جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 5

Sequences And Mathematical Induction

Sequences and Mathematical Induction

- Section 1:
  - *Sequences*
- Section 2:
  - *Mathematical Induction*

Sequences and Mathematical Induction

- Section 1:
  - *Sequences*
- Section 2:
  - *Mathematical Induction*

Sequences

- Part 1: Motivation – sequences in real life
- Part 2: Finding sequences
- Part 3: Summations
- Part 4: Products
- Part 5: Factorials
- Part 6: Properties of Summations, Products, and Factorials
- Part 7: Sequences in Computer Science (i.e. Loops)

Sequences

- Part 1: Motivation – sequences in real life
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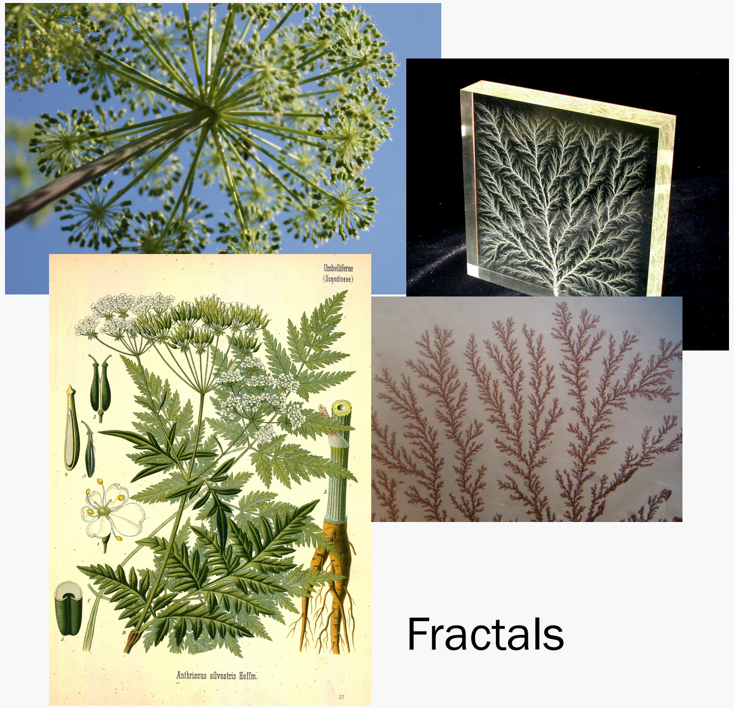
Motivation

Mathematics is the science of patterns, and nature exploits just about every pattern that there is.

- Ian Stewart

Symmetry



Fractals

Spirals

Motivation

Sequences

- How many ancestors do you have?

2 parents

4 grandparents

8 great-grandparents

16 great-great-grandparents

...?

2, 4, 8, 16, 32, 64, 128, ... This is a ***sequence***

Sequences

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Sequences – cont.

2, 4, 8, 16, 32, 64, 128, ….

To express the pattern of the numbers, suppose that each is labeled by an integer giving its position in the row:

Position in the row	1	2	3	4	5	6	7	…
Number of ancestors	2	4	8	16	32	64	128	…

Sequences – cont.

2, 4, 8, 16, 32, 64, 128, ….

To express the pattern of the numbers, suppose that each is labeled by an integer giving its position in the row:

Position in the row	1	2	3	4	5	6	7	…
Number of ancestors	2	4	8	16	32	64	128	…
	2^1	2^2	2^3	2^4	2^5	2^6	2^7	…

So, for a general value of k, A_k will be the number of ancestors you have in the k^{th} generation, which will equal to 2^k.

