جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 3

The Logic of Quantified Statements

Introduction

All men are mortal.

Socrates is a man.

∴ Socrates is mortal.

Even though we can intuitively say that this argument is valid, the methods we learned in the previous chapter cannot help us determine its validity.

First Order Logic

- Predicates and Quantified Statements
- Statements with Multiple Quantifiers

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- Statements with Multiple Quantifiers

Predicates and Quantified Statements

- Introduction and terminology
- Universal and existential quantifiers
- Formalizing and Verbalizing Statements
- Universal Conditional Statements
- Negations of Quantified Statements
- Contrapositive, Converse and Inverse of Quantified Statement
- Necessary and Sufficient Conditions

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Introduction

He is a college student.

This is not a statement, because we don't know who he is.

$x + y > 0$

This is not a statement, because we don't know the values of x and y.

Predicate: the part of the sentence that gives information about the subject.

Example

- Ahmad is a student at Birzeit University.
  - *Ahmad is the subject*
  - *'is a student at Birzeit' is a predicate.*
  - *‘is a student at’ is also a predicate.*

- If we choose P to denote ‘is a student at Birzeit’, then our statements becomes

P(Ahmad) ← Unary predicate

- If we choose Q to denote ‘is a student as’, then our statement becomes

Q(Ahmad, Birzeit) ← Binary predicate

Example – cont.

- We can also use these predicates with symbols:

$P(x)$: x is a student at Birzeit University.

$Q(x, y)$: x is a student at y.

- x and y are called predicate variables.
- When we replace the predicate variables with concrete values, we get a statement.

Terminology

- Predicate:

A sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

- Domain of a predicate variable:

The set of all values that may be substituted in place of the variable.

Terminology

- The symbolic analysis of predicates and quantified statements that we will study in this chapter is usually called first-order logic, **predicate calculus**, **quantificational logic**, and **first-order predicate calculus.**

Terminology

$$P(x_1, x_2, x_3, \dots, x_n)$$

predicate name | predicate variables
predicate symbol | arguments

Predicates name are usually more descriptive than just P or Q, for example:

Student(x), University(y), StudyAtBZU(x), StudyAt(x, y), Even(z), DivisiblyByThree(w), Green(v)

Note that all predicate names are usually capitalized.

Truth Sets of Predicates

- If $P(x)$ is a predicate and x has domain D, the truth set of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.

- The truth set of $P(x)$ is denoted
$$\{x \in D \mid P(x)\}$$

- For example,
$$\{x \in Organization \mid University(x)\}$$

This is read as: the set of all organizations that are universities.

Number Sets and Their Symbols

- Moving on in this course, we will refer to some common number sets. This slide is for your future reference:

Type of Number	Symbol	Definition
Integer Numbers	$\mathbb{Z}$	Whole numbers that can be positive, negative, or 0
	$\mathbb{Z}^+$	All positive whole numbers
	$\mathbb{Z}^-$	All negative whole numbers
Real Numbers	$\mathbb{R}$	All rational and irrational numbers
	$\mathbb{R}^+$	All positive real numbers
	$\mathbb{R}^-$	All negative real numbers
Rational Numbers	$\mathbb{Q}$	All numbers that can be written as a fraction
	$\mathbb{Q}^+$	All positive rational numbers
	$\mathbb{Q}^-$	All negative rational numbers

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The Universal Quantifier ∀

- The symbol ∀ denotes ‘for all’ and is called the universal quantifier.

- Let $Q(x)$ be a predicate and D the domain of x. A universal statement is a statement of the form “$\forall x \in D,\ Q(x)$.” It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D.

- A value for x for which $Q(x)$ is false is called a counterexample to the universal statement.

Example 1

- Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x$$

Show that this statement is true.

To show that, we need to check that it holds true for every possible value of x:

$$x = 1:\ 1^2 \geq 1$$
$$x = 2:\ 2^2 \geq 2$$
$$x = 3:\ 3^2 \geq 3$$
$$x = 4:\ 4^2 \geq 4$$
$$x = 5:\ 5^2 \geq 5$$

Hence, "$\forall x \in D, x^2 \geq x$" is true.

