جامعة بيرزيت

BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233

Discrete Mathematics

CHAPTER 3

The Logic of Quantified Statements

First Order Logic

- Predicates and Quantified Statements
- Statements with Multiple Quantifiers

First Order Logic

- Predicates and Quantified Statements
- Statements with Multiple Quantifiers

Motivation

- Imagine you are visiting a factory that manufactures computer microchips. The factory guide tells you,

  There is a person supervising every detail of the production process.

- Note that this statement contains informal versions of both the existential quantifier *there is* and the universal quantifier *every*. Which of the following best describes its meaning?
  - *There is one single person who supervises all the details of the production process.*
  - *For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details.*

Outline

- Interpreting Multiply-Quantified Statements
- Negations of Multiply-Quantified Statements
- Order of Quantifiers
- Formal Logical Notation
- Some Extra Fun!

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Convention of Interpretation

- When a statement contains more than one quantifier, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur. For instance, consider a statement of the form

  $\forall x$ in set D, $\exists y$ in set E such that x and y satisfy property $P(x, y)$

- To show that such a statement is true, you must be able to meet the following challenge:
  - *Imagine that someone is allowed to choose any element whatsoever from the set D, and imagine that the person gives you that element. Call it x.*
  - *The challenge for you is to find an element y in E so that the person's x and your y, taken together, satisfy property $P(x, y)$.*
- Note that because you do not have to specify the y until after the other person has specified the x, you are allowed to find a different value of y for each different x you are given.

Convention of Interpretation

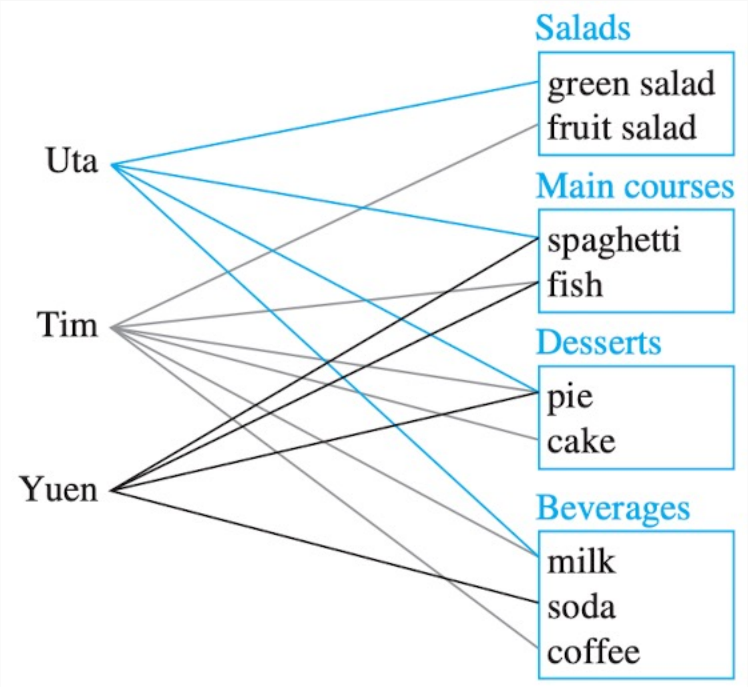
- Now consider a statement containing both ∀ and ∃, where the ∃ comes before the ∀:

  ∃ an x in D such that ∀y in E, x and y satisfy property $P(x\ ,y)$

- To show that a statement of this form is true, you must find one single element (call it x) in D with the following property:
  - *After you have found your x, someone is allowed to choose any element whatsoever from E. The person challenges you by giving you that element. Call it y.*
  - *Your job is to show that your x together with the person's y satisfy property $P(x,y)$.*
- Note that your x has to work for any y the person gives you; you are not allowed to change your x once you have specified it initially.

