-c

Step 6 of 7

(c) $R = \begin{Bmatrix} (0,0)(1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(2,4)(3,1)(3,2) \\ (3,3)(3,4)(4,1)(4,2)(4,3)(4,4) \end{Bmatrix}$

Because $\{0\}$ is a partitions subset 0 *R* 0 since 0 and 0 are in $\{0\}$ -

Comment

Step 7 of 7

Since $\{1,2,3,4\}$ is a subset of partition

1 R 1 since 1 and 1 are in $\{1,2,3,4\}$

1 R 2 since 1 and 2 are in $\{1,2,3,4\}$

1 R 3 since 1 and 3 are in $\{1,2,3,4\}$

1 R 4 since 1 and 4 are in $\{1,2,3,4\}$

Similarly, for 2 R 1, 2 R 3, 2 R 2, 2 R 4 and for 3 R 1, 3 R 2, 3 R 3, 3 R 4 and also 4 R 1, 4 R 2, 4 R 3, 4 R 4 also.

8.3

8.3.4

Step 1 of 2

Consider the set $A=\{a,b,c,d\}$ and an equivalence relation R defined on a set A,

$R=\{(a,a),(b,b),(b,d),(c,c),(d,b),(d,d)\}$.

The objective is to find distinct equivalence classes of R.

Recall the definition of an equivalence class as,

Let A be set and R be an equivalence class on A, then for some element $a \in A$ an equivalence class is defined as, $[a]=\{x \in A \mid x R a\}$.

Comment

Step 2 of 2

Write the equivalence classes for each element in the set A using the above definition.

$$[a]=\{x \in A / x R a\}=\{a\}$$
$$[b]=\{x \in A / x R b\}=\{b,d\}$$
$$[c]=\{x \in A / x R c\}=\{c\}$$
$$[d]=\{x \in A / x R d\}=\{b,d\}$$

Here, observe that $[b]=[d]=\{b,d\}$.

Therefore, the distinct equivalence classes of R are,

$\boxed{\{a\},\{b,d\} \text{ and } \{c\}}$.

8.3

8.3.6

The objective is to find the distinct equivalence classes of R.

The relation R is defined on $A=\{-4,-3,-2,-1,0,1,2,3,4,5\}$ is:

for all $x,y\in A,\ xRy \Leftrightarrow 3\,|\,x-y.$

For each integer a,

$$\begin{aligned}[a] &= \{x\in A\,|\,xRa\}\\ &= \{x\in A\,|\,3\,|\,x-a\}\\ &= \{x\in A\,|\,x=3k+a,\ \text{for some integer } k\}\end{aligned}$$

In particular,

$$\begin{aligned}[1] &= \{x\in A: xR1\}\\ &= \{x\in A: 3|x-1\}\\ &= \{x\in A: x-1=3k,\ \text{for some integer } k\}\\ &= \{x\in A: x=3k+1,\ \text{for some integer } k\}\end{aligned}$$

As $3\,|-2-1, 3\,|\,1-1,$ and $3\,|\,4-1$ follows that the equivalence class is $\{-2,1,4\}$.

Comment

Step 2 of 3

Similarly,

$$\begin{aligned}[2] &= \{x\in A: xR2\}\\ &= \{x\in A: 3|x-2\}\\ &= \{x\in A: x-2=3k,\ \text{for some integer } k\}\\ &= \{x\in A: x=3k+2,\ \text{for some integer } k\}\end{aligned}$$

As $3\,|-4-2, 3\,|-1-2, 3\,|\,2-2$ and $3\,|\,5-2$ follows that the equivalence class is $\{-4,-1,2,5\}$.

8.3

And

$$\begin{aligned}[3] &= \{x \in A : xR3\} \\ &= \{x \in A : 3 | x-3\} \\ &= \{x \in A : x-3 = 3k, \text{ for some integer } k\} \\ &= \{x \in A : x = 3k+3, \text{ for some integer } k\}\end{aligned}$$

As $3|-3-3, 3|0-3, \text{ and } 3|3-3$ follows that the equivalence class is $\{-3,0,3\}$.

Therefore, the distinct equivalence classes of R are $\boxed{\{-2,1,4\},\{-4,-1,2,5\},\{-3,0,3\}}$.

8.3

8.3.9

Step 1 of 5

A relation R on a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Also, for each element a in A, the equivalence class of a (denoted by $[a]$) is the set of all those elements x in A such that x is related to a by R.

