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Mathematics Department

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Math 233

3rd. Hour Exam

1st Semester 6/07

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because 7

1) Mark each of the following statement True a False:

True ✓ $(300)^6 \equiv 1 \pmod 7$

~~True~~ ✗ p is a prime and $a \in N \Rightarrow a^{p-1} \equiv 1 \pmod p$

$\Leftarrow$ True ✓ S is an equivalence relation, then $(x, y) \in S \Leftrightarrow [x] = [y]$

False ✓ R, S are relations on A, then $(R \circ S)^{-1} = R^{-1} \circ S^{-1}$

False ✓ For every a, b, c $\in$ N if $ac \equiv bc \pmod n \Rightarrow a \equiv b \pmod n$

False ✓ Let $f : X \to Y$ be a function, $B \subseteq Y$ then $f(f^{-1}(B)) = B$

True ✓ Let $f : X \to Y$ be a function, $A \subseteq X$ then $A \subseteq f^{-1}(f)A))$

False ✓ The function$\left(f : X \to Y \text{ is one to one}\right) \Leftrightarrow \left(\text{if } x = y \text{ then } f(x) = f(y)\right)$

✓ Then function $f : S \to S$ where $f(s) = s^2$, S is all nonnegative real numbers, is a function from S onto S.

True ✓ Let $f : X \to Y$ be a function, $A \subseteq X$ then if $x \in A \Rightarrow f(x) \in f(A)$

2) Prove or disprove. Give full details.

(a) Let $f : A \to B$ and $g : B \to A$ be two functions such that $g \circ f = i_A$ then f is one to one.

Given: $f : A \to B$, $g : B \to A$, $g \circ f = i_A$

prove f is 1-1: if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Let $x_1, x_2 \in A$ s.t. $f(x_1) = f(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$ (def of $f(D)$)

$\Rightarrow g \circ f(x_1) = g \circ f(x_2)$ (def of $\circ$)

$\Rightarrow i_A(x_1) = i_A(x_2)$ (since $g \circ f = i_A$) (def of i_A)

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is 1-1

(b) If f is a one to one function from N to N (where N is the set of all natural numbers), then f must be onto.

By Counter Example

Let $f(x) = 2x+1$

$f(x)$ is 1-1 $\forall x \in N$

But $f(x)$ is not onto Because only the odd Natural number have a source

$6 \in N$ But $\nexists x \in N$ $f(x) = 6$

(c) $\forall a \in Z, n \in N$ if $a^2 \equiv 1(\text{mod}\, n)$ then $a \equiv 1(\text{mod}\ n)$ or $a \equiv -1(\text{mod}\ n)$

False By Counter Example:

let $a = 3$ and $n = 8$

$\Rightarrow (3)^2 \equiv 1 \pmod 8$ is true

But $3 \not\equiv 1 \pmod 8$

$\wedge\ 3 \not\equiv -1 \pmod 8$

(d) Let f be a function $f : Z \to Z$ (where Z is the set of all integers) defined as $f(x) = 2x+3$. If $a \equiv b\ (\text{mod}\ 5) \Rightarrow f(a) \equiv f(b)\ (\text{mod}\ 5)$

Given $a \equiv b \pmod 5 \Rightarrow \frac{a-b}{5} = k \in \mathbb{Z}$

$f(x) = 2x+3$

prove: $f(a) \equiv f(b) \pmod 5$

prove :- $\frac{f(a)-f(b)}{5}$ = Integer

$f(a) = 2a+3$

$f(b) = 2b+3$

$\Rightarrow \frac{f(a)-f(b)}{5} = \frac{2a+3-2b-3}{5} = \frac{2(a-b)}{5} = 2(\underset{k \in \mathbb{Z}}{k})$ = Integer

3) Let Z be the set of all integers, for $n \in Z$ $B_n = \{m \in Z : \exists q \text{ s.t. } m = 5q + n\}$

(a) Write five elements in B_0.

$B_0 = \{$ 5, 10, 15, 20, 25, ...$\}$

(b) Write five elements in B_1

$B_1 = \{$ ___, ___, 13, 20, 25, ...$\}$

(c) How many distinct B_n are there? List them.

$B_2 = \{7, 12, 17, 22, 27 \ldots\}$

$B_3 = \{8, 13, 18, 23, 28 \ldots\}$

$B_4 = \{9, 14, 19, 24, 29 \ldots\}$

$B_5 = \{10, 15, 20, 25, 30 \ldots\} = B_0$

$B_6 = B_1$

$\Rightarrow$ There are 5 distinct B_n and ...