Terminology

- A **sequence** is a set of elements written in a row:

$$a_m, am_{+\,1}, am_{+\,2}, \ldots, an$$

A **term**

Generalized as

a_k

We read it as

a sub k

subscript

or

index

Terminology

- A **sequence** is a set of elements written in a row:

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n$$

The **initial term** (a_m)

The **final term** (a_n)

Terminology

- This is an **infinite sequence:**

$$a_m, am_{+1}, am_{+2}, \ldots$$

Terminology

- The rule that shows how the values of a_k depend on k is called the **explicit formula** or **general formula.**

Finding Terms of Sequences Given by Explicit Formulas

Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

$$a_k = \frac{k}{k+1} \; for\ all\ integers\ k \ \geq 1$$

$$b_i = \frac{i \ - 1}{i} \; for\ all\ integers\ i \ \geq 2$$

Computer the first five terms of both sequences.

Finding Terms of Sequences Given by Explicit Formulas

$$a_1 = \frac{1}{1+1} = \frac{1}{2}$$
$$a_2 = \frac{2}{2+1} = \frac{2}{3}$$
$$a_3 = \frac{3}{3+1} = \frac{3}{4}$$
$$a_4 = \frac{4}{4+1} = \frac{4}{5}$$
$$a_5 = \frac{5}{5+1} = \frac{5}{6}$$

$$b_2 = \frac{2-1}{2} = \frac{1}{2}$$
$$b_3 = \frac{3-1}{3} = \frac{2}{3}$$
$$b_4 = \frac{4-1}{4} = \frac{3}{4}$$
$$b_5 = \frac{5-1}{5} = \frac{4}{5}$$
$$b_6 = \frac{6-1}{6} = \frac{5}{6}$$

Finding Terms of Sequences Given by Explicit Formulas

- The first five terms of both sequences are ½, $2/3$, ¾, $4/5$, and $5/6$.
- It can be shown that all the terms of both sequences are identical.

Infinite Sequences with Finite Number of Values

- Computer the first six terms of the sequence $c_{0,}$ c_1, c_2, … defined as follows:

$$c_j = (-1)^j \; for\; all\; integers\; j \geq 0$$
$$c_0 = (-1)^0 = 1$$
$$c_1 = (-1)^1 = -1$$
$$c_2 = (-1)^2 = 1$$
$$c_3 = (-1)^3 = -1$$
$$c_4 = (-1)^4 = 1$$
$$c_5 = (-1)^5 = -1$$

- This sequence oscillates endlessly between 1 and -1. It is called an alternating sequence.

Finding an Explicit Formula to Fit Given Initial Terms

- Find an explicit formula for a sequence that has the following initial terms:

$$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \ldots$$

- Denote the general term of the sequences by a_k, and suppose the first term is a_1.
- Observe the two following patterns:
  - *The denominator of each term equals the square of the subscript of that term.*

$$a_k = \frac{\pm 1}{k^2}$$

Finding an Explicit Formula to Fit Given Initial Terms

- Find an explicit formula for a sequence that has the following initial terms:

$$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \dots$$

- Denote the general term of the sequences by a_k, and suppose the first term is a_1.
- Observe the two following patterns:
  - *The denominator of each term equals the square of the subscript of that term.*
  - *The numerator oscillates back and forth between* $+1$ *and* -1*; it is* $+1$ *when* k *is odd, and* -1 *when* k *is even.*

$$a_k = \frac{(-1)^{k+1}}{k^2} \; for\ all\ integers\ k \geq 1$$

$$OR\ a_k = \frac{(-1)^{k}}{(k+1)^2} \; for\ all\ integers\ k \geq 0$$

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Summation Notation

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Upper limit of the summation

Expanded form

This is read as the summation from k equals m to n of a_k

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Lower limit of the summation

Computing the Summations

- Let $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1$, and $a_5 = 2$.

Compute the following:

1. $\sum_{k=1}^{5} a_k$

Computing the Summations

- Let $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1$, and $a_5 = 2$.

Compute the following:

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- Let $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1$, and $a_5 = 2$.

Compute the following:

1. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = -2 + -1 + 0 + 1 + 2 = 0$

Computing the Summations

- Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$.