Comment

Step 2 of 5

Consider the set $X = \{-1, 0, 1\}$.

The set A is the set of all subsets of X.

$$\begin{aligned} A &= P(X) \\ &= \{\phi, \{-1\}, \{0\}, \{1\}, \{-1,0\}, \{0,1\}, \{-1,1\}, \{-1,0,1\}\} \end{aligned}$$

The relation R is an equivalence relation on the set A, for all sets s and t in the power set of A, and is defined as;

$s\ R\ t$ if and only if the sum of the elements in s equals the sum of the elements in t.

Comment

Step 3 of 5

Determine the required equivalence classes of the elements in A as follows;

$$\begin{aligned} [\phi] &= \{s \in R \mid s\ R\ \phi\} \\ &= \phi \end{aligned}$$

And;

$$\begin{aligned} [\{-1\}] &= \{s \in A \mid s\ R\ \{-1\}\} \\ &= \{\{-1, 0\}\} \end{aligned}$$

Similarly;

$$\begin{aligned} [\{0\}] &= \{s \in R \mid s\ R\ \{0\}\} \\ &= \{\{1, -1\}, \{-1, 0, 1\}\} \end{aligned}$$

Similarly obtain;

$$\begin{aligned} [\{1\}] &= \{s \in A \mid s\ R\ \{1\}\} \\ &= \{\{0, 1\}\} \end{aligned}$$

8.3

Step 4 of 5

Proceed in a similar manner to obtain;

$$\begin{aligned}[\{-1,0\}] &= \{s \in A \mid s\ R\{-1,0\}\} \\ &= \{\{-1\}\}\end{aligned}$$

Also then;

$$\begin{aligned}[\{0,1\}] &= \{s \in A \mid s\, R\{0,1\}\} \\ &= \{\{1\}\}\end{aligned}$$

Comment

Step 5 of 5

Continue further to obtain;

$$\begin{aligned}[\{1,-1\}] &= \{s \in A \mid s\, R\{1,-1\}\} \\ &= [\{0\},\{1,-1,0\}]\end{aligned}$$

And;

$$\begin{aligned}[\{-1,0,1\}] &= \{s \in s\, R\{-1,0,1\}\} \\ &= \{\{0\},\{-1,1\}\}\end{aligned}$$

Hence, the required equivalence classes are;

$$\boxed{\{\phi\},\{\{-1\},\{-1,0\}\},\{\{0\},\{1,-1\},\{-1,0,1\}\},\{\{1\},\{0,1\}\}}.$$

8.3

8.3.10

The objective is to find the distinct equivalence classes of $R.$

The relation R is defined on $A=\{-5,-4,-3,-2,-1,0,1,2,3,4,5\}$ is:

for all $(m,n)\in A,\ mRn \Leftrightarrow 3\mid m^2-n^2.$

In particular for $n=-5,$

$$\begin{aligned}[-5]&=\{m\in A: mR(-5)\}\\&=\{m\in A: 3\mid m^2-(-5)^2\}\\&=\{m\in A: 3\mid m^2-25\}\end{aligned}$$

As $3\mid(-5)^2-25, 3\mid(-4)^2-25, 3\mid(-2)^2-25, 3\mid(-1)^2-25, 3\mid 1^2-25, 3\mid 2^2-25, 3\mid 4^2-25,$ and $3\mid 5^2-25$ follows that the equivalence class is $\{-5,-4,-2,-1,1,2,4,5\}.$

Comment

Step 2 of 4

For $n=-4,$

$$\begin{aligned}[-4]&=\{m\in A: mR(-4)\}\\&=\{m\in A: 3\mid m^2-(-4)^2\}\\&=\{m\in A: 3\mid m^2-16\}\end{aligned}$$

As $3\mid(-5)^2-16, 3\mid(-4)^2-16, 3\mid(-2)^2-16, 3\mid(-1)^2-16, 3\mid 1^2-16, 3\mid 2^2-16, 3\mid 4^2-16,$ and $3\mid 5^2-16$ follows that the equivalence class is $\{-5,-4,-2,-1,1,2,4,5\}.$

$$\begin{aligned}[-3]&=\{m\in A: mR(-3)\}\\&=\{m\in A: 3\mid m^2-(-3)^2\}\\&=\{m\in A: 3\mid m^2-9\}\end{aligned}$$