(d) Prove that $\{B_n\}_{n \in z}$ is a partition of Z.

Given:- $B_n = \{m \in Z : \exists q \text{ s.t } m = 5q + n\}$

1. Let $x \in \cup B_n \Rightarrow x = 5q + n$ ... $\Rightarrow \cup B_n = Z$

2. Let $B_{n_1} = [x]$, $B_{n_2} = [y]$ s.t $B_{n_1} \cap B_{n_2} \neq \phi$

we want to prove that $[y] = [x]$

$B_{n_1} \cap B_{n_2} \neq \phi \Rightarrow \exists z \in [y] \wedge z \in [x]$

$\Rightarrow (y, z) \in R \wedge (x, z) \in R$ (def of [ ])

$\Rightarrow (y, x) \in R$ R is eq (Trans)

prove $\{B_n\}_{n \in z}$ is a partition

$\cup B_n = S = Z$

$\forall B_{n_1}, B_{n_2} \in B_n$ then $B_{n_1} = B_{n_2}$ or $B_{n_1} \cap B_{n_2} = \phi$

Cont $[x] = [y]$

Let $a \in [x] \Rightarrow (x, a) \in R$

$\Rightarrow (x, a) \in R \wedge (y, x) \in R$

$\Rightarrow (y, x) \in R \wedge (a, x) \in R$ (sym)

$\Rightarrow (y, a) \in R \Rightarrow a \in [y]$ def of [ ]

Let $a \in [y]$

$(y, a) \in R \wedge (y, x) \in R$

$\Rightarrow (x, a) \in R \Rightarrow a \in [x]$

$\Rightarrow [x] = [y]$

(e) Define the equivalence relation induced by this partition.

$Z/R = \{B_0, B_1, B_2, B_3, B_4\}$

$= \{\{5, 10, 15, 20 \ldots\}, \{6, 11, 16 \ldots\}, \{7, 12, 17 \ldots\}, \{9, 14, 19 \ldots\}, \{8, 13, 18 \ldots\}\}$

R

$R = \{(0, 5q), (1, 5q+1), (2, 5q+2), (3, 5q+3), (4, 5q+4)\}$ q Int

$(x, y) \in R \Longleftrightarrow (x, 5q + x) \in R$

X أي عدد ينتمي لـ Z

Y: أي عدد عبارة عن 5q + x حيث أن q أي عدد ينتمي لـ Z

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4) Let $n \in N$ and $a, b \in z$ where $d = \gcd(a, n)$. prove that if d does not divide b then there is no solution for the equation $ax \equiv b \pmod n$.

Contrapositive

If $d \nmid b \Rightarrow ax \not\equiv b \pmod n$ Given $d = \gcd(a, n)$

Converse: $ax \equiv b \pmod n \Rightarrow d \mid b$ $d \mid a \wedge d \mid n$

$\frac{ax - b}{n} = k \Rightarrow \frac{ax}{n} - \frac{b}{n} = k$

$\Rightarrow \frac{dhx}{dm} - \frac{b}{dm} = k \Rightarrow dhx - b = dmk$

$\Rightarrow b = dhx - dmk \Rightarrow b = d(hx - mk)$ $(hx - mk) \in \mathbb{Z}$

$\Rightarrow b/d \Rightarrow$ If $d \nmid b$, $ax \not\equiv b \pmod n$

5) Let S= set of all positive rational numbers, and $T = Z^* x Z^*$ where Z^* is the set of all positive integers. Define $f : T \to S$ such that $f((m, n)) = \frac{m}{n}$

(a) Prove that f is a function Given $T = \mathbb{Z} \times \mathbb{Z}$

prove $\forall (m,n) \in T$ $\exists y$ s.t $f(m,n) = y$ $f : T \to S$ s.t $f(m,n) = \frac{m}{n}$

Case 1: If $m = n \Rightarrow \exists y = 1$ s.t $f(m,n) = \frac{m}{n} = 1 \in S$

Case 2: If $m > n \Rightarrow \exists y$ s.t $f(m,n) = y = \frac{m}{n} \in S$

Case 3: If $m < n \Rightarrow \exists y$ s.t $f(m,n) = y = \frac{m}{n} \in S$

2. $(m_1,n_1) = (m_2,n_2) \Rightarrow f(m_1,n_1) = f(m_2,n_2)$

let $(m_1,n_1) = (m_2,n_2)$

$\Rightarrow f(m_1,n_1) = \frac{m_1}{n_1} \wedge f(m_2,n_2) = \frac{m_2}{n_2}$

$\Rightarrow \frac{m_1}{n_1} = \frac{m_2}{n_2} \Rightarrow f(m_1,n_1) = f(m_2,n_2)$

$\Rightarrow f((m,n))$ is a function

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(b) Is f one to one? Give full details.

