جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

Comp233
Discrete Mathematics

CHAPTER 7

Functions

Functions

- Introduction to Functions
- One-to-one, onto, and inverse Functions

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Introduction to Functions

- What is a Function?
- Function Equality
- Examples of Functions
- Boolean Functions
- Well-Defined Functions

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Motivation

- Whenever you study mathematics or computer science, you will find functions everywhere.
- So far, we have seen functions in the form of truth tables (Boolean functions), sequences (functions defined on a set of integers), mod and div (functions defined on Cartesian product of integers), and floor and ceiling (functions from $\mathbb{R}$ to $\mathbb{Z}$).
- In this chapter we will cover functions that focus on discrete sets, such as finite sets and sets of integers.

Definition

In ordinary language, the word function indicates the dependence of one varying quantity on another.

Saying that your grade in the course will be a function of your performance on the exams means that there is some rule that translates/converts exams scores into final grades.

Definition – cont.

- A function f from a set X to a set Y is a relation between elements of X, called inputs, and elements of Y, called outputs, with the property that each input is related to one and only one output.

  The notation $f : X \to Y$ means that f is a function from X to Y.

  X is called the domain of f, and Y is called the co-domain of f.

- Given an input element x in X, there is a unique output element y in Y that is related to x by f. We say that "f sends x to y" and write it as $x \xrightarrow{f} y$ or $f : x \to y$.

Definition – cont.

- The unique element y to which f send x is denoted $f(x)$ and it called:
  - *f of x*
  - *The output of f for the input x*
  - *The value of f at x*
  - *The image of x under f*

- The set of all value of f taken together is called the range of f or the image of X under f. Symbolically,

  range of f = image of X under f = $\{y \in Y \mid y = f(x), for\ some\ x\ in\ X\}$.

Definition – cont.

- Given an element y in Y. There may exist elements in X with y as their image. If $f(x) = y$, then x is called a preimage of y or an inverse image of y.

- The set of all inverse images of y is called the inverse image of y. Symbolically,

  inverse image of y = $\{x \in X \mid f(x) = y\}$

Co-domain vs Range

- The co-domain and range are both on the output side of a function but are subtly different.
- The co-domain is the set of values that could possibly come out. The co-domain is part of the definition of the function.
- The range is the set of values that actually do come out.

- The range is a subset of the co-domain.

- Why both? Well, sometimes we don't know the *exact* range (because the function may be complicated or not fully known), but we know the set it *lies in* (such as integers or real numbers). So, we define the co-domain and continue.

Co-domain vs Range – Example

We can define a function

$$f(x) = 2x$$

with a domain of integers, and co-domain of integers, because the product of two integers is an integer.

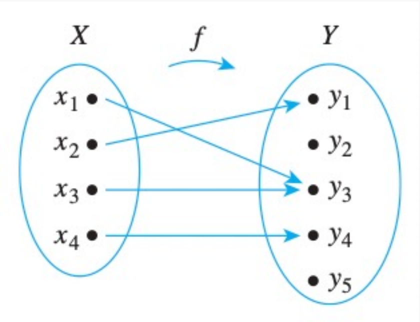
But by thinking about it we can see that the range (actual output values) is just the even integers.

So, the co-domain is integers, but the range is even integers.