Interpreting Multiply-Quantified Statements

- Three students in a college cafeteria go through the line and make the following choices:



Write each of following statements informally and find its truth value:

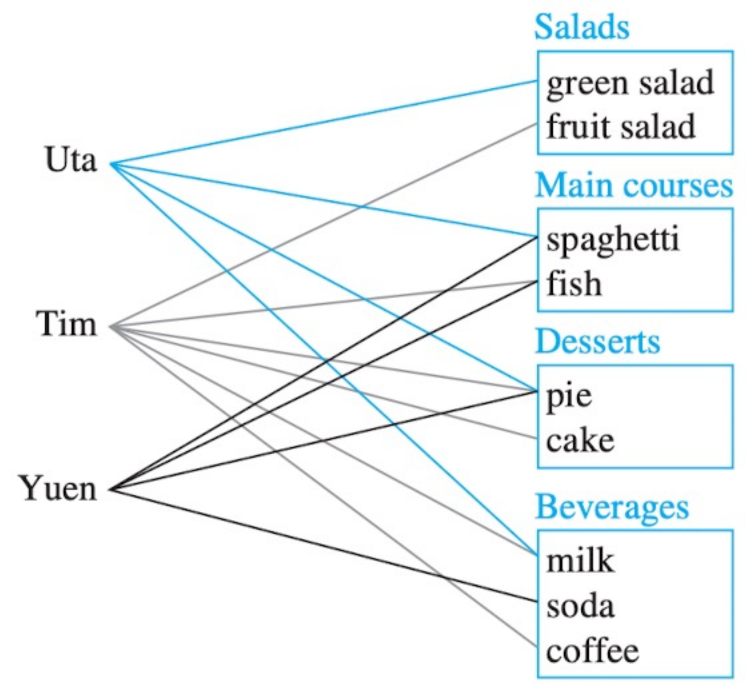
$\exists$ an item I such that $\forall$ students S, S chose I.

There is an item that was chosen by every student.

This is true; every student chose pie

Interpreting Multiply-Quantified Statements

- Three students in a college cafeteria go through the line and make the following choices:



Write each of following statements informally and find its truth value:

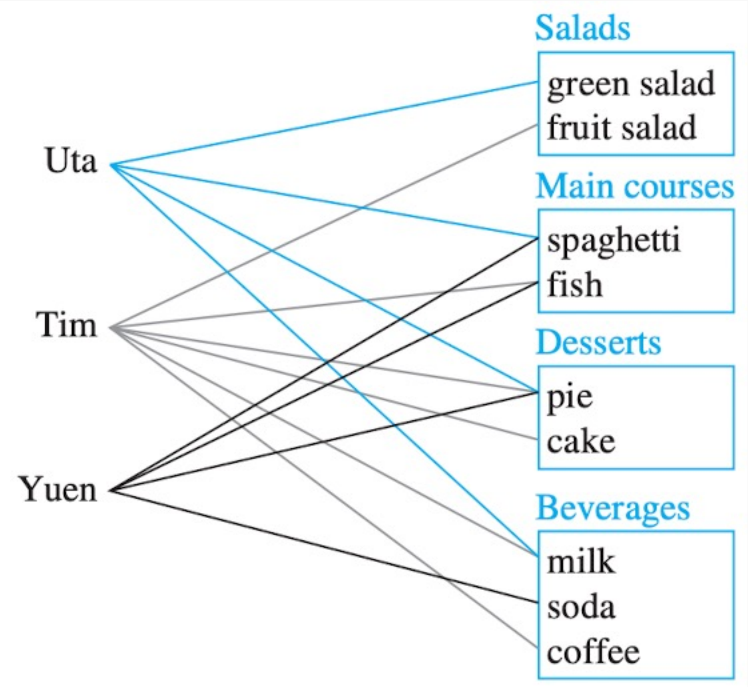
$\exists$ a student S such that $\forall$ items I, S chose I.

There is a student who chose every available item.

This is false; no student chose all nine items.

