جامعة بيرزيت

BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233

Discrete Mathematics

CHAPTER 4

Elementary Number Theory and Methods of Proof

Number Theory

- Direct Proof and Counterexamples
- Rational Numbers
- Divisibility
- Division into Cases and the Quotient-Remainder Theorem

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- Direct Proof and Counterexamples
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Rational Numbers

- Introduction
- Properties of Rational Numbers
  - *Sum of Rational Numbers*
  - *Double of Rational Numbers*
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Definition

نسبي

- A real number r is rational if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator.

مقام

حاصل قسمة

- A real number that is not rational is irrational.

- More formally, if r is a real number, then

$$r \text{ is rational} \leftrightarrow \exists \text{ integers } a \text{ and } b \text{ such that } r = \frac{a}{b} \text{ and } b \neq 0$$

Rational or Irrational

a. Is 10/3 a rational number?

Yes, 10/3 is a quotient of the integers 10 and 3 and hence is rational.

b. Is $-\frac{5}{39}$ a rational number?

Yes, $-\frac{5}{39} = \frac{5}{39}$*, which is a quotient of the integers −5 and 39 and hence is rational.*

c. Is 0.281 a rational number?

Yes, 0.281 = 281/1000.

d. Is 7 a rational number?

Yes, 7 = 7/1

Rational or Irrational – cont.

e. Is 0 a rational number?

Yes, 0 = 0/1

f. Is 2/0 a rational number?

No, 2/0 is not a number (division by 0 is not allowed).

g. Is 2/0 an irrational number?

No, because every irrational number is a number, and 2/0 is not a number

Rational or Irrational – cont.

h. Is 0.12121212… a rational number (where the digits 12 are assumed to repeat forever)?

Yes. Let $x = 0.12121212 \ldots.$

Then $100x = 12.12121212 \ldots.$

Thus $100x - x = 12.12121212 \ldots - 0.12121212 \ldots = 12$

But also

$100x - x = 99x$ by basic algebra

Hence $99x = 12$

and so

$$x = \frac{12}{99}$$

Therefore, $0.12121212 \ldots = 12/99$, which is a ratio of two nonzero integers and thus is a rational number.

Rational or Irrational – cont.

i. If m and n are integers and neither m nor n is zero, is (m + n)/mn a rational number?

Yes, since m and n are integers, so are $m + n$ and mn (because sums and products of integers are integers).

Also, $mn \neq 0$ by the zero product property that says:

If neither of two real numbers is zero, then their product is also not zero

It follows that $(m + n)/mn$ is a quotient of two integers with a nonzero denominator and hence is a rational number.

Integers as Rational Numbers

Every integer is a rational number.

- $7 = \frac{7}{1}$ which is the quotient of integers and hence a rational number.
- $-12 = \frac{-12}{1}$ which is the quotient of integers and hence a rational number.
- $0 = \frac{0}{1}$ which is the quotient of integers and hence a rational number.
- And any particular, but arbitrarily chosen integer $n = \frac{n}{1}$ which is the quotient of integers and hence a rational number.

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Note on This Section

- This section, as well as the remaining sections, will consist of theories that may seem trivial or intuitive to you. However, the main concern of the chapter is proving these theories.

- The best approach to study this chapter is understand the proof that is being presented, then try to use it to prove other theories related to the same topic.

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Sum of Rational Numbers

Prove that the sum of any two rational numbers is rational.

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

Step 1: Begin by mentally or explicitly rewriting the statement to be proved in the form
“∀________ , if ________ then _________.”

∀ real numbers r and s, if r and s are rational then $r + s$ is rational.

Formal restatement

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

Step 2: Ask yourself, “Where am I starting from?” or “What am I supposing?”

Suppose r and s are particular but arbitrarily chosen real numbers such that r and s are rational;

or, more simply,

Suppose r and s are rational numbers.

Starting Point

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

Step 3: Ask yourself, “What must I show to complete the proof?”

Show that $r + s$ is rational.

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

Step 4: Ask yourself, "How do I get from the starting point to the conclusion?"

or

"Why must $r + s$ be rational if both *r* and *s* are rational?"

The answer to this depends on the problem we're trying to solve, or the proof we're trying to make.

In this case, our proof depends essentially on how rational numbers are defined.

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

- Rational numbers are quotients of integers, so to say that r and s are rational means that

$$r = \frac{a}{b}$$

and

$$s = \frac{d}{c}$$

for some integers a, b, c, d where $b \neq 0$ and $d \neq 0$.

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

- It follows by substitution that

$$r + s = \frac{a}{b} + \frac{c}{d}$$

You need to show that $r + s$ is rational, which means that $r + s$ can be written as a single fraction or ratio of two integers with a nonzero denominator.

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

- Rewriting the fraction with a common denominator gives us

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}$$

- By adding fractions with a common denominator, we get

$$\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

$$\frac{ad + bc}{bd}$$

Is this fraction a ratio of integers?

Yes.

Because products and sums of integers are integers, $ad + bc$ and bd are both integers.

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

$$\frac{ad + bc}{bd}$$

Is the denominator $bd \neq 0$?

Yes.

By the zero product property, since $b \neq 0$ and $d \neq 0$, then $bd \neq 0$.

Sum of Rational Numbers – cont.

Prove that the sum of any two rational numbers is rational.

$$\frac{ad + bc}{bd}$$

Since this number is the fraction of integer, with a nonzero denominator, then this number is a rational number.

→ $r + s$ is rational by definition of rational number, which is what was to be shown.

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Double of Rational Numbers

The double of a rational number is rational.

The double of a number is just its sum with itself.

But since the sum of any two rational numbers is rational, the sum of a rational number with itself is rational.

Hence the double of a rational number is rational.

More formally:

Double of Rational Numbers – cont.

The double of a rational number is rational.

Suppose r is any rational number.

Then $2r = r + r$ is a sum of two rational numbers.

So, by our previous proof, $2r$ is rational.

<u>We can use theorems we already proved to prove new theorems.</u>

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Deriving New Mathematics from Old

- Suppose that you have already proved the following properties of even and odd integers:

1. The sum, product, and difference of any two even integers are even.
2. The sum and difference of any two odd integers are even.
3. The product of any two odd integers is odd.
4. The product of any even integer and any odd integer is even.
5. The sum of any odd integer and any even integer is odd.
6. The difference of any odd integer minus any even integer is odd.
7. The difference of any even integer minus any odd integer is odd.

Exercise to do at home!

Use the properties listed above to prove that if a is any even integer and b is any odd integer, then $\frac{a^2+b^2+1}{2}$ is an integer.