جامعة بيرزيت

BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 6

Set Theory

Set Theory

6.1. Basic Definitions of Set Theory

6.2 Properties of Sets

6.3 Algebraic Proofs

6.4 Boolean Algebras

Set Theory

Basic Definitions of Set Theory

- Basic Definitions and Set Properties
- Subsets and Proper Subsets
- Set Equality
- Operations on Sets
- Partitions of Sets
- Cartesian Products

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Basic Definitions

“A set is a collection into a whole M of definite and separate objects of our intuition or our thought. These objects are called the elements of M.”

Georg Cantor

A set is an unordered collection of different elements.

An element (or a member) of a set is any one of the distinct objects that belong to that set.

Notation of Sets

Conventionally, we denote a set by a capital letter and denote the elements of the set by lower-case letters. We usually separate the elements using commas.

For example, we can write the set A that contains the vowels of the English alphabet as:

$$A = \{a, e, i, o, u\}$$

We read this as ‘the set A containing the vowels of the English alphabet’.

Basic Definitions – cont.

By definition:

- The order of elements in a set is not relevant.

For example, $\{a, e, i, o, u\}$ is the same set as $\{e, o, i, u, a\}$

- Elements of a set cannot be repeated.

For example, $\{a, e, i, o, u, i\}$ is not allowed.

Notation of Sets

- Defining a set by listing all the elements of set like this

$$A = \{a, e, i, o, u\}$$

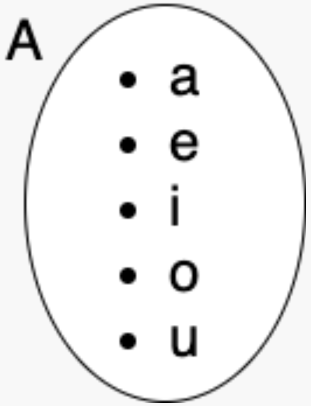
  is called the roster nation.

- Defining a set by its property like this

$$A = \{x \mid x \text{ is a vowel in the Englih language}\}$$

  is called the set-builder notation.

- Sets can also be defined using Venn diagrams:



Notation of Elements

In the previous example,

$$A = \{a, e, i, o, u\}$$

a is an element of the set A.

However, $\{a\}$ is not an element of the set A.

$\{a\} \neq a$

Notation of Elements – cont.

- An element of a set may be a different set.
- For example, $\{1, \{a\}\}$ is a set that contains 2 elements: 1 and $\{a\}$.

Notation of Set Membership

In the example,

$$A = \{a, e, i, o, u\}$$

To say that a is an element (or a member) of the set A, we use the notation $a \in A$.

To say that b is not an element (or a member) of the set A, we use the notation $b \notin A$.

These statements can have a truth value:

$a \in A$ is true

$b \in A$ is false

Empty Set Notation

- A set can potentially have no elements. In this case, it is called an *empty set*.
- An empty set is denoted by { } or the null sign Ø
- In the past, "0" was occasionally used as a symbol for the empty set, but this is now considered to be an improper use of notation. (Wikipedia, Empty set)

Empty Set Notation – cont.

- An empty set is <u>NOT</u> the same as a set containing the element $\emptyset$:

$$\{\,\} \neq \{\emptyset\}$$

- An empty set is <u>NOT</u> the same as a set containing the element 0:

$$\{\,\} \neq \{0\}$$

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Subsets

- If A and B are sets, then A is called a subset of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B.

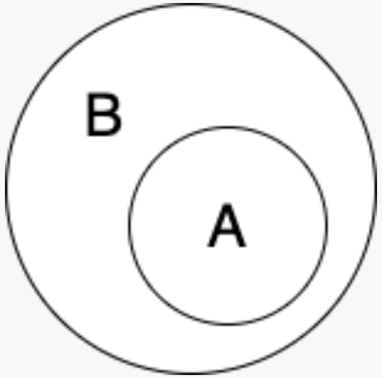
$$A \subseteq B \iff \forall x, \quad x \in A \rightarrow x \in B$$

- This is equivalent to saying

  A is contained in B,

  and

  B contains A.



