جامعة بيرزيت

BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

# COMP233
# Discrete Mathematics

# CHAPTER 2

## THE LOGIC OF COMPOUND STATEMENTS

# The Logic Of Compound Statements

- This chapter presents rules that can be used for deductive reasoning.
- We will learn how what statements are, how to evaluate their truth and falsity, and how to perform simplification, reasoning and validation checks.

# Outline

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# Propositions

- A **statement** (or **proposition**) is a sentence that is true or false but not both.

- "Two plus two equals four" is a statement because it is true.
- "Two plus two equals five" is a statement because it is false.

- "He is a college student"

  This truth or falsity of this sentence depends on the reference for the pronoun *he*.

→ Considered on its own, the sentence is neither true nor false, and so it is not a statement.

# Negation

- Given a statement $p$, the sentence "$\sim p$" is read "not $p$" or "It is not the case that $p$", and is called the **negation** of $p$.

- Negation has opposite truth value from $p$: if $p$ is true, $\sim p$ is false; if $p$ is false, $\sim p$ is true.

**Truth Table for $\sim p$**

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

# Conjunction

- Given a statement $p$ and another statement $q$, the sentence "$p \wedge q$" is read "$p$ and $q$" and is called the **conjunction** of $p$ and $q$.

- Conjunction is true when, and only when, both $p$ and $q$ are true. If either $p$ or $q$ is false, or if both are false, $p \wedge q$ is false.

**Truth Table for $p \wedge q$**

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

# Disjunction

- Given a statement $p$ and another statement $q$, the sentence "$p \vee q$" is read "$p$ or $q$" and is called the disjunction of $p$ and $q$.

- Disjunction is true when either $p$ is true, or $q$ is true, or both $p$ and $q$ are true; it is false only when both $p$ and $q$ are false.

**Truth Table for $p \vee q$**

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Statement Forms

- A **statement form** (or propositional form) is an expression made up of statement variables (such as $p$, $q$, and $r$) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables.

- The **truth table** for a given statement form displays the truth values that correspond to <u>all possible combinations</u> of truth values for its component statement variables.

- Let's construct the truth table for the statement form $(p \vee q) \wedge \sim(p \wedge q)$:

| p | q | p ∨ q | p ∧ q | ~(p ∧ q) | (p ∨ q) ∧~(p ∧ q) |
|---|---|---|---|---|---|
| | | | | | |

- Let’s construct the truth table for the statement form $(p \vee q) \wedge \sim(p \wedge q)$:

| p | q | p ∨ q | p ∧ q | ~(p ∧ q) | (p ∨ q) ∧~(p ∧ q) |
|---|---|---|---|---|---|
| T | T | | | | |
| T | F | | | | |
| F | T | | | | |
| F | F | | | | |

# Exclusive Or

- Let's construct the truth table for the statement form $(p \vee q) \wedge \sim(p \wedge q)$:

| p | q | p ∨ q | p ∧ q | ~(p ∧ q) | (p ∨ q) ∧~(p ∧ q) |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | F | T | F | T | T |
| F | T | T | F | T | T |
| F | F | F | F | T | F |

- This is sometimes abbreviated p ⊕ q or p XOR q.

# Double Negative Property

- Let's construct the truth table for $p$, $\sim p$, and $\sim(\sim p)$:

| $p$ | $\sim p$ | $\sim(\sim p)$ |
|---|---|---|
| T | F | T |
| F | T | F |

- We notice that $p$ and $\sim(\sim p)$ always have the same truth values, so they are logically equivalent.

$$\sim(\sim p) \equiv p$$

# Logical Equivalence

- The statements
  6 is greater than 2
  and
  2 is less than 6
  are two different ways of saying the same thing because of the definition of the phrases greater than and less than.

- The statements
  Dogs bark and cats meow
  and
  Cats meow and dogs bark
  are also two different ways of saying the same thing, because of the logical form of the statements.

