جامعة بيرزيت

BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233

Discrete Mathematics

CHAPTER 6

Set Theory

Set Theory

6.1. Basic Definitions of Set Theory

6.2 Properties of Sets

6.3 Algebraic Proofs

6.4 Boolean Algebras

Set Theory

6.1. Basic Definitions of Set Theory

6.2 Properties of Sets

6.3 Algebraic Proofs

6.4 Boolean Algebras

Properties of Sets

- Subset Relations and Procedural Definitions
- Set Identities
- Proving Set Identities
  - *Distributive Law*
  - *De Morgan's Law*
  - *Intersection and Union with a Subset*
- The Empty Set
- Proof of Conditional Statements

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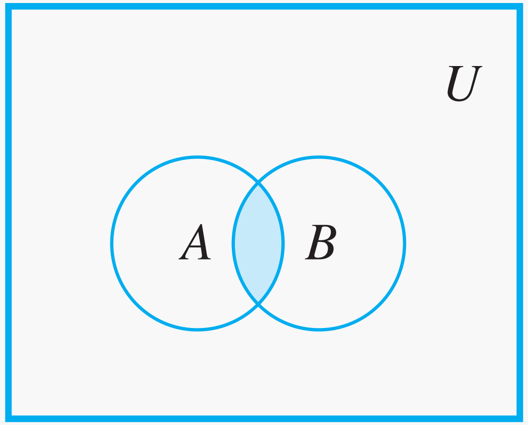
Subset Relations – Inclusion of Intersection

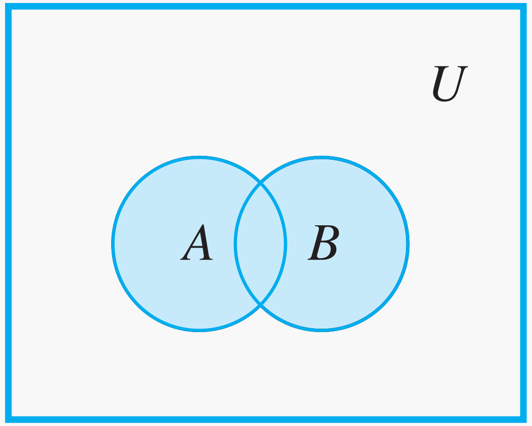
- $A \cap B \subseteq A$
- $A \cap B \subseteq B$



Shaded region represents $A \cap B$.

Subset Relations – Inclusion in Union

- $A \subseteq A \cup B$
- $B \subseteq A \cup B$



Shaded region represents $A \cup B$.

Subset Relations - Transitive Property of Subsets

- If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

- Remember:

$$\mathbb{Z} \subseteq \mathbb{Q}$$
$$\mathbb{Q} \subseteq \mathbb{R}$$
$$\mathbb{Z} \subseteq \mathbb{R}$$

Procedural Versions of Set Definitions

- Let X and Y be subsets of a universal set U, and let x and y be elements of U, then:

- $x \in X \cup Y \quad \leftrightarrow \quad x \in X \quad \vee \quad x \in Y$
  - $X \cup Y = \{x \in U \mid x \in X \ \vee \ x \in Y\}$

- $x \in X \cap Y \quad \leftrightarrow \quad x \in X \quad \wedge \quad x \in Y$
  - $X \cap Y = \{x \in U \mid x \in X \ \wedge \ x \in Y\}$

- $x \in X - Y \quad \leftrightarrow \quad x \in X \quad \wedge \quad x \notin Y$
  - $X - Y = \{x \in U \mid x \in X \ \wedge \ x \notin Y\}$

Procedural Versions of Set Definitions

- Let X and Y be subsets of a universal set U, and let x and y be elements of U, then:

- $x \in X^c \quad \leftrightarrow \quad x \notin X$
  - $X^c = \{x \in U \mid x \notin X\}$

- $(x, y) \in X \times Y \quad \leftrightarrow \quad x \in X \quad \wedge \quad y \in Y$
  - $X \times Y = \{(x, y) \mid x \in X \ \wedge \ y \in Y\}$

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Set Identities

- Commutative Laws: for all sets A and B:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Set Identities

- Associative Laws: for all sets A, B, C:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Set Identities

- Distributive Laws: for all sets A, B, C:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Identities

