جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 6

Set Theory

Set Theory

6.1. Basic Definitions of Set Theory

6.2 Properties of Sets

6.3 Algebraic Proofs

6.4 Boolean Algebras

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Boolean Algebras

- Introduction
- Properties of Boolean Algebras
- Proof of Boolean Algebra Properties

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Introduction

- Logical equivalences that we covered in chapter 2 and set properties that we covered in pervious sections of this chapter are very similar.

- If you let
  - $\vee$ *(or) correspond to* $\cup$ *(union),*
  - $\wedge$ *(and) correspond to* $\cap$ *(intersection),*
  - $\boldsymbol{t}$ *(a tautology) correspond to* U *(a universal set),*
  - $\boldsymbol{c}$ *(a contradiction) correspond to* $\emptyset$ *(the empty set),*
  - $\sim$ *(negation) correspond to* ^{c} *(complementation)*

  you can see that the structures are identical.

Introduction – cont.

- For example,

Logical Equivalence	Set Property
$p \vee q \equiv q \vee p$	$A \cup B = B \cup A$
$p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$	$A \cup (B \cup C) \equiv A \cup (B \cup C)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$
$p \vee \boldsymbol{c} \equiv p$	$A \cup \emptyset = A$
$p \vee \sim p \equiv \boldsymbol{t}$	$A \cup A^c = U$
$\sim (\sim p) \equiv p$	$(A^c)^c = A$
$\sim (p \vee q) \equiv \sim p \wedge \sim q$	$(A \cup B)^c = A^c \cap B^c$
$p \vee (p \wedge q) \equiv p$	$A \cup (A \cap B) \equiv A$
$\sim \boldsymbol{t} \equiv \boldsymbol{c}$	$U^c = \emptyset$

- The full table can be found in the book.

Boolean Algebras

- Introduction
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Boolean Algebra

- A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following properties hold:
  - *Commutative Laws*
  - *Associative Laws*
  - *Distributive Laws*
  - *Identity Laws*
  - *Complement Laws*

Boolean Algebra – cont.

- Commutative Laws: For all a and b in B:
  - $a)$ $a + b = b + a$
  - $b)$ $a \cdot b = b \cdot a$

- Associative Laws: For all a, b, and c in B:
  - $a)$ $(a + b) + c = a + (b + c)$
  - $b)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Boolean Algebra – cont.

- Distributive Laws: For all a, b, and c in B:
  - *a)* $a + (b \cdot c) = (a + b) \cdot (a + c)$
  - *b)* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

- Identity Laws: There exist distinct elements 0 and 1 in B such that for all a in B:
  - *a)* $a + 0 = a$
  - *b)* $a \cdot 1 = a$

Boolean Algebra – cont.

- Complement Laws: For each a in B, there exists an element in B, denoted $\bar{a}$ and called the **complement** or **negation** of a, such that:
  a) $a + \bar{a} = 1$
  b) $a \cdot \bar{a} = 0$

Properties of Boolean Algebra

Let B be any Boolean algebra.

- *Uniqueness of the Complement Law:* For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \bar{a}$.

- *Uniqueness of* 0 *and* 1: If there exists x in B such that $a + x = a$ for all a in B, then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for all a in B, then $y = 1$.

- *Idempotent Law:* For all $a \in B$,
  - *a)* $a + a = a$
  - *b)* $a \cdot a = a$

Properties of Boolean Algebra – cont.

Let B be any Boolean algebra.

- *Double Complement Law:* For all $a \in B$, $\overline{(\bar{a})} = a$.

- *Universal Bound Law:* For all $a \in B$,
  a) $a + 1 = 1$
  b) $a \cdot 0 = 0$

- *De Morgan's Law:* For all $a \in B$,
  a) $\overline{a + b} = \bar{a} \cdot \bar{b}$
  b) $\overline{a \cdot b} = \bar{a} + \bar{b}$

Properties of Boolean Algebra – cont.

Let B be any Boolean algebra.

- *Absorption Law:* For all $a,\ b\ \in\ B$,
  - *a)* $(a + b) \cdot a = a$
  - *b)* $(a \cdot b) + a = a$

- *Complements of 0 and 1:*
  - *a)* $\overline{0} = 1$
  - *b)* $\overline{1} = 0$

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Proving the Uniqueness of the Complement Law

- Suppose a and x are particular, but arbitrarily chosen, elements of B that satisfy the following hypothesis: $a + x = 1$ and $a \cdot x = 0$. Then:

$x = x \cdot 1$	*because 1 is an identity for ·*
$= x \cdot (a + \bar{a})$	*by the complement law of +*
$= x \cdot a + x \cdot \bar{a}$	*by the distributive law of · over +*
$= a \cdot x + x \cdot \bar{a}$	*by the commutative law of ·*
$= 0 + x \cdot \bar{a}$	*by hypothesis*
$= a \cdot \bar{a} + x \cdot \bar{a}$	*by the complement law of ·*
$= (\bar{a} \cdot a) + (\bar{a} \cdot x)$	*by the commutative law ·*
$= \bar{a} \cdot (a + x)$	*by the distributive law of · over +*
$= \bar{a} \cdot 1$	*by hypothesi*
$= \bar{a}$	*because 1 is an identity for ·*