Method of Exhaustion

- The technique we used to show the truth of the previous universal statements is called the method of exhaustion.
- This method consists of showing the truth of the predicate separately for each individual element of the domain.
- This method can be used whenever the domain of the predicate variable is finite.
- Since most mathematical sets are infinite, this method can rarely be used to derive general mathematical results.

Method of Exhaustion – cont.

- A statement such as:

All students in Birzeit are Palestinians.

Can only be proved by going around and asking each student in Birzeit if they were Palestinian or not.

This is very exhausting – but eventually, it is doable.

- On the other hand, statement such as

Every rational number can be written as a fraction of two integers

Cannot be proved by checking every possible combination, because there's an infinite number of possibilities.

We will learn a smarter way to prove universal statements.

Example 2

- Consider the statement

$$\forall x \in \mathbb{R}, x^2 \geq x$$

Find a counterexample to show that this statement is false.

Take $x = \frac{1}{2}$. Then x is in $\mathbb{R}$ since $\frac{1}{2}$ is a real number, and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \ngeq \frac{1}{2}$$

Hence "$\forall x \in \mathbb{R}, x^2 \geq x$" is false.

The Existential Quantifier ∃

- The symbol ∃ denotes "there exists" and is called the existential quantifier.

- Let $Q(x)$ be a predicate and D the domain of x. An existential statement is a statement of the form "$\exists x \in D,\ such\ that\ Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is defined to be false if, and only if, $Q(x)$ is false for all x in D.

Example 1

- Consider the statement

$$\exists m \in \mathbb{Z}^+ \; such \; that \; m^2 = m$$

Show that this statement is true.

Observe that $1^2 = 1$.

Thus "$m^2 = m$" is true for at least one positive integer m.

Hence "$\exists m \in \mathbb{Z}^+ \; such \; that \; m^2 = m$" is true.

Example 2

Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m$$

Show that this statement is false.

To show that, we need to check that the statement $m^2 = m$ is false for all values in E:

$$m = 5:\ 5^2 = 25 \neq 5$$
$$m = 6:\ 6^2 = 36 \neq 6$$
$$m = 7:\ 7^2 = 49 \neq 7$$
$$m = 8:\ 8^2 = 64 \neq 8$$

Thus, "$\exists m \in E \text{ such that } m^2 = m$" is false.

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Formal vs Informal Language

- Formal language is the language that uses symbols.
  - *This is used when we want to think out a complicated problem.*
  - *The question will usually be formulated as:*
    - Formalize the following sentence.
    - Rewrite the following formal statement formally.

- Informal language is the language we use to speak.
  - *This is used when we are trying to make sense of mathematical concepts.*
  - *The question will usually be formulated as:*
    - Verbalize the following sentence.
    - Rewrite the following formal statement informally.

From Formal to Informal

- $\forall x \in \mathbb{R}, x^2 \geq 0$
  - *All real numbers have nonnegative squares*
  - *Every real number has a nonnegative square*
  - *Any real number has a nonnegative square*
  - *The square of each real number is larger than or equal to zero*
  - مربع كل الأعداد الحقيقة موجب أو صفر
  - كل الأعداد الحقيقة لها مربع غير سالب

From Formal to Informal – cont.

- $\forall x \in \mathbb{R}, x^2 \neq -1$
  - *All real numbers have squares that are not equal to* -1*.*
  - *No real numbers have squares equal to* -1*.*
  - كل الأعداد الحقيقية مربعها لا يساوي -1
  - لا يوجد عدد حقيقي مربعه يساوي -1

From Formal to Informal – cont.

- $\exists m \in \mathbb{Z}^{+} \text{ such that } m^2 = m$
  - *There is a positive integer whose square is equal to itself.*
  - *We can find at least one positive integer equal to its own square.*
  - *Some positive integer equals its own square.*
  - هناك عدد صحيح موجب مربعه يساويه نفسه
  - هناك على الأقل عدد صحيح موجب واحد يساوي مربعه

From Informal to Formal

- All triangles have three sides.
  - $\forall$ triangle t, t has three sides
  - $\forall t \in T,$ where T is the set of all triangles, t has three sides
  - $\forall t \in T, HasThreeSides(t)$
  - $\forall t \in T, HasSides(t, 3)$

From Informal to Formal – cont.