Subsets – example

- $\mathbb{Z}$ is the set of all integers
- $\mathbb{Q}$ is the set of all rational numbers
- $\mathbb{R}$ is the set of all real numbers

$$\mathbb{Z} \subseteq \mathbb{Q}$$
$$\mathbb{Q} \subseteq \mathbb{R}$$
$$\mathbb{Z} \subseteq \mathbb{R}$$

Proper Subsets

- If A and B are sets, then A is called a proper subset of B, denoted by $A \subset B$, if, and only if, every element of A is also an element of B, but there is at least on element of B that is not in A.

- For example,

$$A = \{1, 2, 3\}, B = \{1, 2, 3\}$$

A is a subset of B, but it is not a proper subset of B.

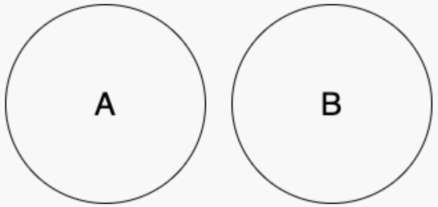
$$A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}$$

A is a proper subset of B.

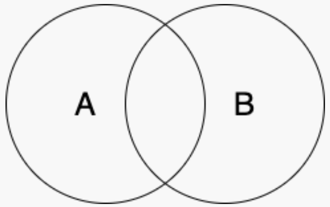
Not-a-Subset Relation

- The relationship 'not a subset', denoted by $A \not\subseteq B$, can happen in three cases:

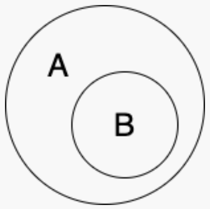
1. When none of the elements in A is in B.



2. When some elements in A are in B, but some elements of A are not in B.



3. When B is a proper subset of A.



$\in$ vs $\subseteq$

Are the following statements true or false?

- $2 \in \{1, 2, 3\}$ ✓
- $\{2\} \in \{1, 2, 3\}$ ✗
- $2 \subseteq \{1, 2, 3\}$ ✗
- $\{2\} \subseteq \{1, 2, 3\}$ ✓
- $\{2\} \subseteq \{\{1\}, \{2\}\}$ ✗
- $\{\{2\}\} \subseteq \{\{1\}, \{2\}\}$ ✓
- $\{2\} \in \{\{1\}, \{2\}\}$ ✓

Proof of Subset Relations

- Define the sets A and B as follows:

$$A = \{m \in \mathbb{Z} \,| m = 6r + 12 \; for \; some \; r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \,| n = 3s \; for \; some \; s \in \mathbb{Z}\}$$

Prove that $A \subseteq B$.

- Suppose x is an arbitrarily chosen element of A.
- Show that $x \in B$, meaning show that $x = 3s \; for \; some \; s \in \mathbb{Z}$.

Proof of Subset Relations – cont.

Suppose $x \in A$

Then, by definition,

$$x = 6r + 12$$

$$x = 3(2r + 4)$$

Let $2r + 4 = s$, which is an integer because it's the result of the multiplication and addition of integers

$$x = 3s$$

Therefore, $x \in B$, and as such, $A \subseteq B$

Power Sets

- Given a set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

- For example,

$$\mathcal{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$$

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Set Equality

Given sets A and B, A equals B, written $A = B$, if, and only if, every element of A is in B and every element of B is in A.

$B \subseteq A$

$A \subseteq B$

$$A = B \iff A \subseteq B \text{ } and \text{ } B \subseteq A$$

Proof of Set Equality

- Define the sets A and B as follows:

$$A = \{m \in \mathbb{Z} \,|m = 2r \; for \; some \; r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \,|n = 2s - 2 \; for \; some \; s \in \mathbb{Z}\}$$

Prove that $A = B$.