- Identity Laws: for any set A:

$$A \cup \emptyset = A$$

$$A \cap U = A$$

Set Identities

- Complement Laws: for any set A:

$$A \cup A^c = U$$

$$A \cap A^c = \emptyset$$

Set Identities

- Double Complement Law: for any set A:

$$(A^c)^c = A$$

Set Identities

- Idempotent Laws: for any set A:

$$A \cap A = A$$

$$A \cup A = A$$

Set Identities

- Universal Bound Laws: for any set A:

$$A \cap \emptyset = \emptyset$$

$$A \cup U = U$$

Set Identities

- De Morgan's Laws: for all sets A and B:

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Set Identities

- Absorption Laws: for all sets A and B:

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Set Identities

- Complements of U and $\emptyset$:

$$U^c = \emptyset$$

$$\emptyset^c = U$$

Set Identities

- Set Difference Law: for all sets A and B:

$$A - B = A \cap B^c$$

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Proving Set Identities

- We have seen that:

  Two sets are equal $\leftrightarrow$ each is a subset of the other.

To prove that two sets, X and Y, are equal:

Prove that $X \subseteq Y$

Prove that $Y \subseteq X$

You need to show both because set equality is a biconditional statement.

Proof of Distributive Law

- Show that for all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

We must show that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

And

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

Proof of Distributive Law – cont.

- To show that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

We need to show that

$$\forall x, if\ x \in A \cup (B \cap C), then\ x \in (A \cup B) \cap (A \cup C)$$

- And to show that

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

We need to show that

$$\forall x, if\ x \in (A \cup B) \cap (A \cup C), then\ x \in A \cup (B \cap C)$$

Proof of Distributive Law – cont.

- To show that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

Suppose that $x \in A \cup (B \cap C)$

By definition of union, $x \in A \lor x \in (B \cap C)$

Case 1: $x \in A$

Then, by definition of union, $x \in (A \cup B)$ and $x \in (A \cup C)$

And, by definition of intersection, $x \in (A \cup B) \cap (A \cup C)$

Proof of Distributive Law – cont.

- Case 2: $x \in B \cap C$

Then, by definition of intersection, $x \in B \ \wedge \ x \in C$

And, by definition of union, since $x \in B$, then $x \in (A \cup B)$,

and since $x \in C$, then $x \in (A \cup C)$

And, by definition of intersection, $x \in (A \cup B) \cap (A \cup C)$

Therefore,

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

Proof of Distributive Law – cont.

- And to show that

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

Suppose $x \in (A \cup B) \cap (A \cup C)$

By definition of intersection, $x \in (A \cup B) \land x \in (A \cup C)$

Case 1: $x \in A$

By definition of union, we can immediately conclude that $x \in A \cup (B \cap C)$

Proof of Distributive Law – cont.

Case 2: $x \notin A$

Since $x \in A \cup B$, then, by definition of union, $x \in A \vee x \in B$.

Since $x \notin A$, then by elimination, $x \in B$.

And since $x \in A \cup C$, then, by definition of union, $x \in A \vee x \in C$.

Since $x \notin A$, then by elimination, $x \in C$.

Since $x \in B$ and $x \in C$, then, by definition of intersection, $x \in (B \cap C)$

And, by definition of union, $x \in A \cup (B \cap C)$.

Proof of Distributive Law – cont.

Since we showed that in both cases ($x \in A$ and $x \notin A$)

$$x \in A \cup (B \cap C)$$

We can conclude that

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

Proof of Distributive Law – cont.

And since we've shown that both

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

and

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$$

then we can conclude that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof of Distributive Law– cont.

- Show that for all sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

We must show that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

And

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

An exercise to do at home

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Proof of De Morgan's Law for Sets

- For all sets A and B:

$$(A \cup B)^c = A^c \cap B^c$$

We must show that

$$(A \cup B)^c \subseteq A^c \cap B^c$$

And

$$A^c \cap B^c \subseteq (A \cup B)^c$$

Proof of De Morgan's Law for Sets – cont.

To prove that $(A \cup B)^c \subseteq A^c \cap B^c$, suppose $x \in (A \cup B)^c$.

By definition of complement, $x \notin A \cup B$

Which is equivalent to saying that it is false that (x is in A or x is in B)

By De Morgan's laws of logic, this means

x is not in A and x is not B

Which can be written as

$$x \notin A \ \wedge \ x \notin B$$

Proof of De Morgan's Law for Sets – cont.

