جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 5

Sequences And Mathematical Induction

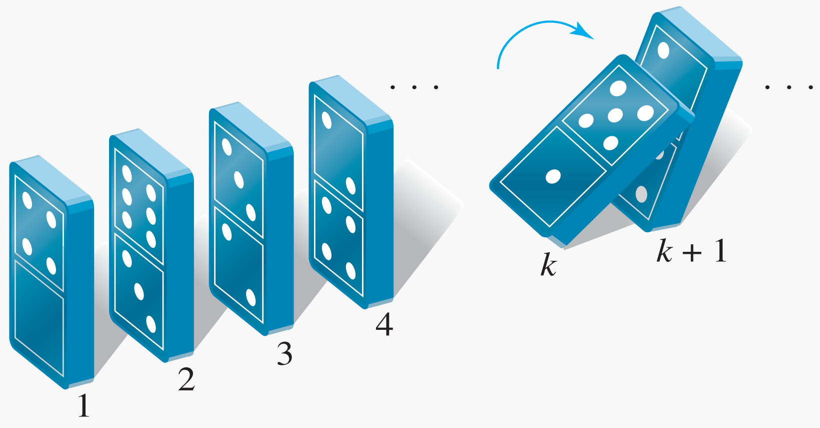
Sequences and Mathematical Induction

- Section 1:
  - *Sequences*
- Section 2:
  - *Mathematical Induction*

Mathematical Induction

What is Mathematical Induction

- "Mathematical induction is the standard proof technique in computer science."
  - *Anthony Ralston*
- It is one of the more recently developed techniques of proof in the history of mathematics.
- The basis of this principle is to generalize specific cases, through observing patterns, to produce a general rule.
  - *E.g. if I know that if the first domino falls, then the the second domino will fall, and if I can prove that if the k^{th} domino falls, then the $(k+1)^{st}$ domino falls, then I can induce that if the first domino falls, then all the dominos in the train of dominos will fall too.*



Principle of Mathematical Induction

- Let $P(n)$ be a property that is defined for integers n, and let *a* be a fixed integer.

Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

for all integers $n \geq a, P(n)$

is true.

It's called a principle because it uses conjuncture to generate a rule instead of laws.

Mathematical Induction

Method of Proof by Mathematical Induction

- Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k + 1)$ is true. To perform this step:

1. suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

Then

2. show that $P(k + 1)$ is true.

We call this the inductive hypothesis

Example – cont.

- Show that for all integers $n \geq 8$, n¢ can be obtained using any combination of 3¢ and 5¢ coins only.

- Let $P(n)$ be the sentence: n¢ *can be obtained using* 3¢ *and* 5¢ *coins.*

- Step 1 (basis step): Show $P(8)$ is true.
  - 8¢ *can be obtained using* 3¢ *and* 5¢ *coins.*

Example – cont.

- Step 2 (inductive step): Show for all $k \geq 8$, if $P(k)$ is true then $P(k+1)$ is true.
  - *Inductive hypothesis: Suppose $P(k)$ is true:*

  Case 1: $P(k)$ contains a 5¢ coin: replace the 5¢ coin with two 3¢ coins, and you have $P(k+1)$.

  Case 2: $P(k)$ doesn't contain a 5¢ coin: since $k \geq 8$, it must contain at least three 3¢ coins, which can be replaced by two 5¢ coins, and you have $P(k+1)$.

- In either case, $P(k+1)$ can be obtained using 3¢ and 5¢ coins.

Mathematical Induction

- Part 1: Introduction
- Part 2: Induction is a Method of Proof
- Part 3: Proving Sum of Integers
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Sum of the First n Integers

- What is result of the following summation:

$$\sum_{k=1}^{100} k$$

$$1 + 2 + 3 + \dots + 50 + 51 + \dots + 98 + 99 + 100$$

sum is 101

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Sum of the First n Integers

- What is result of the following summation:

$$\sum_{k=1}^{100} k$$

1 + 2 + 3 + … + 50 + 51 + … + 98 + 99 + 100

= 101 + 101 + … + 101 (50 times)

= 5050

$$= \frac{100 \times 101}{2} = \frac{n\,(n+1)}{2}$$

Terminology

- A ***closed form***: is an expression that can be computed by applying a fixed number of familiar operations to the arguments.
- This means that we don't have to iterate (loop) over an unknown number of calculations.
- $\frac{n\,(n+1)}{2}$ is the closed form of the summation $\sum_{i=0}^{n} n$.

Sum of the First *n* Integers Theorem

- For all integers $n \geq 1$, $1 + 2 + 3 + \ \ldots + n = \frac{n(n+1)}{2}$
- Show that P(1) is true: $P(1) = \frac{1\times(1+1)}{2} = \frac{2}{2} = 1$
- Show for all integers $k \geq 1$, if P(k) is true then P(k+1) is also true.