Interpreting Multiply-Quantified Statements

- Three students in a college cafeteria go through the line and make the following choices:



Write each of following statements informally and find its truth value:

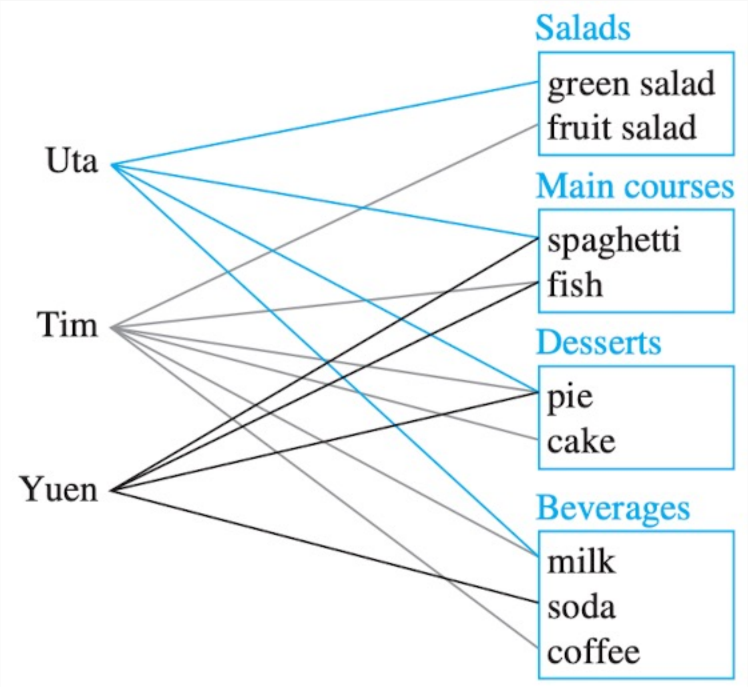
∃ a student S such that ∀ stations Z, ∃ an item I in Z such that S chose I.

There is a student who chose at least one item from every station

This is true; both Uta and Tim chose at least one item from every station

Interpreting Multiply-Quantified Statements

- Three students in a college cafeteria go through the line and make the following choices:



Write each of following statements informally and find its truth value:

∀ students S and ∀ stations Z, ∃ an item I in Z such that S chose I

Every student chose at least one item from every station.

This is false; Yuen did not choose a salad.

Translating from Informal to Formal Language

- Consider the statement

  "There is a smallest positive integer."

  Write this statement formally using both symbols ∃ and ∀.

- To say that there is a smallest positive integer means that there is a positive integer m with the property that no matter what positive integer n a person might pick, m will be less than or equal to n.

  ∃ a positive integer m such that ∀ positive integers n, $m \leq n$

- This statement is true because 1 is a positive integer that is less than or equal to every positive integer.

Translating from Informal to Formal Language

- Consider the statement

  "There is no smallest positive real number."

  Write this statement formally using both symbols ∃ and ∀.

- To say that there is no smallest positive real number means that no matter what positive real number x that a person might pick, there will be another positive real number y that is less than x.

  ∀ positive real numbers x, ∃ a positive real number y such that $y < x$

- This statement is true because if you imagine any positive real number x on the real number line, These numbers correspond to all the points to the right of 0. Observe that no matter how small x is, the number x /2 will be both positive and less than x.

Outline

- Interpreting Multiply-Quantified Statements
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Negations of Multiply-Quantified Statements

- Recall that

$$\sim (\forall x \; in \; D, P(x)) \; \equiv \; \exists x \; in \; D \; such \; that \; \sim P(x)$$

- and

$$\sim (\exists x \; in \; D \; such \; that \; P(x)) \; \equiv \; \forall x \; in \; D, \sim P(x)$$

- We apply these laws to find

$$\sim (\forall x \; in \; D, \exists y \; in \; E \; such \; that \; P(x, y))$$

- by moving in stages from left to right along the sentence.

First version of negation: $\exists x \; in \; D \; such \; that \; {\sim}(\exists y \; in \; E \; such \; that \; P(x, y))$.

Final version of negation: $\exists x \; in \; D \; such \; that \; \forall y \; in \; E \; , {\sim}P(x, y)$.