Compute the following:

1. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = -2 + -1 + 0 + 1 + 2 = 0$
2. $\sum_{k=2}^{2} a_k$

Computing the Summations

- Let $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1$, and $a_5 = 2$.

Compute the following:

1. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = -2 + -1 + 0 + 1 + 2 = 0$
2. $\sum_{k=2}^{2} a_k = a_2$

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2. $\sum_{k=2}^{2} a_k = a_2 = -1$
3. $\sum_{k=1}^{2} a_{2k}$

Computing the Summations

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Compute the following:

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2. $\sum_{k=2}^{2} a_k = a_2 = -1$
3. $\sum_{k=1}^{2} a_{2k} = a_{2.1} + a_{2.2}$

Computing the Summations

- Let $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1$, and $a_5 = 2$.

Compute the following:

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2. $\sum_{k=2}^{2} a_k = a_2 = -1$
3. $\sum_{k=1}^{2} a_{2k} = a_{2.1} + a_{2.2} = a_2 + a_4$

Computing the Summations

- Let $a_1 = -2, a_2 = -1, a_3 = 0, a_4 = 1$, and $a_5 = 2$.

Compute the following:

1. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = -2 + -1 + 0 + 1 + 2 = 0$
2. $\sum_{k=2}^{2} a_k = a_2 = -1$
3. $\sum_{k=1}^{2} a_{2k} = a_{2.1} + a_{2.2} = a_2 + a_4 = -1 + 1 = 0$

Computing the Summation

- However, it is more common to have the terms of a summation expressed using an explicit formula. For example:

$$\sum_{k=1}^{5} k^2$$

or

$$\sum_{i=0}^{8} \frac{(-1)^i}{i+1}$$

Computing the Summation

- However, it is more common to have the terms of a summation expressed using an explicit formula. For example:

  $\sum_{k=1}^{5} k^2 =$

Computing the Summation

- However, it is more common to have the terms of a summation expressed using an explicit formula. For example:

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

From Summation to Expanded Form

- Write the following summation in expanded form:

$$\sum_{i=0}^{8} \frac{(-1)^i}{i+1}$$

From Summation to Expanded Form

- Write the following summation in expanded form:

$$\sum_{i=0}^{8} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \frac{(-1)^4}{4+1} + \frac{(-1)^5}{5+1} + \frac{(-1)^6}{6+1} + \frac{(-1)^7}{7+1} + \frac{(-1)^8}{8+1}$$

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- Write the following summation in expanded form:

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$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$$

From Summation to Expanded Form

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$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \cdots + \frac{(-1)^n}{n+1}$$

From Summation to Expanded Form

- Write the following summation in expanded form:

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$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

From Expanded Form to Summation

- Express the following using summation notation:

$$\frac{1}{n}+\frac{2}{n+1}+\frac{3}{n+2}+\cdots+\frac{n+1}{2n}$$

From Expanded Form to Summation

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$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

Value of k	term	Simplification of term
0	$\frac{1}{n}$	
1	$\frac{2}{n+1}$	
2	$\frac{3}{n+2}$	
…	…	
n	$\frac{n+1}{2n}$	

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$\dots$	$\dots$	$\dots$
n	$\frac{n+1}{2n} = \frac{n+1}{n+n}$	$\frac{n+1}{n+n}$

From Expanded Form to Summation

- Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}$$

Value of k	term	Simplification of term
0	$\frac{1}{n}$	$\frac{0+1}{n+0}$
1	$\frac{2}{n+1}$	$\frac{1+1}{n+1}$
2	$\frac{3}{n+2}$	$\frac{2+1}{n+2}$
…	…	…
n	$\frac{n+1}{2n} = \frac{n+1}{n+n}$	$\frac{n+1}{n+n}$

Final Terms

- When solving problems, it is often useful to rewrite a summation using the recursive form of the definition, either by separating off the final term of a summation or by adding a final term to a summation.