- No dogs have wings
  - $\forall \text{dogs } d, d$ does not have wings
  - $\forall d \in D,$ where D is the set of all dogs, d does not have wings
  - $\forall d \in D, HasNoWings(d)$
  - $\forall d \in D, HasWings(d, 0)$

From Informal to Formal – cont.

- Some programs are structured
  - $\exists$ a program p such that p is structured.
  - $\exists p \in P$, where P is the set of all programs, such that p is structured
  - $\exists p \in P, Structured(p)$

Expressing Quantified Statements

- As we have seen, there is no one correct way to formalize or verbalize a statement.
- However, the clearer and more explicit you are, the better your answer is.

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Universal Conditional Statements

- A universal conditional statements is a statement that looks like this:

$$\forall x, \text{if } P(x) \text{ then } Q(x)$$

- This is considered one of the most important form of statement in mathematics.

Examples

- Rewrite the following statements in the form of

$$\forall \_\_\_\_\_ , if \_\_\_\_\_\_\_\_\_ \quad then \_\_\_\_\_\_\_\_\_.$$

- If a real number is an integer, then it is a rational number.
  - $\forall$ real numbers x, if x is an integer, then x is a rational number.
  - $\forall x \in \mathbb{R}$, if $x \in \mathbb{Z}$ then $x \in \mathbb{Q}$
  - $\forall x \in \mathbb{R}, x \in \mathbb{Z} \longrightarrow x \in \mathbb{Q}$

Examples – cont.

- Rewrite the following statements in the form of

$$\forall \_\_\_\_\_,\ if \_\_\_\_\_\_\_\_\_\_\ \ then \_\_\_\_\_\_\_\_\_\_.$$

- All bytes have eight bits.
  - $\forall x$, if x is a byte, then x has eight bits.
  - $\forall x, Byte(x) \rightarrow HasBits(x, 8)$

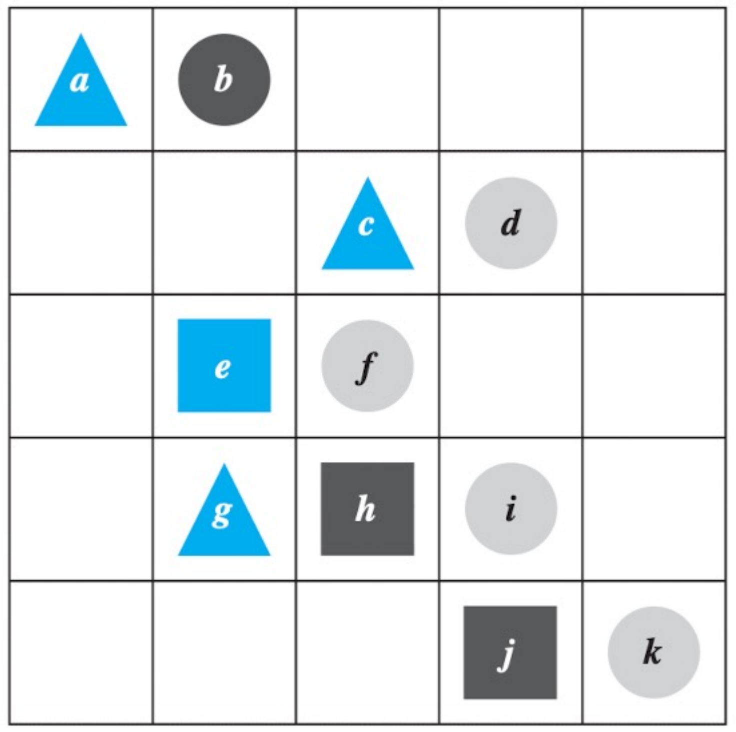
Examples – cont.