Negations of Multiply-Quantified Statements

- Similarly, to find

$$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E\ , P(x\ , y))$$

- we have

First version of negation: $\forall x \text{ in } D, \sim(\forall y \text{ in } E\ , P(x, y))$.

Final version of negation: $\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$.

Negations of Multiply-Quantified Statements

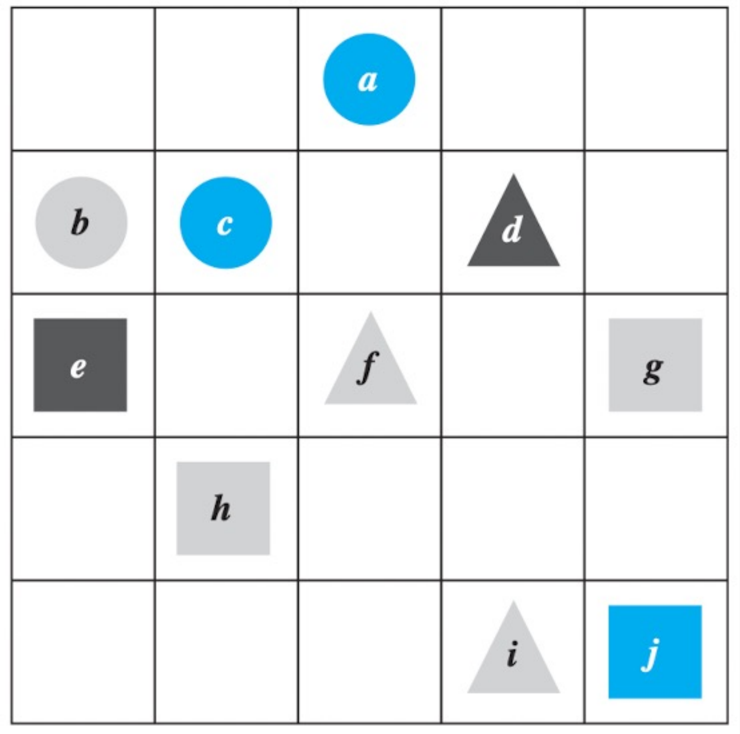
- In summary,

$$\sim (\forall\ x\ in\ D, \exists y\ in\ E\ such\ that\ P(x\ , y))\ \equiv \exists x\ in\ D\ such\ that\ \forall y\ in\ E\ , \sim P(x\ , y).$$

$$\sim (\exists x\ in\ D\ such\ that\ \forall y\ in\ E\ , P(x\ , y))\ \equiv \forall x\ in\ D, \exists y\ in\ E\ such\ that\ \sim P(x\ , y).$$

Negating Statements in a Tarski World

- Write a negation for the following statements, and determine which is true, the given statement or its negation.



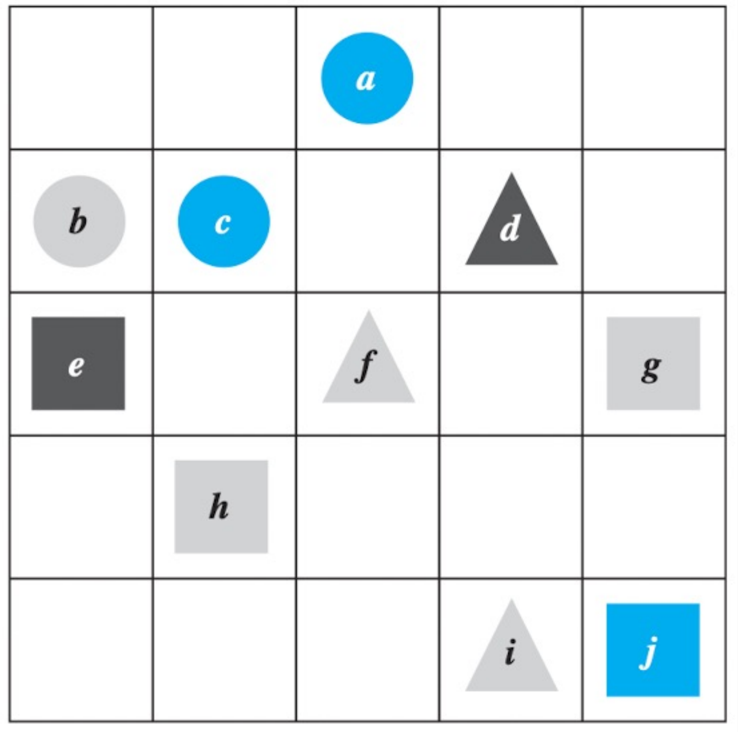
For all squares x, there is a circle y such that x and y have the same color.