Suppose P(k) is true: $1 + 2 + 3 + \ \ldots + k = \frac{k(k+1)}{2}$, and show that $P(k+1) = \frac{(k+1)(k+2)}{2}$

$$P(k+1) = P(k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k^2+k}{2} + \frac{2k+2}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

Separating off the last term

Sum of the First *n* Integers in Programming

Prove that the following code segments produce the same result for value of n $\geq$ 1:

```
int s = 0;
for (int i = 0; i <=n; i++) {
        s = s + i;
}
```

```
int s = (n * (n + 1) ) / 2;
```

The proof of this question is the same as the proof of the statement:

For all integers $n \geq 1, 1 + 2 + 3 + \ \ldots + n = \frac{n(n+1)}{2}$

Examples of Sums

- Evaluate the following expressions:

1. $2 + 4 + 6 + \cdots + 500$

Examples of Sums

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1. $2 + 4 + 6 + \cdots + 500 = 2\,(1 + 2 + 3 + \ \ldots + 250)$

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Examples of Sums

- Evaluate the following expressions:

1. $2+4+6+\cdots+500=2\,(1+2+3+\ \ldots+250)=2\times\frac{250\times251}{2}=62{,}750$
2. $5+6+7+\ \ldots+50$

Examples of Sums

- Evaluate the following expressions:

1. $2 + 4 + 6 + \cdots + 500 = 2\,(1 + 2 + 3 + \ \ldots + 250) = 2\times\frac{250\times251}{2} = 62{,}750$
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$$= \frac{50\times 51}{2} - 10$$

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3. $For\ an\ integer\ h \geq 2, write\ 1 + 2 + 3 + \cdots + (h - 1)\ in\ closed\ form$

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2. $5 + 6 + 7 + \ \ldots + 50 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots + 50 - (1 + 2 + 3 + 4)$

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3. $For\ an\ integer\ h \geq 2, write\ 1 + 2 + 3 + \cdots + (h - 1)\ in\ closed\ form$

   $= \frac{(h-1)((h-1)+1)}{2} = \frac{(h-1)h}{2}$

Mathematical Induction

Sum of a Geometric Sequence

- For any real number *r* other than 1, and any integer $n \geq 0$:

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

Prove this using mathematical induction.

Sum of a Geometric Sequence

- $P(0) = \sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$ basis step
- $P(k) = \sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1}$ inductive hypothesis
- $P(k+1) = \sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}$ we need to show that this is true.

Sum of a Geometric Sequence

- $P(k+1) = P(k) + \ r^{(k+1)}$

Sum of a Geometric Sequence

- $P(k+1) = P(k) + \ r^{(k+1)} = \sum_{i=0}^{k} r^i + r^{(k+1)}$

$$= \frac{r^{(k+1)}-1}{r-1} + r^{(k+1)}$$

$$= \frac{r^{(k+1)}-1}{r-1} + \frac{r^{(k+1)}(r-1)}{r-1}$$

$$= \frac{r^{(k+1)}-1+r\times r^{(k+1)}-r^{(k+1)}}{r-1}$$

$$= \frac{r^{(k+1)}-1+r^{(k+2)}-r^{(k+1)}}{r-1}$$

$$= \frac{r^{(k+2)}-1}{r-1}$$

which we wanted to prove

Food for Thought

- Can you write two different algorithms that produce the sum of a geometric sequences, one that uses a loop and one that doesn't?

- Which one is cheaper* for the computer?

  *cheaper means less resources (e.g. memory) and/or less time

Examples of Sums of a Geometric Sequence

- Assuming that $m \geq 3$, write the following summations in closed forms:

1. $1 + 3 + 9 + 27 + \ldots + 3^{m-2}$

Examples of Sums of a Geometric Sequence

- Assuming that $m \geq 3$, write the following summations in closed forms:

1. $1 + 3 + 9 + 27 + \ldots + 3^{m-2} = 3^0 + 3^1 + 3^2 + \cdots + 3^{m-2}$

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$$= \frac{3^{(m-2)+1} - 1}{3 - 1}$$

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2. $9 + 27 + 81 + \cdots + 3^m$

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   $= \frac{3^{(m-2)+1}-1}{3-1} = \frac{3^{m-1}-1}{2}$

2. $9 + 27 + 81 + \cdots + 3^m = 3^2 + 3^3 + 3^4 + \cdots + 3^m$

   $= 3^2 \times (3^0 + 3^1 + 3^2 + \cdots + 3^{(m-2)})$

Examples of Sums of a Geometric Sequence

- Assuming that $m \geq 3$, write the following summations in closed forms:

1. $1 + 3 + 9 + 27 + \ldots + 3^{m-2} = 3^0 + 3^1 + 3^2 + \cdots + 3^{m-2}$

$$= \frac{3^{(m-2)+1} - 1}{3-1} = \frac{3^{m-1} - 1}{2}$$

2. $9 + 27 + 81 + \cdots + 3^m = 3^2 + 3^3 + 3^4 + \cdots + 3^m$

$$= 3^2 \times (3^0 + 3^1 + 3^2 + \cdots + 3^{(m-2)}) = 9 \times \frac{3^{m-1} - 1}{2}$$

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Proving a Divisibility Property

- Use mathematical induction to prove that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Basis step: $P(1) = 2^{2 \cdot 1} - 1 = 4 - 1 = 3$, which is divisible by 3.