Arrow Diagrams

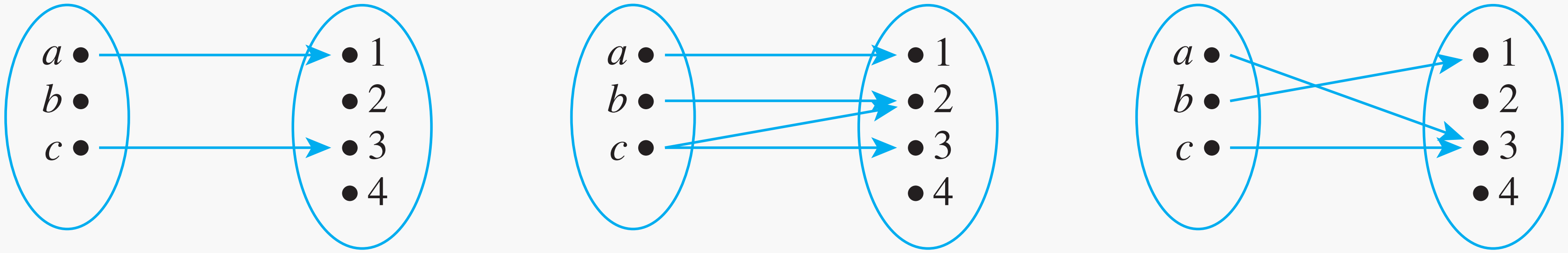
- If X and Y are finite sets, you can define a function f from X to Y by making a list of elements in X and a list of elements in Y and drawing an arrow from each element in X to the corresponding element in Y.

- The definition of function implies that the arrow diagram for a function f has the following two properties:
  - *Every element of X has an arrow coming out of it.*
  - *No element of X has two arrows coming out of it that point to two different elements of Y.*



Example

- Which of the arrow diagrams below define a function from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?



(a) (b) (c)

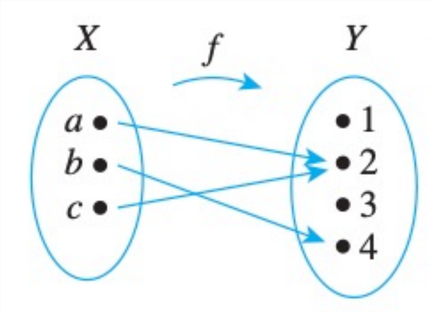
Not a function, because because there is an element of X, namely b, that is not sent to any element of Y

Not a function because there is an element of X, namely c, that is not sent to a unique element of Y.

A function, because each element of X is sent to a unique element of Y.

Example

- Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram below:



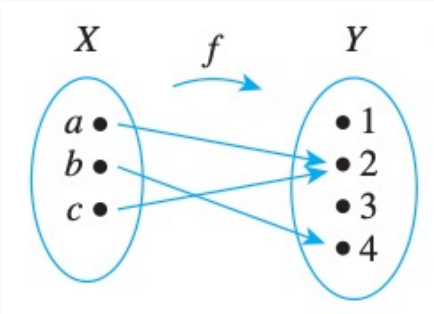
- Write the domain and co-domain of f.

Domain of $f = \{a, b, c\}$

Co-domain of $f = \{1, 2, 3, 4\}$

Example

- Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram below:

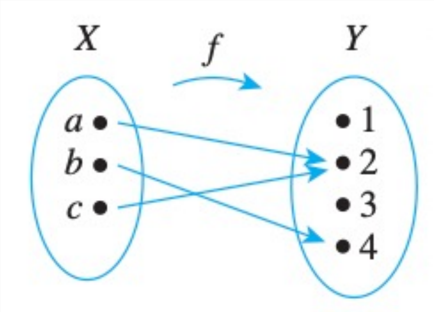


- Find $f(a), f(b)$, and $f(c)$.

$$f(a) = 2$$
$$f(b) = 4$$
$$f(c) = 2$$

Example

- Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram below:

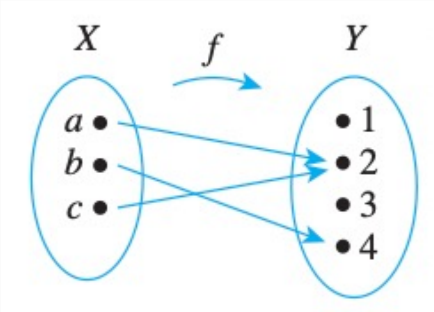


- What is the range of f?