Negation: ∃ a square x such that ∀ circles y, x and y do not have the same color.

The negation is true. Square e is black and no circle is black, so there is a square that does not have the same color as any circle.

Negating Statements in a Tarski World

- Write a negation for the following statements, and determine which is true, the given statement or its negation.



There is a triangle x such that for all squares y, x is the right of y.

Negation: ∀ triangles x, ∃ a square y such that x is

not to the right of y.

The negation is true because no matter what triangle is chosen, it is not to the right of square g (or square j).

Outline

Order of Quantifiers

- $\forall x \, \exists y, Loves(x, y)$
  - *Everything loves something*
  - *Each thing loves one or more things*
  - *Each thing loves at least one thing*
- $\exists y \, \forall x, Loves(x, y)$
  - *Something is loved by everything.*
  - *Everything love the same thing.*
  - *There exists something that everything loves.*

Let's Expand – Verbalize the Following:

- $\forall x \, \exists y, Loves(x, y)$
  - *Everything loves something*
- $\forall y \, \exists x, Loves(x, y)$
  - *Everything is loved by something*
- $\exists y \, \forall x, Loves(x, y)$
  - *Everyone love the same thing*
- $\forall x \, \forall y, Loves(x, y)$
  $\forall x, y, Loves(x, y)$
  - *Everything loves everything*
- $\exists x \, \forall y, Loves(x, y)$
  - *Something loves everything*
- $\forall y \, \exists x, Loves(y, x)$
  - *Everything loves something*
- $\forall x \, \exists y, Loves(x, y), x \neq y$
  - *Everything loves something but not itself*
- $\exists x \, \exists y, Loves(x, y)$
  $\exists x, y, \; Loves(x, y)$
  - *Something loves something*

Let's Expand – Formalize the Following:

- Everyone loves all movies.
  - $\forall p \in person, \forall m \in movies$
    $, \; Loves(p, m)$
- There is a movie that everyone loves.
  - $\exists m \in movies, \forall p \in person$
    $, \; Loves(p, m)$
- Everyone loves some movies.
  - $\forall p \in person, \exists m \in movies$
    $, \; Loves(p, m)$
- Some people love some movies.
  - $\exists p \in person, \exists m \in movies,$
    $Loves(p, m)$
- Some people love all movies.
  - $\exists p \in person, \forall m \in movies,$
    $Loves(p, m)$
- All movies are loved by someone.
  - $\forall m \in movies, \exists p \in person,$
    $Loves(p, m)$

Multiple Quantifiers with Negated Predicates

- $\exists x, y \in person, \sim Loves(x, y)$
  - *Somebody does not love somebody*
- $\forall x, y \in person, \sim Loves(x, y)$
  - *Nobody loves anybody*
- $\exists x \in person, \forall y \in person, \sim Loves(x, y)$
  - *Somebody does not love anybody*
- $\forall x \in person, \exists y \in person, \sim Loves(y, x)$
  - *Everybody is not loved by somebody*