Inductive hypothesis: $2^{2k} - 1$ is divisible by 3.

show that $P(k+1) = 2^{2(k+1)} - 1$ is divisible by 3.

$$
\begin{aligned}
2^{2(k+1)} - 1 &= 2^{2k+2} \\
&= 2^{2k} \cdot 2^2 - 1 \\
&= 2^{2k} \cdot 4 - 1 = 2^{2k} \cdot (3+1) - 1 \\
&= \underbrace{3 \cdot 2^{2k}}_{\text{Divisible by 3 because } 2^{2k} \text{ is an integer}} + 2^{2k} - 1
\end{aligned}
$$

Divisible by 3 from the inductive hypothesis

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Proving an Inequality

- Use mathematical induction to prove that for all integers $n \geq 3$, $2n + 1 < 2^n$.
- What's different about this property is that it is an inequality, but inequalities work the same as any other property.

- Basis step: $\mathrm{P}(3)$: $2 \times 3 + 1 < 2^3$

  $7 < 8$, which is true.
- Inductive Hypothesis: $\mathrm{P}(\mathrm{k})$: $2 \cdot k + 1 < 2^k$ (we assume this is true)
- We need to show that $\mathrm{P}(\mathrm{k}+1)$: $2(k+1) + 1 < 2^{(k+1)}$

Proving an Inequality – cont.

- Use mathematical induction to prove that for all integers $n \geq 3$, $2n + 1 < 2^n$.

$2(k+1) + 1 = 2k + 2 + 1 = 2k + 3 = (2k+1) + 2$

Since we know that $2k + 1 < 2^k$ from our inductive hypothesis we can use the inequality $2 \cdot k + 1 < 2^k$

And since $n \geq 3$, then by default $k > 2$, which makes $2 < 2^k$

And by using these two inequalities:

$$(2k+1) + 2 < 2^k + 2^k$$

$$(2k+1) + 2 < 2^{k+1}$$ which is what we needed to show

Mathematical Induction

Proving a Property of a Sequence

- We have a sequence that is defined as follows:

$a_1 = 2$

$a_k = 5a_{k-1}$ for all integers $k \geq 2$

- We need to use mathematical induction to show that the terms of the sequence satisfy the property

$a_n = 2 \times 5^{n-1}$ for all integers $\mathrm{n} \geq 1$

Proving a Property of a Sequence – cont.

- For practice, let's write the first five terms of the sequence:

$a_1 = 2$
$a_2 = 5a_{2-1} = 5a_1 = 5\times 2 = 10$
$a_3 = 5a_{3-1} = 5a_2 = 5\times 10 = 50$
$a_4 = 5a_{4-1} = 5a_3 = 5\times 50 = 250$
$a_5 = 5a_{5-1} = 5a_4 = 5\times 250 = 1250$

Proving a Property of a Sequence – cont.

- We want to use mathematical induction to prove that $a_n = 2\times5^{n-1}$ for all integers $\mathrm{n} \geq 1$

- Basis step: $P(1) = 2\times5^{1-1} = 2\times5^0 = 2\times1 = 2$, which matches the first term in our definition.
- Inductive hypothesis: we assume that $P(k) = 2\times5^{k-1}$ is true

  We need to show that

  $P(k+1) = 2\times5^{(k+1)-1} = 2\times5^k$

Proving a Property of a Sequence – cont.

- According to the definition of the sequence:

$a_{k+1} = 5a_{(k+1)-1} = 5a_k$

And through our inductive hypothesis, we know that $a_k = 2\times 5^{k-1}$

Therefore:

$$
\begin{aligned}
a_{k+1} &= 5\times\left(2\times 5^{k-1}\right) \\
&= 2\times\left(5\times 5^{k-1}\right) \\
&= 2\times\left(5^{k}\right) \qquad \text{which is what we needed to show}
\end{aligned}
$$

Mathematical Induction

Deduction vs Induction vs Abduction

Deductive reasoning, or *deduction*, is making an inference based on widely accepted facts or premises. If a beverage is defined as "drinkable through a straw," one could use deduction to determine soup to be a beverage.

Inductive reasoning, or *induction*, is making an inference based on an observation, often of a sample. You can induce that the soup is tasty if you observe all of your friends consuming it.

Abductive reasoning, or *abduction*, is making a probable conclusion from what you know. If you see an abandoned bowl of hot soup on the table, you can use abduction to conclude the owner of the soup is likely returning soon.

Merriam-Webster Dictionary