جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 2

THE LOGIC OF COMPOUND STATEMENTS

Outline

Conditional Statements

- Logical inference or logical deduction is reasoning from a *hypothesis* to a *conclusion*.

- "If such and such is known, then something or other must be the case.”

  hypothesis: "such and such is known" — conclusion: "something or other must be the case"

- The hypothesis may be called an *antecedent.*
- The conclusion may be called a *consequent.*

Conditional Statements – example

If 4,686 is divisible by 6, then 4,686 is divisible by 3.

- What part is the hypothesis?

  4,686 is divisible by 6

- What part is the conclusion?

  4,686 is divisible by 3

Notation

- The notation of the statement *if p then q* is done using $\longrightarrow$

$$p \longrightarrow q$$

- If 4,686 is divisible by 6, then 4,686 is divisible by 3.

$$4{,}686 \; is \; divisible \; by \; 6 \longrightarrow 4{,}686 \; is \; divisible \; by \; 3$$

Conditional Statements Truth Table

- Let's try to drive the truth table logically for this statement:

If x is divisible by 6, it is divisible by 3.

Let's assume that p: x is divisible by 6, and q: x is divisible by 3.

	p	q	$p \longrightarrow q$
x = 18, x is divisible by 6, x is divisible by 3	T	T	T
x = ??, x is divisible by 6, x is not divisible by 3	T	F	F
x = 9, x is not divisible by 6, x is divisible by 3	F	T	T
x = 7, x is not divisible by 6, x is not divisible by 3	F	F	T

Note on Truth Tables

- The values in a truth table refer to the truth of the statement, not its correctness.

- Saying:

If *7 is divisible by 6, the 7 is divisible by 3*

is true, even if saying that 7 is divisible by 6 or 7 is divisible by 3 is incorrect.

- True and correct are not equivalent.

Constructing Truth Tables for Conditional Statements

- Construct the truth table for the following statement, and label the conclusion and hypothesis columns:

$$p \vee \sim q \longrightarrow \sim p$$

p	q	$\sim p$ (conclusion)	$\sim q$	$p \vee \sim q$ (hypothesis)	$p \vee \sim q \longrightarrow \sim p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Division into Cases

$p \vee q \longrightarrow r$ is equivalent to $(p \longrightarrow r) \wedge (q \longrightarrow r)$

p	q	r	$p \vee q$	$p \vee q \longrightarrow r$	$(p \longrightarrow r)$	$(q \longrightarrow r)$	$(p \longrightarrow r) \wedge (q \longrightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Division into Cases – cont.

- $(p \land q \longrightarrow r)$ is equivalent to $(p \longrightarrow r) \lor (q \longrightarrow r)$
- Exercise: construct the truth table to show this equivalency.

Representing If-Then as Or

- $p \longrightarrow q$ is equivalent to $\sim p \vee q$

- Let's prove it by examples:

If you don't get to work on time, then you are fired.

p: you don't get to work on time

q: you are fired

~p: you get to work on time

You get to work on time, or you're fired.

Representing If-Then as Or – cont.

- $p \longrightarrow q$ is equivalent to $\sim p \vee q$
- Let's prove it by truth table:

p	q	$\sim p$	$p \longrightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The Negation of a Conditional Statement

- If $p \longrightarrow q$ is equivalent to $\sim p \vee q$
- And by De Morgan's Law:

$$\sim(\sim p \vee q) \text{ is equivalent to } \sim(\sim p) \wedge \sim q$$

- And by the double negative law it is equivalent to

$$p \wedge \sim q$$

Then we can conclude that

$$\sim(p \longrightarrow q) \text{ is equivalent to } p \wedge \sim q$$

The Negation of a Conditional Statement – cont.

- What is the negation of the following statement:

If my car is in the garage, then I cannot get to class

p: my car is in the garage

q: I cannot get to class

~q: I can get to class

My car is in the garage, <u>and</u> I can get to class

Contraposítives المكافئ العكسي

- The contrapositive of a conditional statement of the form “If p then q” is

  If ~q then ~p.

- Symbolically, the contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

- A conditional statement is logically equivalent to its contrapositive.

Contrapositives – cont.

- What is the contrapositive of the following statements:

If you study, you will pass.

If you didn’t pass, then you didn’t study.

If today is Easter, the tomorrow is Monday.

If tomorrow isn’t Monday, then today is not Easter.

Converse المقلوب

- Suppose a conditional statement of the form "If p then q" is given. The converse of this statement is "If q then p."

- Symbolically, the converse of $p \rightarrow q$ is $q \rightarrow p$.

- The converse of a statement is <u>not</u> logically equivalent to it.

