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FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

# COMP233
# Discrete Mathematics

# CHAPTER 6

Set Theory

# Set Theory

6.1. Basic Definitions of Set Theory

6.2 Properties of Sets

6.3 Algebraic Proofs

6.4 Boolean Algebras

# Set Theory

# Algebraic Proofs

- Disproving Set Properties
- Problem-Solving Strategy
- The Number of Subsets of a Set
- Algebraic Proofs of Set Identities
  - *Deriving a Set Difference Property*
  - *Deriving a Set Identity Using Properties of $\emptyset$*

# Algebraic Proofs

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# Disproving Set Properties

- In the previous sections, we saw proofs of set properties that were true.
- Occasionally, a proposed set property is false.
- So, how do we prove a set property?

# Disproving Set Properties – cont.

- To show that a universal statement is false, it suffices to find one example (called a counterexample) for which it is false.

- For example, if we have the following property that we want to disprove:

  For all sets, $A$, $B$, and $C$, $(A - B) \cup (B - C) = A - C$

- We need to think of what elements can we find in each of the set, and what conditions needs to be satisfied so that the equality above will not hold.

# Disproving Set Properties – cont.

- To find an element that is in $(A - B) \cup (B - C)$ and not in $A - C$:
  - *$A - C$ contains all the elements that are in $A$ and not $C$.*
  - *$(A - B) \cup (B - C)$ contains all the elements that are in $A$ but not in $B$, together with all the element that re in $B$ but not in $C$.*
  - *An element can be in $(A - B) \cup (B - C)$ if it is in $A$ and not in $B$.*
  - *If that element is also in $C$ then it will not be in $A - C$.*

$\rightarrow$ *if an element is in $A$ and $C$ and not in $B$, this property will be false.*

# Disproving Set Properties – cont.

- For all sets, $A$, $B$, and $C$, $(A - B) \cup (B - C) = A - C$

Counterexample:

Let $A = \{1, 2\}$, $B = \{2\}$, $C = \{1\}$

$$A - B = \{1\}$$
$$B - C = \{2\}$$
$$(A - B) \cup (B - C) = \{1, 2\}$$
$$A - C = \{2\}$$

Since $1 \in (A - B) \cup (B - C)$ but $1 \notin A - C$, then $(A - B) \cup (B - C) \neq A - C$

# Disproving Set Properties – cont.

- To find an element that is in $(A - B) \cup (B - C)$ and not in $A - C$:
  - $A - C$ *contains all the elements that are in* $A$ *and not* $C$.
  - $(A - B) \cup (B - C)$ *contains all the elements that are in* $A$ *but not in* $B$*, together with all the element that re in* $B$ *but not in* $C$.
  - *An element can be* $(A - B) \cup (B - C)$ *if it is in B but not in C or A.*
  - *Such an element will be in* $(A - B) \cup (B - C)$ *but it won't be in* $A - C$

→ *if an element is in* $A$ *and* $C$ *and not in* $B$*, this property will be false.*

# Disproving Set Properties – cont.

- For all sets, $A$, $B$, and $C$, $(A - B) \cup (B - C) = A - C$

Counterexample:

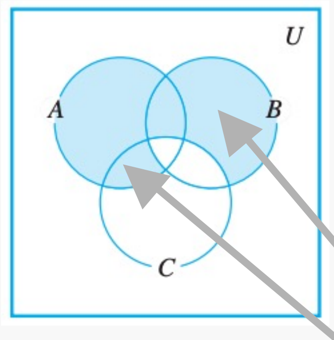
Let $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1\}$

$$A - B = \{2\}$$
$$B - C = \{3\}$$
$$(A - B) \cup (B - C) = \{2, 3\}$$
$$A - C = \{2\}$$

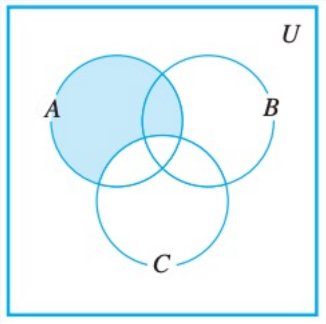
Since $3 \in (A - B) \cup (B - C)$ but $3 \notin A - C$, then $(A - B) \cup (B - C) \neq A - C$

# Disproving Set Properties – cont.

- Another approach to solving this problem is to picture sets $A$, $B$, and $C$ by drawing a Venn diagram:




$(A - B) \cup (B - C)$

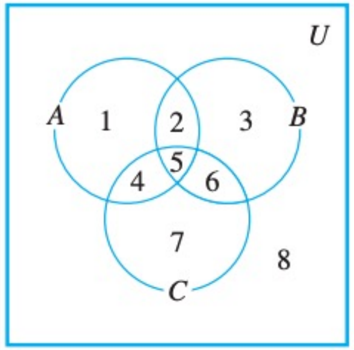



$A - C$

Any element in these shaded areas will not be in $A - C$

# Disproving Set Properties – cont.

- You can take it further and use the diagram construct a concrete counterexample by assigning elements to each area of the diagram, and creating the sets based on those elements:




Using this example, let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 5, 6\}$, $C = \{4, 5, 6, 7\}$
$(A - B) \cup (B - C) = \{1, 4\} \cup \{2, 3\} = \{1, 2, 3, 4\}$
$(A - C) = \{1, 2\}$

$3 \in \{1, 2, 3, 4\}$ but $3 \notin \{1, 2\}$ (and 4 too).