- For example, $\sum_{k=m}^{n} a_k$ can be rewritten as $\sum_{k=m}^{n-1} a_k + a_n$ for all integers $n > m$

Separating Off a Final Term

- Rewrite the following by separating off the final term:

$$\sum_{k=1}^{n} \frac{1}{k^2}$$

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$$\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{n-1} \frac{1}{k^2} + \frac{1}{n^2}$$

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$$\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{n-1} \frac{1}{k^2} + \frac{1}{n^2}$$

$$\sum_{k=1}^{n+1} \frac{1}{k^2}$$

Separating Off a Final Term

- Rewrite the following by separating off the final term:

$$\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{n-1} \frac{1}{k^2} + \frac{1}{n^2}$$

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2}$$

Adding On a Final Term

- Write the following as a single summation.

$$\sum_{k=0}^{n-1} 2^k + 2^n$$

Adding On a Final Term

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$$\sum_{k=0}^{n-1} 2^k + 2^n = \sum_{k=0}^{n} 2^k$$

Adding On a Final Term

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$$\sum_{k=0}^{n-1} 2^k + 2^n = \sum_{k=0}^{n} 2^k$$

$$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

Telescoping Sums

- Telescoping sums are finite sums in which pairs of consecutive terms cancel each other, leaving only the initial and final terms. These sums can be rewritten as a simple expression.

Telescoping Sums

- If you know that

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$$

Use this identity to find a simple expression for the following sum:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

Telescoping Sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

Telescoping Sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

Telescoping Sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Telescoping Sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Telescoping Sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

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Product Notation

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_n$$

This is read as
the product
from k equals m
to n
of a_k

Computing the Product

- Computer the following products:

1. $\prod_{k=1}^{1} \frac{k}{k+1}$

Computing the Product

- Computer the following products:

1. $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1}$

Computing the Product

- Computer the following products:

1. $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$

Computing the Product

- Computer the following products:

1. $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$
2. $\prod_{k=1}^{5} k$

Computing the Product

- Computer the following products:

1. $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$
2. $\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

Computing the Product

- Computer the following products:

1. $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$
2. $\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

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Factorials

- For each positive integer n, the quantity n factorial denoted $n!$, is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot n - 1 \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

- Zero factorial, denoted $0!$, is defined to be 1:

$$0! = 1$$

Note: $n!$ really means: how many distinct ways there are to arrange n things?

The Recursive Definition of Factorials

- Factorials can be defined recursively as follows:

Given any nonnegative integer, n,

$$n! = \begin{cases} 1 & if\ n = 0 \\ n \cdot (n-1)! & if\ n \geq 1 \end{cases}$$

The First Ten Factorials

$0! = 1$
$1! = 1 = 1 \cdot 0! = 1 \cdot 1 = 1$
$2! = 2 \cdot 1 = 2 \cdot 1! = 2 \cdot 1 = 2$
$3! = 3 \cdot 2 \cdot 1 = 3 \cdot 2! = 3 \cdot 2 = 6$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3! = 4 \cdot 6 = 24$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4! = 5 \cdot 24 = 120$
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6 \cdot 5! = 6 \cdot 120 = 720$
$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7 \cdot 6! = 7 \cdot 720 = 5{,}040$
$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8 \cdot 7! = 8 \cdot 5{,}040 = 40{,}320$
$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 9 \cdot 8! = 9 \cdot 40{,}320 = 362{,}880$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3! . 3!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8{\cdot}7!}{7!} = 8$

2. $\frac{5!}{2!{\cdot}3!} = \frac{5{\cdot}4{\cdot}3!}{2!{\cdot}3!} = \frac{5{\cdot}4}{2{\cdot}1} = \frac{20}{2} = 10$