- Rewrite the following statements in the form of
  $$\forall \_\_\_\_\_, if \_\_\_\_\_\_\_\_\_ \quad then \_\_\_\_\_\_\_\_\_.$$

- No fire trucks are green.
  - $\forall x$, if x is a fire truck, then x is not green.
  - $\forall x, Firetruck(x) \rightarrow \sim Green(x)$

Example – Tarski's World

- Tarski's World is a computer program developed by information scientists Jon Barwise and John Etchemendy to help teach the principles of logic.



Determine the truth or falsity of each of the following statements:

a. $\forall t, Triangle(t) \rightarrow Blue(t)$ true

b. $\forall x, Blue(x) \rightarrow Triangle(x)$ false

c. $\exists y$ such that $Square(y) \wedge RightOf(d, y)$. true

d. $\exists z$ such that $Square(z) \wedge Gray(z)$. false

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Negation of Universal Statements

- Consider the statement “All mathematicians wear glasses.”

- Which of those statements do you think is its negation?
  - ✗ a. *No mathematicians wear glasses.*
  - ✓ b. *There exists at least one mathematician who does not wear glasses.*

- The negation of a universal statement (“all are”) is logically equivalent to an existential statement (“some are not” or “there is at least one that is not”)

Negation of Universal Statements

- The negation of a statement of the form

$$\forall x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \; in \; D \text{ such that } {\sim}Q(x)$$

- Symbolically,

$$\sim (\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } {\sim}Q(x)$$

Negation of Existential Statements

- Now consider the statement "Some snowflakes are the same."

- Which of those statements do you think is its negation?
  - ✓ a. *"No snowflakes are the same."*
  - ✓ b. *"All snowflakes are different."*

- The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Negation of Existential Statements

- The negation of a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

  is logically equivalent to a statement of the form

$$\forall x \in D, \sim Q(x)$$

- Symbolically,

$$\sim (\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x)$$

Examples

- Write formal negations for the following statements:

- ∀ primes p, p is odd.
  - ∃ a prime p such that p is not odd.

- ∃ a triangle T such that the sum of the angles of T equals $200°$.
  - ∀ triangles T, the sum of the angles of T does not equal $200°$.

Examples – cont.

- Rewrite the following statement formally. Then write formal and informal negations.

- No politicians are honest.
  - *Formal version:* ∀ politicians x , x is not honest.
  - *Formal negation:* ∃ a politician x such that x is honest.
  - *Informal negation:* Some politicians are honest.

Examples – cont.

- Write informal negations for the following statements:

- All computer programs are finite.
  - *There is a computer program that is not finite.*
  - *Some computer programs are infinite.*

- Some computer hackers are over 40.
  - *No computer hackers are over 40.*
  - *All computer hackers are 40 or under.*

Examples – cont.

- Rewrite the following statement formally. Then write formal and informal negations.

- The number 1,357 is divisible by some integer between 1 and 37.
  - *Formal version:* ∃ an integer n between 1 and 37 such that 1,357 is divisible by n.
  - *Formal negation:* ∀ integers n between 1 and 37; 1,357 is not divisible by n.
  - *Informal negation:* The number 1,357 is not divisible by any integer between 1 and 37.

Ambiguous Negation

- Just inserting the word ‘not’ to negate a quantified statements can result in an ambiguous statement.
- A possible negation of “All mathematicians wear glasses” is “All mathematicians do not wear glasses.”
- The problem is that this sentence has two meanings, either that not all mathematicians wear glasses, or all mathematicians are not glass-wearers.
- So, be careful as to how you word your sentences, because ambiguous statements are not correct.

Negations of Universal Conditional Statements

- The form of such negations can be derived from facts that have already been established.
- By definition of the negation of a for all statement,

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x))$$

- But the negation of an if-then statement is logically equivalent to an and statement.

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x).$$

- This leads to

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x))$$

Examples

- Write a formal negation for the following statement:
  - ∀ people p, if p is blond then p has blue eyes.

    ∃ a person p such that p is blond and p does not have blue eyes.