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# Problem-Solving Strategy

- How can you discover whether a given universal statement about sets is true or false?

- The optimistic approach: you start trying to prove the statement by asking yourself:
  - *What do I need to show?*
  - *How do I show it?*
- The pessimistic approach: you start by searching your mind for a set of conditions that must be fulfilled to construct a counterexample.

- Either way, you may be immediately successful, or you may run into difficulty. This is why you need to be ready to switch to the other approach if the one you are are trying does not look promising.

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# The Number of Subsets of a Set

- If a set has $n$ elements, then its power set has $2^n$ elements.

- The proof of this theorem uses mathematical induction:

  For all integers $n \geq 0$, if a set $X$ has $n$ elements, then $\mathcal{P}(X)$ has $2^n$ elements.

Basis step: show that P(0) is true:

if a set has zero elements, its power set will only contain the empty set, which means the size of its power set is 1

since $2^0 = 1$, then the theorem is true for $n = 0$

# The Number of Subsets of a Set – cont.

- Inductive hypothesis:

Let $k$ be any integer with $k \geq 0$, and suppose that any set with $k$ elements has $2^k$ subsets.

We must show that any set with $k + 1$ elements has $2^{k+1}$

Let $X$ be a set with $k + 1$ elements, and pick element $z$ in $X$.

Observe that any subset of $X$ either contains $z$ or not.

Any subset of $X$ that does not contain $z$ is a subset of $X - \{z\}$.

# The Number of Subsets of a Set – cont.

- Any subset $A$ of $X - \{z\}$ can be matched up with a subset $B$, equal to $A \cup \{z\}$, of $X$ that contains $z$.
  - *For example, if $X = \{x, y, z\}$, then*

| Subset A | Subset B |
|---|---|
| $\emptyset$ | $\emptyset \cup \{z\} = \{z\}$ |
| $\{x\}$ | $\{x\} \cup \{z\} = \{x, z\}$ |
| $\{y\}$ | $\{y\} \cup \{z\} = \{y, z\}$ |
| $\{x, y\}$ | $\{x, y\} \cup \{z\} = \{x, y, z\}$ |

- Consequently, there are as many subsets of $X$ that contain $z$ as do not, thus there are twice as many subsets of $X$ than there are subsets of $X - \{z\}$.

# The Number of Subsets of a Set – cont.

- Since $X$ has $k+1$ elements, then $X-\{z\}$ has $k$ elements.

  $\rightarrow$ the number of subsets of $X-\{z\}=2^k$ by our inductive hypothesis.

Therefore,

the number of subsets of $X = 2 \cdot (the\ number\ of\ subsets\ of\ X-\{z\})$

$$= 2 \cdot 2^k = 2^{k+1}$$

- Since we proved the basis step and the inductive step, we conclude that the theorem is true.

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# Algebraic Proofs of Set Identities

- Once we established a number of identities and properties, new properties can be derived from them algebraically.
- To use known properties to derive new ones, you need to use the fact that such properties are universal statements.

  → Like the laws of algebra for real numbers, they apply to a wide variety of different situations.

# Algebraic Proofs of Set Identities – cont.

- For example, one of the distributive laws states that

$$\text{for all sets } A, B, \text{and } C, \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- And this can be a general template that would apply to the following example:

$$\text{for all sets } W, X, Y, Z, \qquad (W \cap X) \cap (Y \cup Z) = ((W \cap X) \cap Y) \cup ((W \cap X) \cap Z)$$

Where $(W \cap X)$ plays the role of $A$, $Y$ plays the role of $B$, and $Z$ plays the role of $C$.

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# Deriving a Set Difference Property

- Use the properties we learned in the previous section to construct an algebraic proof that for all sets $A$, $B$, and $C$:

$$(A \cup B) - C = (A - C) \cup (B - C)$$

| | |
|---|---|
| $(A \cup B) - C = (A \cup B) \cap C^c$ | by the set difference law |
| $= C^c \cap (A \cup B)$ | by the commutative law for $\cap$ |
| $= (C^c \cap A) \cup (C^c \cap B)$ | by the distributive law |
| $= (A \cap C^c) \cup (B \cap C^c)$ | by the commutative law for $\cap$ |
| $= (A - C) \cup (B - C)$ | by the set difference law |

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# Deriving a Set Identity Using Properties of ∅

- Use the properties we learned in the previous section to construct an algebraic proof that for all sets $A$, $B$, and $C$:

$$A - (A \cap B) = A - B$$

| | | |
|---|---|---|
| $A - (A \cap B)$ | $= A \cap (A \cap B)^c$ | by the set difference law |
| | $= A \cap (A^c \cup B^c)$ | by De Morgan's law |
| | $= (A \cap A^c) \cup (A \cap B^c)$ | by the distributive law |
| | $= \emptyset \cup (A \cap B^c)$ | by the complement law |
| | $= (A \cap B^c) \cup \emptyset$ | by the commutative law for ∪ |
| | $= A \cap B^c$ | by the identity law for ∪ |
| | $= A - B$ | by the set difference law |