As $3\mid(-3)^2-9, 3\mid 0^2-9,$ and $3\mid 3^2-9$ follows that the equivalence class is $\{-3,0,3\}.$

8.3

Step 3 of 4

For $n=-2$,

$$[-2]=\{m\in A: mR(-2)\}$$
$$=\{m\in A: 3\mid m^2-(-2)^2\}$$
$$=\{m\in A: 3\mid m^2-4\}$$

As $3\mid(-5)^2-4, 3\mid(-4)^2-4, 3\mid(-2)^2-4, 3\mid(-1)^2-4, 3\mid 1^2-4, 3\mid 2^2-4, 3\mid 4^2-4$, and $3\mid 5^2-4$ follows that the equivalence class is $\{-5,-4,-2,-1,1,2,4,5\}$.

For $n=0$,

$$[0]=\{m\in A: mR0\}$$
$$=\{m\in A: 3\mid m^2-0^2\}$$
$$=\{m\in A: 3\mid m^2\}$$

As $3\mid(-3)^2-0, 3\mid 0^2-0$, and $3\mid 3^2-0$ follows that the equivalence class is $\{-3,0,3\}$.

Comment

Step 4 of 4

As the square of any integer is always positive, so $[-5]=[5],[-4]=[4]$, $[-3]=[3],[-2]=[2]$, and $[-1]=[1]$.

From the above result, the equivalence classes of $[-5]=[-2]=[-4]$ and $[-3]=[0]$

Therefore, the distinct equivalence classes of R are: $\boxed{\{-5,-4,-2,-1,1,2,4,5\},\{-3,0,3\}}$.

8.3

8.3.18

a.

The objective is to give an example of two sets that are distinct but not disjoint.

The distinct sets contain one or more different elements.

Disjoint sets are those sets whose intersection is an empty set.

Example of two sets which are distinct but not disjoint are,

Let $A=\{2,3\}$ and $B=\{3,4\}$

So, sets A and B are distinct because $A \neq B$, but A and B are not disjoint since $A \cap B=\{3\} \neq \phi.$

Comment

Step 2 of 2

b.

Consider the values of x, y, z as,

$x=2, y=3,$ and $z=4.$

The set A_1 contains the elements x and $y.$

Therefore, the set $A_1=\{2,3\}.$

The set A_2 contains the elements y and $z.$

Hence, the set $A_2=\{3,4\}.$

Here, the element $2=x \in A_1$, but it is not in the set A_2 that is $x \notin A_2.$

Similarly, the element $4=z \in A_2$ but it is not in the set A_1 that is $z \notin A_1.$

However, neither of the sets A_1 and A_2 contain both 2 and 4.

Hence, the required sets are, $A_1=\{2,3\}$ and $A_2=\{3,4\}.$

8.3

8.3.20

Step 1 of 4

Consider the relation E defined on $\mathbb{Z}$ as follows:

$mEn \Leftrightarrow 2|(m-n)$ For all $m,n \in \mathbb{Z}$.

(1)

The objective is to prove the given relation is equivalence relation.

The equivalence relation satisfies reflexive, symmetric, and transitive properties.

Show that E is reflexive:

For all $m \in z$, $m \in m$

$2|(m-m) \Rightarrow 2|0$, which is true.

Hence, the relation E is reflexive.

Comment

Step 2 of 4

Show that E is symmetric:

For all $m,n \in z, m \in n$

$= 2|(m-n)$

$= 2|(-(n-m)) \Rightarrow n \in m$.

It is true.

Hence, the relation E is symmetric.

8.3

Step 3 of 4

Show that E is transitive:

For all $m, n, p \in z$ if $m\mathrm{E}n$ and $n\mathrm{E}p$

$$2 \mid (m-n) \text{ and } 2 \mid (n-p) \Rightarrow 2 \mid ((m-n)+(n-p))$$

$$= 2 \mid (m-p) \Rightarrow m\mathrm{E}p,$$

This is true.

Hence, the relation E is transitive.

Therefore, E is an equivalence relation since *E* is reflexive, symmetric and transitive.

Comment

Step 4 of 4

(2)

The objective is to describe the distinct equivalence classes of the relation E.