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Formal Logical Notation

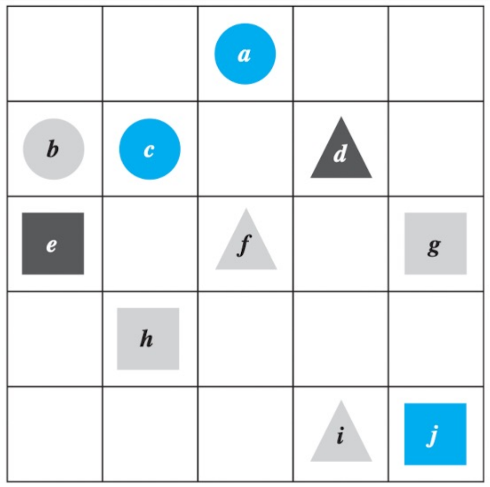
- The formalism depends on the following facts:

"$\forall x\ in\ D, P(x)$" can be written as "$\forall x\ (x\ in\ D\ \rightarrow\ P(x))$."

"$\exists x\ in\ D\ such\ that\ P(x)$" can be written as "$\exists x\ (x\ in\ D\ \wedge\ P(x))$."

Tarski World Formalisms

- Let the common domain D of all variables be the set of all the objects in the Tarski world. Use formal, logical notation to write each of the following statements, and write a formal negation for the following statement:



For all circles x, x is above f.

Statement: $\forall x\ (Circle(x)\ \rightarrow\ Above(x, f))$.

Negation: $\sim(\forall x\ (Circle(x)\ \rightarrow\ Above(x, f)))$

$\equiv \exists x \sim(Circle(x)\ \rightarrow\ Above(x, f))$ by negation of ∀

$\equiv \exists x\ (Circle(x)\ \wedge \sim Above(x, f))$ by negation of if-then

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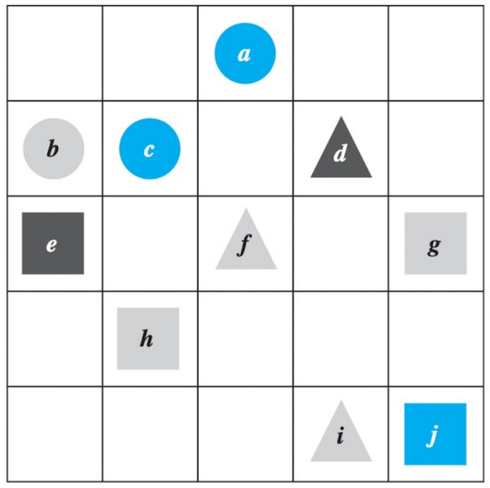

There is a square x such that x is black.

Statement: $\exists x(Square(x) \wedge Black(x))$.

Negation: $\sim(\exists x\ (Square(x) \wedge Black(x)))$

$\equiv \forall x \sim(Square(x) \wedge Black(x))$ by negation of ∃

$\equiv \forall x\ (\sim Square(x) \vee \sim Black(x))$ by De Morgan's Law

Tarski World Formalisms

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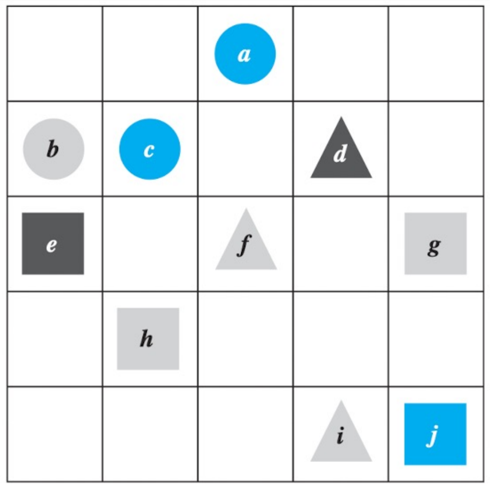


For all circles x, there is a square y such that x and y have the same color.