Inverse المعكوس

- Suppose a conditional statement of the form "If p then q" is given. The inverse of this statement is "If ~p then ~q."

- Symbolically, the inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

- The inverse of a statement is <u>not</u> logically equivalent to it.

Converse and Inverse – cont.

- What is the converse and inverse of the following statements:

If you study, you will pass.

Converse: if you passed, then you studied

Inverse: If you don’t study, you won’t pass.

If today is Easter, the tomorrow is Monday.

Converse: if tomorrow is Monday, then today is Easter.

Inverse: If today isn’t Easter, the tomorrow is not Monday.

Converse and Inverse – cont.

- What do you notice about the relation between the converse and the inverse?

Biconditional

- Given statement variables p and q, the **biconditional of $\boldsymbol{p}$ and $\boldsymbol{q}$** is

  "p if, and only if, q"

  and is denoted

  $$p \leftrightarrow q$$

- It is true if both p and q have the same truth values and is false if p and q have opposite truth values.

- The words *if and only if* are sometimes abbreviated **iff.**

Biconditional Statements Truth Table

- Let's try to drive the truth table logically for this statement:

You will pass the course if and only if your average is at least 60

Let's assume that p: you pass the course, and q: your average is at least 60.

	p	q	$p \leftrightarrow q$
You passed the course, your average is at least 60	T	T	T
You passed the course, your average is less than 60	T	F	F
You fail the course, your average is at least 60	F	T	F
You fail the course, your average is less than 60	F	F	T

Biconditional – cont.

- Saying

“p if, and only if, q”

should mean the same as saying both

“p if q” and “p only if q.”

Biconditional Statements Truth Table

- Let's compare the truth table of $p \longleftrightarrow q$ with the truth table of $(p \longrightarrow q) \wedge (q \longrightarrow p)$

p	q	$p \longleftrightarrow q$	$(p \longrightarrow q)$	$(q \longrightarrow p)$	$(p \longrightarrow q) \wedge (q \longrightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Order of Operations for Logical Operators

1. Negation

   $\sim$

2. Conjunction and disjunction

   $\wedge, \vee$

   if both are preset, we will need parentheses.

3. Conditional and biconditional

   $\rightarrow, \leftrightarrow$

   if both are preset, we will need parentheses.

Necessary and Sufficient Conditions

If r and s are statements:

- r is a **sufficient condition** for s means "if r then s." كافي
  - *if r happened this enough for us to know that s will happen*
  - *if r didn't happen, s might or might not have happened*

- r is a **necessary condition** for s means "if not r then not s." ضروري
  - *if r did not happen, then we know that s wouldn't happen.*
  - *the occurrence of r is necessary to obtain the occurrence of s.*
  - *it is possible for r to happen without s happening too.*

Necessary and Sufficient Conditions – cont.

If r and s are statements:

- r is a **sufficient condition** for s means "if r then s."
- r is a **necessary condition** for s means "if not r then not s."

  and its contrapositive, "if s then r."

- r is both a **sufficient** and **necessary** condition for s means "if r then s," and "if s then r."

  which means "r if, and only if, s."

Necessary and Sufficient Conditions – cont.

- A shape being a square is sufficient for it to have 4 sides

  $$\text{Square} \longrightarrow \text{4 sides}$$

- But it is not necessary, because it could have 4 sides and not be a square.

- A shape having 4 sides is necessary for it to be a square

  $$\sim \text{4 sides} \longrightarrow \sim \text{square}$$

- But it is not sufficient.

Necessary and Sufficient Conditions – cont.

- x's being a square is a sufficient condition for x's being a rectangle. ✓
- x's being a square is a necessary condition for x's being a rectangle. ✗
- x's being a rectangle is a sufficient condition for x's being a square. ✗
- x's being a rectangle is a necessary condition for x's being a square. ✓

Necessary and Sufficient Conditions – cont.

- x's being greater than 15 is a sufficient condition for x's being less than 20. ✗
- x's being greater than 15 is a necessary condition for x's being less than 20. ✗
- x's being less than 20 is a sufficient condition for x's being greater than 15. ✗
- x's being less than 20 is a necessary condition for x's being greater than 15. ✗
- x's being less than 12 is a sufficient condition for x's being less than 20. ✓
- x's being less than 12 is a necessary condition for x's being less than 20. ✗
- x's being less than 20 is a sufficient condition for x's being less than 12. ✗
- x's being less than 20 is a necessary condition for x's being less than 12. ✓

Necessary and Sufficient Conditions – cont.

- x's having two arms is a sufficient condition for x's being a human being. ✗
- x's having two arms is a necessary condition for x's being a human being. ✗