3. $\frac{1}{2!{\cdot}4!} + \frac{1}{3!.3!} = \frac{1}{2!{\cdot}4!} \cdot \frac{3}{3} + \frac{1}{3!.3!} \cdot \frac{4}{4}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3!.3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3!.3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3! . 3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3! . 3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3! . 3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3! . 3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!} = \frac{7}{3! \cdot 4!} = \frac{7}{144}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3! . 3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3! . 3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!} = \frac{7}{3! \cdot 4!} = \frac{7}{144}$
4. $\frac{(n+1)!}{n!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3!.3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3!.3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!} = \frac{7}{3! \cdot 4!} = \frac{7}{144}$
4. $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3!.3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3!.3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!} = \frac{7}{3! \cdot 4!} = \frac{7}{144}$
4. $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n + 1$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3!.3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3!.3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!} = \frac{7}{3! \cdot 4!} = \frac{7}{144}$
4. $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n + 1$
5. $\frac{n!}{(n-3)!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3!.3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3!.3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!} = \frac{7}{3! \cdot 4!} = \frac{7}{144}$
4. $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n + 1$
5. $\frac{n!}{(n-3)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)!}{(n-3)!}$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3! . 3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3! . 3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!} = \frac{7}{3! \cdot 4!} = \frac{7}{144}$
4. $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n + 1$
5. $\frac{n!}{(n-3)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)!}{(n-3)!} = n \cdot (n-1) \cdot (n-2)$

Computing with Factorials

- Simplify the following expressions:

1. $\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$
2. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$
3. $\frac{1}{2! \cdot 4!} + \frac{1}{3! . 3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3! . 3!} \cdot \frac{4}{4} = \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!} = \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!} = \frac{7}{3! \cdot 4!} = \frac{7}{144}$
4. $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n + 1$
5. $\frac{n!}{(n-3)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)!}{(n-3)!} = n \cdot (n-1) \cdot (n-2) = n^3 - 3n^2 + 2n$

Sequences

- Part 1: Motivation – sequences in real life
- Part 2: Finding sequences
- Part 3: Summations
- Part 4: Products
- Part 5: Factorials
- Part 6: Properties of Summations, Products, and Factorials
- Part 7: Sequences in Computer Science (i.e. Loops)

Properties of Summations and Products

- If a_m, a_{m+1}, a_{m+2},… and b_m, b_{m+1}, b_{m+2},… are sequences of real numbers and c is any real number, then the following equations hold for any integer n ≥ m:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
2. $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ *generalized distributive law*
3. $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$

These properties will be helpful when you program loops

Using Summation Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single summation:

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$$

Using Summation Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single summation:

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(k+1) + 2 \cdot \sum_{k=m}^{n}(k-1)$$

Using Summation Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single summation:

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(k+1) + 2 \cdot \sum_{k=m}^{n}(k-1)$$

$$= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2 \cdot (k-1)$$

Using Summation Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single summation:

$$\begin{aligned}\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k &= \sum_{k=m}^{n}(k+1) + 2 \cdot \sum_{k=m}^{n}(k-1) \\ &= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2 \cdot (k-1) \\ &= \sum_{k=m}^{n}((k+1) + 2 \cdot (k-1))\end{aligned}$$

Using Summation Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single summation:

$$\begin{aligned}\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k &= \sum_{k=m}^{n} (k+1) + 2 \cdot \sum_{k=m}^{n} (k-1) \\ &= \sum_{k=m}^{n} (k+1) + \sum_{k=m}^{n} 2 \cdot (k-1) \\ &= \sum_{k=m}^{n} ((k+1) + 2 \cdot (k-1)) \\ &= \sum_{k=m}^{n} (3k-1)\end{aligned}$$

Using Product Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single product:

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right)$$

Using Product Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single product:

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \left(\prod_{k=m}^{n}(k+1)\right) \cdot \left(\prod_{k=m}^{n}(k-1)\right)$$

Using Product Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single product:

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \left(\prod_{k=m}^{n}(k+1)\right) \cdot \left(\prod_{k=m}^{n}(k-1)\right)$$

$$= \prod_{k=m}^{n}(k+1)(k-1)$$

Using Product Properties

- Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expression as a single product:

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \left(\prod_{k=m}^{n}(k+1)\right) \cdot \left(\prod_{k=m}^{n}(k-1)\right)$$

$$= \prod_{k=m}^{n}(k+1)(k-1)$$

$$= \prod_{k=m}^{n}(k^2-1)$$

Sequences

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Change of Variables

Notice the following two summations:

$$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$$

and

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$$

Change of Variables

We can conclude that

$$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2$$

→ The symbol used to represent the index of a summation can be replaced by any other ***symbol as long as the replacement is made in each location where the symbol occurs***.

→ The index of a summation is called a ***dummy variable***. Outside of its local context (both before and after), the symbol may have another meaning entirely.

Change of Variables

- The appearance of a summation can be altered by more complicated changes of variable as well.

$$\sum_{j=2}^{4}(j-1)^2 = (2-1)^2 + (3-1)^2 + (4-1)^2$$

$$= 1^2 + 2^2 + 3^2$$

$$= \sum_{k=1}^{3} k^2$$

Change of Variables

- Transform the following summation by making the specified change of variable:

$$summation\text{: } \sum_{k=0}^{6} \frac{1}{k+1}, \qquad change\ of\ vairable\text{: } j = k + 1$$

When $k = 0, j = k + 1 = 0 + 1 = 1$

When $k = 6, j = k + 1 = 6 + 1 = 7$

Since $j = k + 1$, then $k = j – 1$

Hence,

$$\frac{1}{k+1} = \frac{1}{(j-1)+1} = \frac{1}{j}$$

Change of Variables

- Transform the following summation by making the specified change of variable:

$$summation: \sum_{k=0}^{6} \frac{1}{k+1}, \qquad change\ of\ vairable: j = k + 1$$

Putting all the previous steps together:

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}$$

Change of Variables

- Transform the following summation by making the specified change of variable:

$$summation: \sum_{k=1}^{n+1} \frac{k}{n+k}, \qquad change\ of\ vairable: j = k - 1$$

When $k = 1, j = k - 1 = 1 - 1 = 0$

When $k = n + 1, j = k - 1 = (n + 1) - 1 = n$

Since $j = k - 1$, then $k = j + 1$

Hence,

$$\frac{k}{n+k} = \frac{j+1}{n+(j+1)} = \frac{j+1}{n+j+1}$$

Change of Variables

- Transform the following summation by making the specified change of variable:

$$summation: \sum_{k=1}^{n+1} \frac{k}{n+k}, \qquad change\ of\ vairable: j = k - 1$$

Putting all the previous steps together:

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{j=0}^{n} \frac{j+1}{n+j+1}$$

Dummy Variables in Programming Loops

- Is there a difference between the output of the following algorithm segments?

```
for i := 1 to n
    print a[i]
next i
```

```
for j := 0 to n - 1
    print a[j + 1]
next j
```

```
for k := 2 to n + 1
    print a[k - 1]
next k
```

- No. All three algorithm segments produce the same output.

Dummy Variables in Programming Loops

- Is there a difference between the output of the following algorithm segments?

```
s := a[1]
for k := 2 to n
   s := s + a[k]
next k
```

```
s := 0
for k := 1 to n
   s := s + a[k]
next k
```

- No, both algorithm segments produce the value of $\sum_{k=1}^{n} a[k]$.

The difference is the initializing of the sum to equal a[1] in the first segment, and initializing it to equal 0 in the second segment.

Floor and Ceiling

- The floor of a number, n, denoted by $\lfloor n \rfloor$ is the largest integer variable smaller than n.
- The ceiling of a number, n, denoted by $\lceil n \rceil$ is the smallest integer variable larger than n.

Exercise

- Consider the sequence defined by $a_n = \frac{2n+(-1)^n-1}{4}$ for all integers $n \geq 0$. Find an alternative explicit formula for a_n that uses the floor notation.