- Write an informal negation for the following statement:
  - If a computer program has more than 100,000 lines, then it contains a bug.

    There is at least one computer program that has more than 100,000 lines and does not contain a bug.

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Variants of Universal Conditional Statements

- Recall that a conditional statement has a contrapositive, a converse, and an inverse. The definitions of these terms can be extended to universal conditional statements.

- Consider a statement of the form:

$$\forall x \in D, if\ P(x)\ then\ Q(x)$$

Its contrapositive is the statement: $\forall x \in D, if\ {\sim}Q(x)\ then\ {\sim}P(x)$.

Its converse is the statement: $\forall x \in D, if\ Q(x)\ then\ P(x)$.

Its inverse is the statement: $\forall x \in D, if\ {\sim}P(x)\ then\ {\sim}Q(x)$.

Equivalency of These Variants

- Remember that:

A statement and its contrapositive are equivalent.

$$\forall x \in D, if\ P(x)\ then\ Q(x) \equiv \forall x \in D, if\ {\sim}Q(x)\ then\ {\sim}P(x)$$

A converse and an inverse of a statement are equivalent.

$$\forall x \in D, if\ Q(x)\ then\ P(x) \equiv \forall x \in D, if\ {\sim}P(x)\ then\ {\sim}Q(x)$$

Neither the converse or the inverse of a statement are equivalent to it.

$$\forall x \in D, if\ P(x) then\ Q(x) \not\equiv \forall x \in D, if\ Q(x) then\ P(x)$$

$$\forall x \in D, if\ P(x)\ then\ Q(x) \not\equiv \forall x \in D, if\ {\sim}P(x)\ then\ {\sim}Q(x)$$

Example

- Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

The formal version of this statement is $\forall x \in \mathbb{R}, if\ x > 2\ then\ x^2 > 4$.

<u>**Contrapositive:**</u> $\forall x \in \mathbb{R},\ if\ x^2 \leq 4\ then\ x \leq 2$.

Or: If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

<u>**Converse:**</u> $\forall x \in \mathbb{R},\ if\ x^2 > 4\ then\ x > 2$.

Or: If the square of a real number is greater than 4, then the number is greater than 2.

<u>**Inverse:**</u> $\forall x \in \mathbb{R},\ if\ x \leq 2\ then\ x^2 \leq 4$.

Or: If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

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Necessary and Sufficient Conditions, Only If

- The definitions of necessary, sufficient, and only if can also be extended to apply to universal conditional statements.

- "$\forall x, r(x)$ is a sufficient condition for $s(x)$" means "$\forall x, if\ r(x)\ then\ s(x)$."
- "$\forall x\ , r(x)$ is a necessary condition for $s(x)$" means "$\forall x, if\ {\sim}r(x)\ then\ {\sim}s(x)$" or, equivalently, "$\forall x, if\ s(x)\ then\ r(x)$."
- "$\forall x, r(x)\ only\ if\ s(x)$" means "$\forall x, if\ {\sim}s(x)\ then\ {\sim}r(x)$" or, equivalently, "$\forall x, if\ r(x)\ then\ s(x)$."

Examples

- Rewrite the following statement as quantified conditional statements. Do not use the word *necessary* or *sufficient*:

- Squareness is a sufficient condition for rectangularity.
  - *$\forall x$, if x is a square, then x is a rectangle.*
  - *If a figure is a square, then it is a rectangle.*

Examples – cont.

- Rewrite the following statement as quantified conditional statements. Do not use the word *necessary* or *sufficient*:

- Being at least 35 years old is a necessary condition for being President of the United States.
  - *∀ people x, if x is younger than 35, then x cannot be President of the United States.*
  - *∀ people x, if x is President of the United States, then x is at least 35 years old.*

Examples – cont.

- Rewrite the following as a universal conditional statement:

- A product of two numbers is 0 only if one of the numbers is 0.
  - *If neither of two numbers is 0, then the product of the numbers is not 0.*
  - *If a product of two numbers is 0, then one of the numbers is 0.*

Exercises

There are some exercise on the covered topic on the lecture’s ITC page. Have fun!