The range of $f = \{2, 4\}$

Example

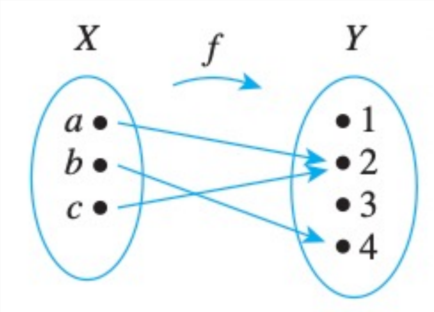
- Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram below:



- Is c an inverse image of 2? Yes
- Is b an inverse image of 3? No

Example

- Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram below:



- Find the inverse images of 2, 4, and 1.

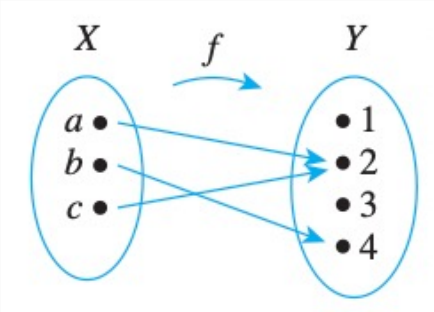
The inverse image of 2 = $\{a, c\}$

The inverse image of 4 = $\{b\}$

The inverse image of 1 = ∅

Example

- Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram below:



- Represent f as a set of ordered pairs.

$$\{(a, 2), (b, 4), (c, 2)\}$$

Introduction to Functions

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Function Equality – Definition

- Suppose f and g are functions from X to Y. Then f equals g, written $f = g$, if, and only if,

$$f(x) = g(x) \quad for\ all\ x \in X$$

Function Equality – Proof Using Tables

- Let $X = \{0,1,2,\}$, and define the following functions $f: X \rightarrow X$, and $g: X \rightarrow X$ as follows:

$$f(x) = (x^2 + x + 1)\ mod\ 3$$
$$g(x) = (x + 2)^2\ mod\ 3$$

Does $f = g$?

$\boldsymbol{x}$	$(x^2 + x + 1)$	$(x^2 + x + 1)\ mod\ 3$	$(x + 2)^2$	$(x + 2)^2\ mod\ 3$	$\boldsymbol{f(x)}$	$\boldsymbol{g(x)}$
0	1	1 mod 3 = 1	4	4 mod 3 = 1	1	1
1	3	3 mod 3 = 0	9	9 mod 3 = 0	0	0
2	7	7 mod 3 = 1	16	16 mod 3 = 1	1	1

Function Equality – Proof

- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $\mathrm{g}: \mathbb{R} \rightarrow \mathbb{R}$ be functions. Define new functions $f + g: \mathbb{R} \rightarrow \mathbb{R}$ and $\mathrm{g} + f: \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$(f + g)(x) = f(x) + g(x) \ \ for\ all \ \ x \in \mathbb{R}$$
$$(g + f)(x) = g(x) + f(x) \ \ for\ all \ \ x \in \mathbb{R}$$

  Does $f + g = g + f$?

$(f + g)(x) = f(x) + g(x)$ by definition of $f + g$

$= g(x) + f(x)$ by the commutative law for addition of real numbers

$= (g + f)(x)$ by definition of $g + f$

Hence $f + g = g + f$.

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The Identity Function on a Set

- Given a set X, define a function i_x from X to X by
$$i_x(x) = x \ \ for\ all\ x\ in\ X$$

The function i_x is called the identity function on X because it sends each element of X to the element that is identical to it.

Example:

Let X be any set and suppose that a_{ij}^{k} and $\phi(z)$ are elements of X. Find $i_x\left(a_{ij}^{k}\right)$ and $i_x(\phi(z))$.

Since i_x is an identity function, the input comes out unchanged:
$$i_x\left(a_{ij}^{k}\right) = a_{ij}^{k}$$
$$i_x(\phi(z)) = \phi(z)$$

Sequences

- The formal definition of sequences specifies that a sequence is a function defined on the set of integers that are greater than or equal to a particular integer.