- *Prove that $A \subseteq B$.*
- Prove that $B \subseteq A$.

An exercise to do at home

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Universal Sets

- A universal set, usually denoted by U, is a set which has elements of all the related sets, without any repetition of elements.

- If we talk about people, a universal set in this context is a set of all the people in the world.

- If we talk about numbers, a universal set in this case is the set of all real numbers.

- Universal sets are also called a universe of discourse.

Operations on Sets

- **Definition**

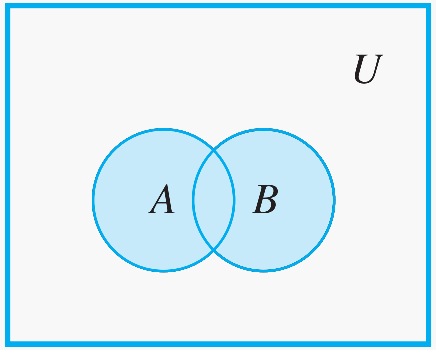
Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$, is the set of all elements that are in B and not A.
4. The **complement** of A, denoted $\boldsymbol{A^c}$, is the set of all elements in U that are not in A.

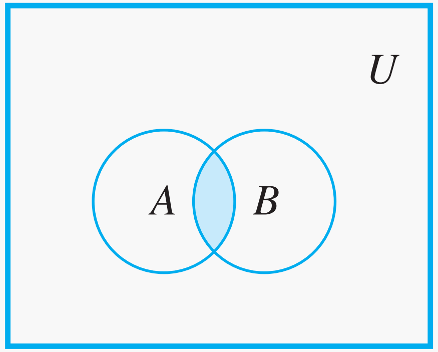
Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

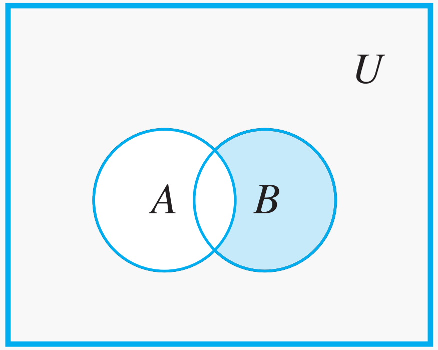
Venn Diagrams of Set Operations



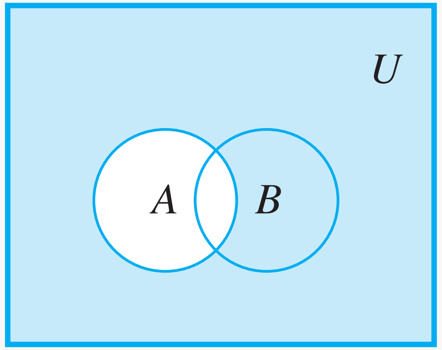
Shaded region represents $A \cup B$.



Shaded region represents $A \cap B$.



Shaded region represents $B - A$.



Shaded region represents A^c.

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Disjoint Sets

- Two sets are called disjoint if, and only if, they have no elements in common.

$$A \; and \; B \; are \; disjoint \;\; \Leftrightarrow A \cap B = \emptyset$$

- For example,

$$A = \{1, 3, 5\}$$
$$B = \{2, 4, 6\}$$

A and B are disjoint because

$$\{1, 3, 5\} \cap \{2, 4, 6\} = \{\,\}$$

Mutually Disjoint Sets

- Sets $A_1, A_2, \dots, A_n$ are mutually disjoint (or pairwise disjoint, or nonoverlapping) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common.
- More precisely, for all $i, j = 1, 2, \dots, n$

$$A_i \cap A_j = \emptyset \ \ whenever\ i \neq j$$

Mutually Disjoint Sets – examples

- Let

$$A_1 = \{3, 5\}$$
$$A_2 = \{1, 4, 6\}$$
$$A_3 = \{2\}$$

Are A_1, A_2, and A_3 mutually disjoint? Yes, they are.