Statement: $\forall x(Circle(x) \rightarrow \exists y(Square(y) \wedge SameColor(x,y)))$

Negation: $\sim(\forall x(Circle(x) \rightarrow \exists y(Square(y) \wedge SameColor(x,y))))$

$\equiv \exists x \sim (Circle(x) \rightarrow \exists y(Square(y) \wedge SameColor(x,y)))$ by negation of $\forall$

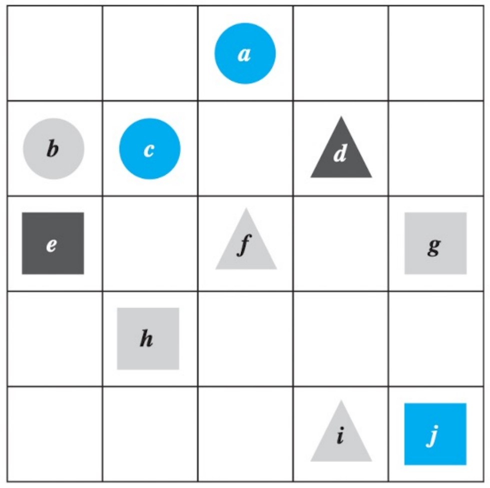
$\equiv \exists x(Circle(x) \wedge \sim(\exists y(Square(y) \wedge SameColor(x,y))))$ by negation of if-then

$\equiv \exists x(Circle(x) \wedge \forall y(\sim(Square(y) \wedge SameColor(x,y))))$ by negation of $\exists$

$\equiv \exists x(Circle(x) \wedge \forall y(\sim Square(y) \vee \sim SameColor(x,y)))$ by De Morgan's Law

Tarski World Formalisms

- Let the common domain D of all variables be the set of all the objects in the Tarski world. Use formal, logical notation to write each of the following statements, and write a formal negation for the following statement:



There is a square x such that for all triangles y, x is to right of y.

Statement: $\exists x(Square(x) \wedge \forall y(Triangle(y) \rightarrow RightOf(x,y)))$.

Negation: $\sim(\exists x(Square(x) \wedge \forall y(Triangle(y) \rightarrow RightOf(x,y))))$

$\equiv \forall x \sim (Square(x) \wedge \forall y(Triangle(y) \rightarrow RightOf(x,y)))$ by negation of $\exists$

$\equiv \forall x(\sim Square(x) \vee \sim (\forall y(Triangle(y) \rightarrow RightOf(x,y))))$ by De Morgan's Law

$\equiv \forall x(\sim Square(x) \vee \exists y(\sim (Triangle(y) \rightarrow RightOf(x,y))))$ by negation of $\forall$

$\equiv \forall x(\sim Square(x) \vee \exists y(Triangle(y) \wedge \sim RightOf(x,y)))$ by negation of if-then

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Something Extra - Prolog

- The programming language Prolog (short for ***pro***gramming in ***log***ic) was developed in France in the 1970s by A. Colmerauer and P. Roussel to help programmers working in the field of artificial intelligence. A simple Prolog program consists of a set of statements describing some situation together with questions about the situation. Built into the language are search and inference techniques needed to answer the questions by deriving the answers from the given statements. This frees the programmer from the necessity of having to write separate programs to answer each type of question.

- I encourage you to give it a look and try to write a simple program to solve a discrete math problem. It can be fun.

A Problem to Solve in Prolog – or by Pen

- There are 5 houses in five different colors.
- In each house lives a person with a different nationality.
- These five owners drink a certain type of beverage, smoke a certain brand of cigar and keep a certain pet.
- No owners have the same pet, smoke the same brand of cigar or drink the same beverage.
- The question is: *Who owns the fish?*

A Problem to Solve in Prolog – or by Pen

- the Brit lives in the red house
- the Swede keeps dogs as pets
- the Dane drinks tea
- the green house is on the left of the white house
- the green house's owner drinks coffee
- the person who smokes Pall Mall rears birds
- the owner of the yellow house smokes Dunhill
- the man living in the center house drinks milk
- the Norwegian lives in the first house
- the man who smokes blends lives next to the one who keeps cats
- the man who keeps horses lives next to the man who smokes Dunhill
- the owner who smokes BlueMaster drinks beer
- the German smokes Prince
- the Norwegian lives next to the blue house
- the man who smokes blend has a neighbor who drinks water