- For example, the sequence denoted

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots, \frac{(-1)^n}{n+1}$$

can be thought of as the function f from the nonnegative integers to the real numbers that associates $0 \to 1$, $1 \to -\frac{1}{2}$, $2 \to \frac{1}{3}$, $3 \to -\frac{1}{4}$, $4 \to \frac{1}{5}$, and, in general, $n \to \frac{(-1)^n}{n+1}$.

- In other words, $f: \mathbb{Z}^{nonneg} \to \mathbb{R}$ is the function defined as follows:

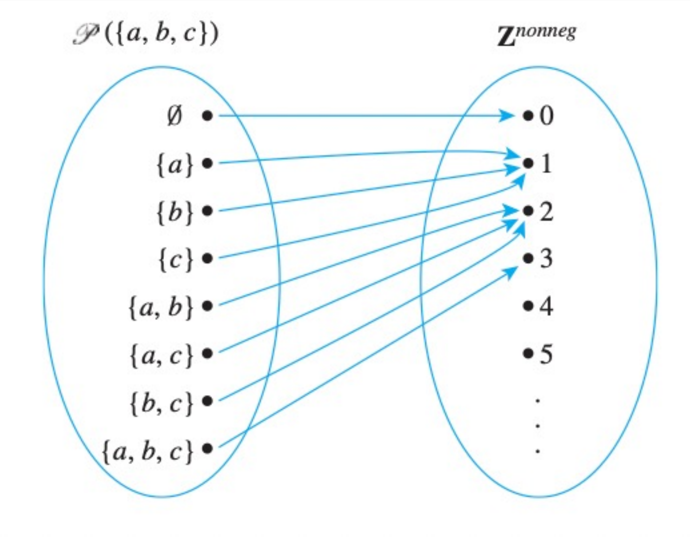
Send each integer $n \geq 0$ to $f(n) = \frac{(-1)^n}{n+1}$

A Function Defined on a Power Set

- Recall that $\mathcal{P}(A)$ denotes the set of all subsets of the set A. Define a function $f\colon \mathcal{P}(\{a,b,c\} \longrightarrow \mathbb{Z}^{nonneg}$ as follows:

  For each $X \in \mathcal{P}(\{a,b,c\}$, $f(x) =$ the number of elements in X.

  Draw an arrow diagram for f.



Functions Defined on a Cartesian Product

- Define the functions $m: \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}$ and $n: \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$ as follows: for all ordered pairs (a, b) of integers:

$$m(a, b) = ab$$
$$n(a, b) = (-a, b)$$

Find the following:

$m(-1, -1)$ $m\left(\frac{1}{2}, \frac{1}{2}\right)$ $m(\sqrt{2}, \sqrt{2})$

$n(2,5)$ $n(-2, 5)$ $n(3, -4)$

Functions Defined on a Cartesian Product

- Define the functions $m: \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}$ and $n: \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$ as follows: for all ordered pairs (a, b) of integers:

$$m(a, b) = ab$$
$$n(a, b) = (-a, b)$$

Find the following:

$m(-1, -1) = 1$ $\qquad$ $m\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}$ $\qquad$ $m\left(\sqrt{2}, \sqrt{2}\right) = 2$

$n(2,5) = (-2,5)$ $\qquad$ $n(-2, 5) = (2,5)$ $\qquad$ $n(3, -4) = (-3, -4)$

String Functions

- Let S be the set of all strings of a’s and b’s, and let ϵ represent the null string (the string with no characters.
- Define a function $g: S \to \mathbb{Z}$ as follows: for each string $s \in S$,
$$g(s) = the\ number\ of\ a's\ in\ s.$$

Find the following:

$g(\epsilon) =$

$g(bb) =$

$g(ababb) =$

$g(bbbaa) =$

String Functions

- Let S be the set of all strings of a's and b's, and let ϵ represent the null string (the string with no characters.
- Define a function $g: S \rightarrow \mathbb{Z}$ as follows: for each string $s \in S$,
$$g(s) = the\ number\ of\ a's\ in\ s.$$