Let $f((m_1,n_1)) = f((m_2,n_2))$ we want to show that $(m_1,n_1) = (m_2,n_2) \Rightarrow m_1 = m_2$, $n_1 = n_2$

$\frac{m_1}{n_1} = \frac{m_2}{n_2} \Rightarrow$ ~~$(m_1 n_2) = (n_1 m_2)$~~

$\Rightarrow m_1 = k m_2$

$n_1 = k n_2$

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(c) Is f onto? Give full details.

(15) $\forall y \in S$ $\exists (m,n) \in T$ s.t $f(m,n) = y = \frac{m}{n}$

Case 1: $0 < y < 1 \Rightarrow \exists (m,n)$ s.t $f(m,n) = y$ s.t $m < n$

Case 2 $y = 1 \Rightarrow \exists (m,n)$ s.t $f(m,n) = y$ s.t $m = n$

Case 3 $y > 1 \Rightarrow \exists (m,n)$ s.t $f(m,n) = y$ s.t $m > n$

$\Rightarrow \forall y \in S$ $\exists (m,n) \in T$ s.t $f(m,n) = y$

$\Rightarrow f$ is onto

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6) Let $f : X \to Y$ be a one to one function from X onto Y

(a) Prove that $f^{-1} : Y \to X$ is a function.

1. $\forall y \in Y \; \exists x \in X$ s.t. $f^{-1}(y) = x$??

$\forall y \in Y \; \exists x \in X$ s.t. $f(x) = y$ (?) f is onto

$\forall y \in Y \; \exists x$ s.t. $x = f^{-1}(y)$ (def of f^{-1})

2. $\forall y_1 = y_2 \Rightarrow f^{-1}(y_1) = f^{-1}(y_2)$??

$y_2 = y_1 = f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (Given ?)

$\Rightarrow x_1 = f^{-1}(y_1), \; x_2 = f^{-1}(y_2)$, (def of f^{-1}) $\Rightarrow y_1 = y_2 \Rightarrow f^{-1}(y_1) = f^{-1}(y_2)$ def of f^{-1}

(b) Prove that $f^{-1} : Y \to X$ is one to one. given f is func

Let $f^{-1}(y_1) = f^{-1}(y_2) = x$

$\Rightarrow x = f^{-1}(y_1), \; x = f^{-1}(y_2)$

$\Rightarrow f(x) = y_1, \; f(x) = y_2$

$\Rightarrow y_1 = y_2 \Rightarrow f^{-1}$ is 1-1

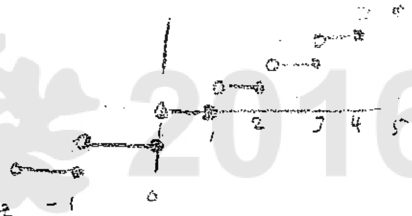
(c) Prove that $f^{-1} : Y \to X$ is onto.

given f is fun $\Rightarrow \forall x \in X \; \exists y \in Y$ s.t. $f(x) = y$

$\forall x \in X \; \exists y \in Y$ s.t. $f^{-1}(y) = x$ def of f^{-1}

$\Rightarrow f^{-1}$ is onto

Bonus Let $f : R \to Z$ f(x) = [x] (greatest integer)



(a) Find $f(A)$ where $A = [0,2] \cup (3,5)$

$f(A) = \{0, 1, 2, 4, 3\}$

(b) Find $f(B)$ where $B = \{0,1\}$

$f(B) = \{0, 1\}$

(c) Find $f(f^{-1}(B))$ where $B = \{0,1\}$

$f(f^{-1}(B)) =$

$\hookrightarrow f^{-1}(B) = (-1, 0] \cup [0, 1]$ $-\frac{2}{2}$

$f(f^{-1}(B)) = \{0, 1\}$

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