Since $x \notin A \;\; \wedge \;\; x \notin B$, then,

by definition of complement, $x \in A^c \; \wedge x \in B^c$

And, by definition of intersection, $x \in A^c \cap B^c$

Therefore,

$$(A \cup B)^c \subseteq A^c \cap B^c$$

Proof of De Morgan's Law for Sets – cont.

To prove that $A^c \cap B^c \subseteq (A \cup B)^c$, suppose $x \in A^c \cap B^c$

By definition of intersection, $x \in A^c \ \wedge x \in B^c$

And, by definition of complement, $x \notin A \ \wedge x \notin B$

In other word, x is not in A and x is not in B

By De Morgan's laws of logic that means

It is false that (x is in A or x is in B)

Proof of De Morgan's Law for Sets – cont.

It is false that (x is in A or x is in B)

which, by definition of union, can be written as

$$x \notin (A \cup B)$$

And, by definition of complement,

$$x \in (A \cup B)^c$$

Therefore,

$$A^c \cap B^c \subseteq (A \cup B)^c$$

Proof of De Morgan's Law for Sets – cont.

Since we showed that

$$(A \cup B)^c \subseteq A^c \cap B^c$$

and

$$A^c \cap B^c \subseteq (A \cup B)^c$$

Then, we can conclude that

$$(A \cup B)^c = A^c \cap B^c$$

Proof of De Morgan's Law for Sets

- For all sets A and B:

$$(A \cap B)^c = A^c \cup B^c$$

We must show that

$$(A \cap B)^c \subseteq A^c \cup B^c$$

And

$$A^c \cup B^c \subseteq (A \cap B)^c$$

Another exercise to do at home

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Intersection and Union with a Subset

- For any sets A and B, if $A \subseteq B$, then

$$\text{(a) } A \cap B = A$$

and

$$\text{(b) } A \cup B = B$$

Intersection and Union with a Subset Part (a)

- Show that for any sets A and B, if $A \subseteq B$, then
$$A \cap B = A$$

We must show that
$$A \cap B \subseteq A$$
and
$$A \subseteq A \cap B$$

Intersection and Union with a Subset Part (a) – cont.

To show that

$$A \cap B \subseteq A$$

We already know that it is true by definition of the intersection property.

Intersection and Union with a Subset Part (a) – cont.

To show that

$$A \subseteq A \cap B$$

We suppose that $x \in A$, and we want to show that $x \in A \cap B$.

Since $A \subseteq B$, then, by definition of subsets, $x \in B$.

Since $x \in A \wedge x \in B$, then, by definition of intersection, $x \in A \cap B$.

Therefore,

$$A \subseteq A \cap B$$

Intersection and Union with a Subset Part (a) – cont.

Since we showed that

$$A \cap B \subseteq A$$

and

$$A \subseteq A \cap B$$

then we can conclude that

$$A \cap B = A$$

Intersection and Union with a Subset

- For any sets A and B, if $A \subseteq B$, then

$$(a)\ A \cap B = A$$

and

$$(b)\ A \cup B = B$$

Show that part (b) is true
Another exercise to do at home

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The Empty Set is a Subset of Every Set

- If E is a set with no elements, and A is any set, then $E \subseteq A$.

- This is easier proved by contradiction.

Let's suppose that there exists a set E with no elements, and a set A, and that $E \nsubseteq A$.

By definition of subsets, this means that there is an element of E that is not an element of A.

However, there is no such element since E has no elements because it's an empty set.

We have a contradiction, making our assumption false, therefore $E \subseteq A$.

Uniqueness of the Empty Set

- There is only one set with no elements.

Suppose E_1 and E_2 are two sets with no elements.

We have already proved that $E_1 \subseteq E_2$ since E_1 is an empty set.

We have also already proved that $E_2 \subseteq E_1$ since E_2 is an empty set.

And, by definition of equality, since $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$, then $E_1 = E_2$.

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Proof for a Conditional Statement

- Show that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Suppose not, meaning suppose there is an element x in $A \cap C$.

By definition of intersection, $x \in A$ and $x \in C$.

Since $A \subseteq B$ and $x \in A$, then $x \in B$ by definition of subsets.

Also, since $B \subseteq C^c$ and $x \in B$, then $x \in C^c$ by definition of subsets.

Since $x \in C^c$, it follows, by definition of complements, that $x \notin C$, which is a contradiction to our assumption.

Hence, our supposition that there is an element x in $A \cap C$ is wrong, and $A \cap C = \emptyset$.

ADVICE

Prove all the subset relations and set identities in this section.