- Let

$$B_1 = \{2, 4, 6\}$$
$$B_2 = \{3, 7\}$$
$$B_3 = \{4, 5\}$$

Are B_1, B_2, and B_3 mutually disjoint? No, they are not.

Partitions of Sets

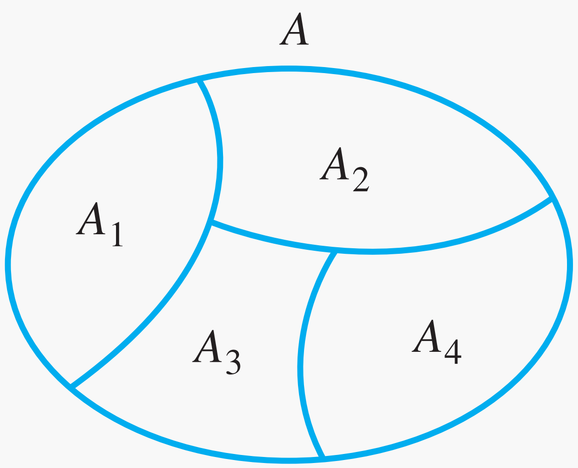
- A **partition of a set** is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset.

- For example, if we divide all the students in this class to groups, where no student is in two groups at the same time, then these groups will be a partition of the class.

Partition of Sets – Formal Definition

- A finite or infinite collection of non-empty sets $\{A_1, A_2, A_3, \ldots\}$ is a partition of a set A if, and only if,

1. The set A is the union of all the $A_i s$
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint



Partition of Sets - Example

Let $\mathbb{Z}$ be the set of all integers, and let

$T_0 = \{n \in \mathbb{Z} \mid n = 3k,\ for\ some\ integer\ k\}$,

$T_1 = \{n \in \mathbb{Z} \mid n = 3k + 1,\ for\ some\ integer\ k\}$,

$T_2 = \{n \in \mathbb{Z} \mid n = 3k + 2,\ for\ some\ integer\ k\}$.

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

Yes. By the quotient-remainder theorem, every integer n can be represented in exactly one of the three forms

n=3k or n=3k+1 or n=3k+2

It also implies that every integer is in one of the sets T_0, T_1, T_2 , so $\mathbb{Z} = T_0 \cup T_1 \cup T_2$.

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n-tuples

Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

n-tuples – example

- (1, 2) = (2, 1) ?
  - *No, by definition of equality and order of pairs:*

    (1, 2) = (2, 1) ⇔ 1 = 2 and 2 = 1

    But 1 ≠ 2, and so the ordered pairs are not equal.

- $(3, (-2)^2, ½) = (\sqrt{9}, 4, 3/6)$?
  - *Yes, by definition of equality and order of pairs:*

    $(3, (-2)^2, ½) = (\sqrt{9}, 4, 3/6) \Leftrightarrow 3 = \sqrt{9}$ *and* $(-2)^2 = 4$ *and* $½ = 3/6$.

    Because all these equations are true, the two ordered triples are equal.

Cartesian Product

- **Definition**

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Cartesian Product – example

- Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

1. Find $A \times B$

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$$

2. Find $A \times B \times C$

$$A \times B \times C = \left\{ \begin{matrix} (x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a), (y, 3, a), \\ (x, 1, b), (x, 2, b), (x, 3, b), (y, 1, b), (y, 2, b), (y, 3, b) \end{matrix} \right\}$$

3. Find $(A \times B) \times C$

$$(A \times B) \times C = \{(u, v) \mid u \in (A \times B) and\ v \in C\}$$

$$(A \times B) \times C = \left\{ \begin{matrix} \big((x, 1), a\big), \big((x, 2), a\big), \big((x, 3), a\big), \big((y, 1), a\big), \big((y, 2), a\big), \big((y, 3), a\big), \\ ((x, 1), b), ((x, 2), b), ((x, 3), b), ((y, 1), b), ((y, 2), b), ((y, 3), b) \end{matrix} \right\}$$