Here, the relation E has two distinct equivalence classes, they are,

$\{x \in \mathbb{Z} \mid x = 2k, \text{ for some integer } k\}$ and $\{x \in \mathbb{Z} \mid x = 2k+1, \text{ for some integer } k\}$

Since the difference of two even numbers is divestible by 2, and the difference of two odd numbers is divisible by 2 by base algebra and $2k$ is an even number and $2k+1$ is an odd number.

8.3

8.3.26

(1)

The objective is to determine whether the following relation D defined on $\mathbf{Z}$ is an equivalence or not:

For all $(m,n) \in \mathbf{Z}$, $m\,D\,n \Leftrightarrow 3 \mid (m^2 - n^2)$.

A relation is said to be an equivalence relation if it satisfies reflexive, symmetric, and transitive properties.

Let's check whether relation D satisfies all the three properties or not.

Comment

Step 2 of 6

Reflexive:

For $m \in \mathbf{Z}$, $m^2 - m^2 = 0$.

As $3 \mid 0$, it follows that $3 \mid (m^2 - m^2) \Rightarrow m\ D\ m$.

Therefore, D is reflexive.

Comment

Step 3 of 6

Symmetric:

For $m, n \in \mathbf{Z}$, let $m\ D\ n$ then by the definition of relation, $3 \mid (m^2 - n^2)$.

Rewrite the expression $3 \mid (m^2 - n^2)$ as:

$3 \mid (m^2 - n^2) \Rightarrow 3 \mid -(n^2 - m^2)$.

It follows that $3 \mid (n^2 - m^2)$.

Thus, $n\ D\ m$.

Therefore, D is symmetric.

8.3

Transitive:

For $m, n, p \in \mathbf{Z}$, let $m\ D\ n$ and $n\ D\ p$.

Then by the definition of relation, $3 \mid (m^2 - n^2)$ and $3 \mid (n^2 - p^2)$.

Combine the expression $3 \mid (m^2 - n^2)$ and $3 \mid (n^2 - p^2)$.

$3 \mid (m^2 - n^2) + (n^2 - p^2) \Rightarrow 3 \mid (m^2 - p^2)$.

It follows that, $m\ D\ p$.

Therefore, D is transitive.

As D being reflexive, symmetric and transitive, so D is an **equivalence relation**.

Comment

Step 5 of 6

(2)

The objective is to describe the distinct equivalence classes of each relation.

To find the equivalence classes, use division algorithm.

For any $m \in \mathbf{Z}$, there exists unique q, r such that $m = 3q + r$, for $0 \le r < 3$.

For $r = 0$, $m = 3q$.

Squaring on both sides,

$$m^2 = (3q)^2$$
$$m^2 - 0^2 = 3(3q^2)$$

For $r = 1$, $m = 3q + 1$

Squaring on both sides,

$$m^2 = (3q+1)^2$$
$$m^2 = (3q)^2 + 2(3q) + 1^2$$
$$m^2 - 1^2 = 3(3q^2 + 2q)$$

8.3

Step 6 of 6

For $r = 2,\ \ m = 3q + 2$

Squaring on both sides,

$$m^2 = (3q+2)^2$$
$$m^2 = (3q)^2 + 2(3q)(2) + 3 + 1^2$$
$$m^2 - 1^2 = 3(3q^2 + 4q + 1)$$

From all the three expressions, $3 \mid m^2 - 0^2$ or $3 \mid m^2 - 1^2$

Therefore, there are **two** equivalence classes $\boxed{[0] \text{ and } [1]}$.

8.3

8.3.37

Step 1 of 4

Consider R is equivalence relation on a set A and $a,b \in A$.

The objective is to prove that *a R b* whenever $b \in [a]$.

Comment

Step 2 of 4

Recall that *R* is an equivalence relation when it is reflexive, symmetric and transitive.

Also recall the definition of equivalence class:

Suppose *R* is equivalence relation on a set A. The equivalence class of $a \in A$ is the set of all elements of *A* that are related to *a*.

Symbolically:

$[a] = \{x \in A \mid xRa\}$.

8.3

Step 3 of 4

Consider a and b be two elements in A such that $b \in [a]$.

The equivalence class of *a* in *R* is given by,

$[a] = \{x \in A \mid x R a\}$.

Since $b \in [a]$.

This implies, $b R a$.

Comment

Step 4 of 4

Recall the definition of symmetric:

Suppose *R* is a relation on a set A. The *R* is said to be symmetric if for $a, b \in A$, $(b, a) \in R$ whenever $(a, b) \in R$.