# Logical Equivalence – cont.

- Two statement forms are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables.
- The logical equivalence of statement forms $P$ and $Q$ is denoted by writing $P \equiv Q$.

- Two statements are called logically equivalent if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

# Testing Logical Equivalence

To test whether two statement forms $P$ and $Q$ are logically equivalent

1. Construct a truth table with one column for the truth values of $P$ and another column for the truth values of $Q$.
2. Check each combination of truth values of the statement variables to see whether the truth value of $P$ is the same as the truth value of $Q$.
   a. *If in each row the truth value of $P$ is the same as the truth value of $Q$, then $P$ and $Q$ are logically equivalent.*
   b. *If in some row $P$ has a different truth value from $Q$, then $P$ and $Q$ are not logically equivalent.*

# Showing Nonequivalence

- Show that the statement forms $\sim(p \land q)$ and $\sim p \land \sim q$ are not logically equivalent.

| p | q | ~p | ~q | p ∧ q | ~(p ∧ q) | ~p ∧~q |
|---|---|---|---|---|---|---|

# Showing Nonequivalence

- Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

| p | q | ~p | ~q | p ∧ q | ~(p ∧ q) | ~p ∧~q |
|---|---|---|---|---|---|---|
| T | T | | | | | |
| T | F | | | | | |
| F | T | | | | | |
| F | F | | | | | |

# Showing Nonequivalence

- Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

| p | q | ~p | ~q | p ∧ q | ~(p ∧ q) | ~p ∧~q |
|---|---|---|---|---|---|---|
| T | T | F | F | T | | |
| T | F | F | T | F | | |
| F | T | T | F | F | | |
| F | F | T | T | F | | |

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- Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

| p | q | ~p | ~q | p ∧ q | ~(p ∧ q) | ~p ∧~q |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | F |
| F | T | T | F | F | T | F |
| F | F | T | T | F | T | T |

# Showing Nonequivalence

- Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

| p | q | ~p | ~q | p ∧ q | ~(p ∧ q) | | ~p ∧~q |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | | F |
| T | F | F | T | F | T | ≠ | F |
| F | T | T | F | F | T | ≠ | F |
| F | F | T | T | F | T | | T |

- $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

# De Morgan’s Laws

- De Morgan’s Laws are for the negations of And and Or statements.

- To put it simply:
  - *The negation of an and statement is logically equivalent to the or statement in which each component is negated.*
  - *The negation of an or statement is logically equivalent to the and statement in which each component is negated.*

# De Morgan's Laws – cont.

- Symbolically, given the statements $p$ and $q$:

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

and

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

# De Morgan's Laws – cont.

- Show that

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

- We can prove this by constructing the truth table for both statements

| p | q | ~p | ~q | p ∧ q | ~(p ∧ q) | ~p ∨ ~q |
|---|---|---|---|---|---|---|
| | | | | | | |

# De Morgan’s Laws – cont.

- Show that

$$\sim(p \land q) \equiv \sim p \lor \sim q$$

- We can prove this by constructing the truth table for both statements

| p | q | ~p | ~q | p ∧ q | ~(p ∧ q) | ~p ∨ ~q |
|---|---|---|---|---|---|---|
| T | T | | | | | |
| T | F | | | | | |
| F | T | | | | | |
| F | F | | | | | |

# De Morgan’s Laws – cont.

- Show that

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

- We can prove this by constructing the truth table for both statements

| p | q | ~p | ~q | p ∧ q | ~(p ∧ q) | ~p ∨ ~q |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T |

# Exercise

- Show that

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Do this exercise at home and submit it on the course's ITC page.

# Discussion

- As a programmer, can you imagine a scenario where it is very important to know how De Morgan’s laws work?

# Tautologies

- A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables.
- A statement whose form is a tautology is a tautological statement.