Find the following:

$g(\epsilon) = 0$

$g(bb) = 0$

$g(ababb) = 2$

$g(bbbaa) = 2$

The Logarithmic Function with Base b

• Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x,** written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

$\log_3 9 =$

$\log_2 \frac{1}{2} =$

$\log_{10} 1 =$

$\log_2 2^m =$

The Logarithmic Function with Base b

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$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

$\log_3 9 = 2$ because $3^2 = 9$

$\log_2 \frac{1}{2} = -1$ because $2^{-1} = \frac{1}{2}$

$\log_{10} 1 = 0$ because $10^0 = 1$

$\log_2 2^m = m$ because $2^m = 2^m$

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Boolean Functions

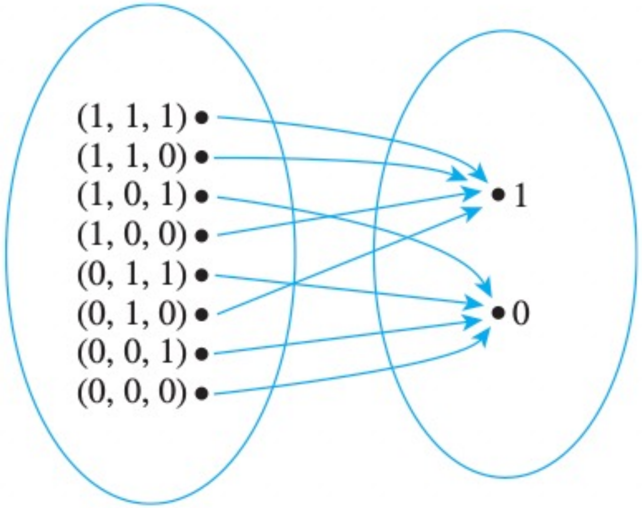
- **Definition**

An (***n*-place**) **Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \to \{0, 1\}$.

Boolean Functions – cont.

Input			Output
P	*Q*	*R*	*S*
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

(a)



(b)

Figure 7.1.2 Two Representations of a Boolean Function

Boolean Functions – example

- Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to $\{1,0\}$ as follows: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)\ mod\ 2$$

Describe f using an input/output table.

Input			Output
x_1	x_2	x_3	$(x_1 + x_2 + x_3)\ mod\ 2$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

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Checking Whether a Function Is Well-Defined – Definition

- A function is not well-defined if it fails to satisfy at least one of the requirements of being a function

- This means that the function either:
  - *For at least one value of* x *there is no value of* y *that satisfies the function definition.*
  - *For at least one value of* x *there are multiple possible values of* y*.*

- You need to find one single value of x that fails to satisfy one of the conditions to call the function not well-defined.

Checking Whether a Function Is Well Defined – Example

- Let's define a function $f : \mathbf{R} \to \mathbf{R}$ by specifying that for all real numbers x, $f(x)$ is the real number y such that $x^2 + y^2 = 1$.

- There are two reasons why this function is not well defined:

  For almost all values of x either:

  (1) there is no y that satisfies the given equation

  Consider when $x = 2$

  (2) there are two different values of y that satisfy the equation

  Consider when $x = 0$

Checking Whether a Function Is Well Defined – Example

- Let's define the following function $f: \mathbb{Q} \rightarrow \mathbb{Z}$ that's defined by the following formula:

$$f\left(\frac{m}{n}\right) = m, for\ all\ integers\ m\ and\ n, with\ n \neq 0$$

  Is f a well-defined function?

If we represent 0.5 as both $\frac{1}{2}$ and $\frac{3}{6}$, the function will give us

$$f\left(\frac{1}{2}\right) = 1$$

$$f\left(\frac{3}{6}\right) = 3$$

And since $1 \neq 3$ then the function is not producing unique images for the same value of x.