Since the relation R is an equivalence relation.

So, the relation R is symmetric.

Hence, by the definition of symmetric relation $b R a$ implies $\boxed{a R b}$.

8.3

8.3.42

Step 1 of 4

(a) *R* is reflexive

Because $(a,b)\in A,(a,b)\Rightarrow ab=ba$ by the definition of *R*. This is true by commutation property of multiplication of integers, so *R* is reflexive.

Comment

Step 2 of 4

(b) *R* is symmetric

Because for all $(a,b),(c,d)\in A$ if $(a,b)R(c,d)$ $\Rightarrow ad=bc$ by the definition of *R*, which implies $bc=ad$.

By the equality of integers, which gives $(c,d)R(a,b)$.

This is true, so *R* is symmetric.

Comment

8.3

(c) The distinct four elements in $[(1,3)]$ are

$(2,6)(-2,-6)(3,9)(-3,-9)$

Since $(2,6)R(1,3) \Rightarrow 2\times 3 = 6\times 1$, which is true

$(-2,-6)R(1,3) \Rightarrow -2\times 3 = -6\times 1$, which is true

$(3,9)R(1,3) \Rightarrow 3\times 3\times = 9\times 1$, which is true

$(-3,-9)R(1,3) \Rightarrow -3\times 3 = -9\times 1$, which is true.

Comment

Step 4 of 4

(d) The distinct four elements in $[(2,5)]$ are

$(-2,-5)(4,10)(-4,-10)(6,15)$

Since $(-2,-5)R(2,5) \Rightarrow -2\times 5 = -5\times 2$, which is true.

$(4,10)R(2,5) \Rightarrow 4\times 5 = 10\times 2$, which is true.

$(-4,-10)R(2,5) \Rightarrow -4\times 5 = -10\times 2$, which is true.

$(6,15)R(2,5) \Rightarrow 6\times 5 = 15\times 2$, which is true.

8.3

8.3.44-c

Step 3 of 11

(c)

The objective is to prove that R is transitive.

For $(a,b),(c,d),(e,f)\in A$, and $(a,b)R(c,d),(c,d)R(e,f)$.

Then by definition of R,

$(a,b)R(c,d)\Leftrightarrow a+d=c+b \quad \cdots\cdots(1)$

$(c,d)R(e,f)\Leftrightarrow c+f=e+d \quad \cdots\cdots(2)$

Add (1) and (2) to get

$a+d+c+f=c+b+e+d$

Subtract $c+d$ from both sides.

$a+f=b+e$

Thus, by definition of R, $(a,b)R(e,f)$.

Therefore, R is **transitive**.

8.3

8.3.44-f

The objective is to list 5 elements of $[(1,2)]$.

For $(1,2) \in A$ then $[(1,2)] = \{(a,b) \in A \mid (a,b) R (1,2)\}$.

For $(2,3) R (1,2)$, then by definition of R:

$$2+2 = 1+3$$
$$4 = 4$$

This is true.

For $(3,4) R (1,2)$, then by definition of R:

$$3+2 = 1+4$$
$$5 = 5$$

This is true.

Comment

Step 10 of 11

For $(4,5) R (1,2)$, then by definition of R:

$$4+2 = 1+5$$
$$6 = 6$$

This is true.

For $(5,6) R (1,2)$, then by definition of R:

$$5+2 = 1+6$$
$$7 = 7$$

This is true.

For $(6,7) R (1,2)$, then by definition of R:

$$6+2 = 1+7$$
$$8 = 8$$

This is true.

Therefore, the five possible elements of $[(1,2)]$ are $\boxed{(2,3),(3,4),(4,5),(5,6),(6,7)}$.

8.3

8.3.46

Step 1 of 1

Consider the relation R on a set $A.$

Suppose R is symmetric and transitive.

The objective is to prove that if for every x in a set A there is a y in A such that $xRy,$ then R is an equivalence relation.

The equivalence relation satisfies reflexive, symmetric and transitive properties.

If for every *x* in *A* there is a *y* in *A* such that $xRy \Rightarrow yRx$ (since R is symmetric).

If xRy and yRx, then xRx since by R is transitive.

So, xRx it is given that R is reflexive.

For $x \in A$, then $xRx.$

Hence, the relation R is reflexive.

Therefore, the relation R is an equivalence relation.