- For example, the statement $p \vee \sim p$ is always true. Let's show that:

| p | ~p | p ∨ ~p |
|---|---|---|
| | | |

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|---|---|---|
| T | F | T |
| F | T | T |

# Contradictions

- A **contradiction** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables.
- A statement whose form is a contradiction is a contradictory statement.

- For example, the statement $p \wedge \sim p$ is always false. Let's show that:

| p | ~p | p ∧ ~p |
|---|---|---|
| | | |

# Contradictions

- A **contradiction** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables.
- A statement whose form is a contradiction is a contradictory statement.

- For example, the statement $p \wedge \sim p$ is always false. Let's show that:

| p | ~p | p ∧ ~p |
|---|---|---|
| T | F | F |
| F | T | F |

# Tautologies and Contradictions

- If $t$ is a tautology and $c$ is a contradiction, show that $p \wedge t \equiv p$ and $p \vee c \equiv p$.

| p | t | p ∧ t | p | c | p ∨ c |
|---|---|---|---|---|---|
| | | | | | |

# Tautologies and Contradictions

- If $t$ is a tautology and $c$ is a contradiction, show that $p \wedge t \equiv p$ and $p \vee c \equiv p$.

| p | t | p ∧ t | p | c | p ∨ c |
|---|---|---|---|---|---|
| T | T | T | T | F | T |
| F | T | F | F | F | F |

- These are called ***identity laws***.

# Summary of Logical Equivalences

**Theorem 2.1.1 Logical Equivalences**

Given any statement variables $p, q$, and $r$, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

| | | |
|---|---|---|
| 1. *Commutative laws:* | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| 2. *Associative laws:* | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| 3. *Distributive laws:* | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| 4. *Identity laws:* | $p \wedge \mathbf{t} \equiv p$ | $p \vee \mathbf{c} \equiv p$ |
| 5. *Negation laws:* | $p \vee \sim p \equiv \mathbf{t}$ | $p \wedge \sim p \equiv \mathbf{c}$ |
| 6. *Double negative law:* | $\sim(\sim p) \equiv p$ | |
| 7. *Idempotent laws:* | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| 8. *Universal bound laws:* | $p \vee \mathbf{t} \equiv \mathbf{t}$ | $p \wedge \mathbf{c} \equiv \mathbf{c}$ |
| 9. *De Morgan's laws:* | $\sim(p \wedge q) \equiv \sim p \vee \sim q$ | $\sim(p \vee q) \equiv \sim p \wedge \sim q$ |
| 10. *Absorption laws:* | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 11. *Negations of* **t** *and* **c**: | $\sim \mathbf{t} \equiv \mathbf{c}$ | $\sim \mathbf{c} \equiv \mathbf{t}$ |

# Summary of Logical Equivalences

- Do I need to learn these laws?

  Yes!
- Do I need to learn their names?

  Yes!
- How can I do that?

  Practice, practice, practice!

  Do as many exercises as possible. I recommend starting from the book's exercises.
- How will the questions look like?

  Let's take a look at an example.

# Simplifying Statement Forms

- Verify the logical equivalence

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

$\sim(\sim p \wedge q) \wedge (p \vee q)$

| | |
|---|---|
| $\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$ | by De Morgan's laws |
| $\equiv (p \vee \sim q) \wedge (p \vee q)$ | by the double negative law |
| $\equiv p \vee (\sim q \wedge q)$ | by the distributive law |
| $\equiv p \vee (q \wedge \sim q)$ | by the commutative law for ∧ |
| $\equiv p \vee c$ | by the negation law |
| $\equiv p$ | by the identity law. |

# Simplifying Statement Forms

- Do I need to write the law's name?

  Yes! If you don't, your solution will be incomplete, and you will lose points.

- Can I merge steps?
- The step I made has two laws?

  Every step you take must simplify or modify one single part of the statement.

  Go back and split the step into two steps.

- The question may ask you to simplify the statement form without giving you a target. When you practice enough, you will know when you reached the simplest form.