جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 7

Functions

Functions

- Introduction to Functions
- One-to-one, onto, and inverse Functions

Outline

- One-to-one Functions
  - *Application: Hash Functions*
- Onto Functions
  - *Application: Exponential and Logarithmic Functions*
- One-to-one Correspondence Functions
- Inverse Functions

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One-to-One Functions – Definition

- Let f be a function from a set X to a set Y. f is one-to-one (or injective) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

Or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).$$

Symbolically,

$$f: X \to Y \text{ is one-to-one} \leftrightarrow \forall x_1, x_2 \in X, \text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Which also means

$$f: X \to Y \text{ is not one-to-one} \leftrightarrow \exists x_1, x_2 \in X, \text{with } f(x_1) = f(x_2) \text{ and } x_1 \neq x_2.$$

One-to-One Functions in Arrow Diagrams

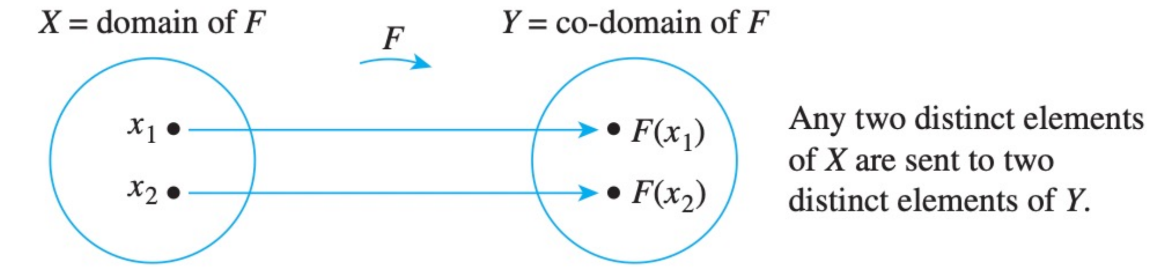


Figure 7.2.1(a) A One-to-One Function Separates Points

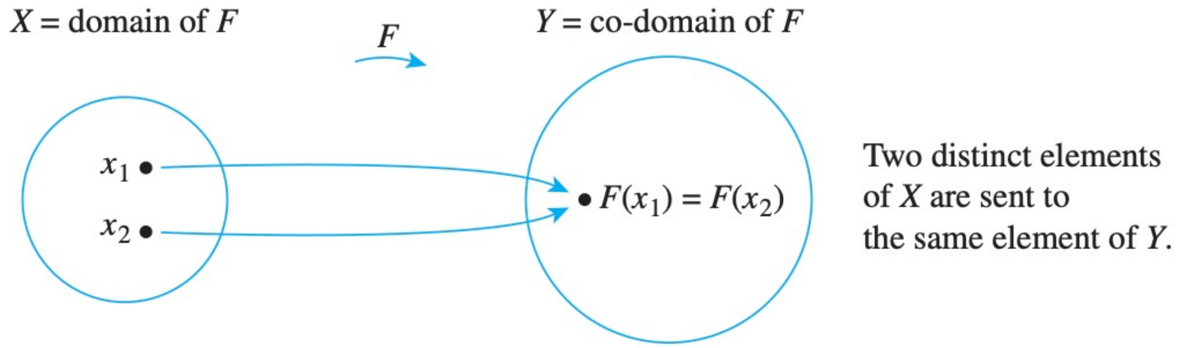
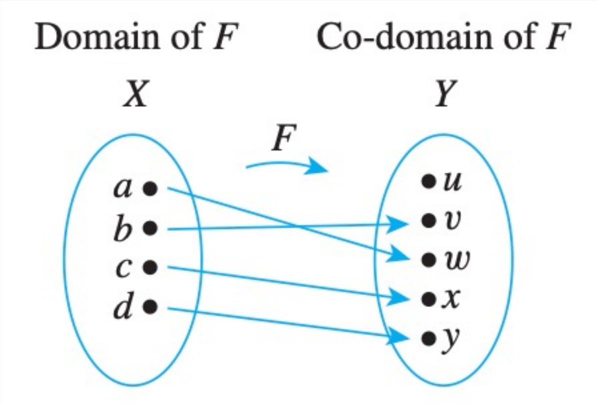


Figure 7.2.1(b) A Function That Is Not One-to-One Collapses Points Together

One-to-One Functions – examples

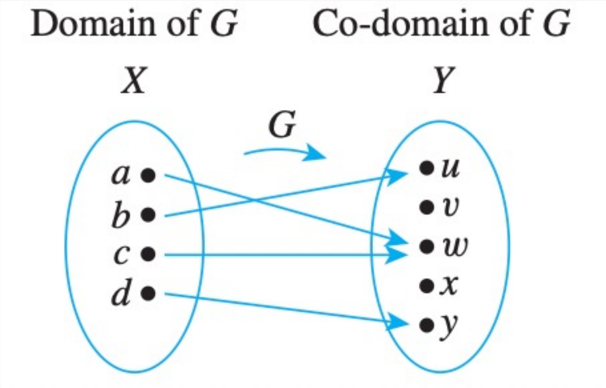
- Is the following function one-to-one?



- Yes, because no two different elements in X are sent to the same element in Y.

One-to-One Functions – examples

- Is the following function one-to-one?



- No, because the elements a and c in X are sent to the same element w in Y. So, $G(a) = G(c)$ but $a \neq c$.

One-to-One Functions – examples

- Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $h: X \rightarrow Y$ as follows:

$$h(1) = c,\ h(2) = a,\ h(3) = d$$

And define $k: X \rightarrow Y$ as follows:

$$k(1) = d,\ k(2) = b,\ k(3) = d$$

- Is either of these functions one-to-one?

➢ h is one-to-one because for the three distinct elements $1, 2,$ and 3, $h(1) \neq h(2)$, $h(2) \neq h(3)$, and $h(3) \neq h(1)$.

➢ k is not one-to-one because $k(1) = k(3)$ but $1 \neq 3$.

Proving that Functions are One-to-One

- Previous examples show proving and disproving that functions are one-to-one on finite sets. But if we have infinite sets, we need to use direct proof.

- To prove that a function is one-to-one, you need to use direct proof to show that if two elements have the same image, then both elements are equal:
  - *Suppose* x_1 *and* x_2 *are elements of* X *such that* $f(x_1) = f(x_2)$
  - *Show that* $x_1 = x_2$

- To show that a function is not one-to-one, you need a counterexample:
  - *Find elements* x_1 *and* x_2 *in* X *so that* $f(x_1) = f(x_2)$ *and* $x_1 \neq x_2$

Proof on Infinite Sets

- Define $f: \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$.
  Is f one-to-one? Prove or give a counterexample.

- Suppose x_1 and x_2 are elements of X such that $4x_1 - 1 = 4x_2 - 1$

  $4x_1 - 1 = 4x_2 - 1$ *by definition of f*

  $4x_1 = 4x_2$ *by adding one to both sides of the function*

  $x_1 = x_2$ *by dividing both sides of the function by 4*

Which is what we needed to show.

Proof on Infinite Sets

- Define $f: \mathbb{Z} \rightarrow \mathbb{Z}$ by the rule $f(x) = x^2$ for all $x \in \mathbb{Z}$.
  Is f one-to-one? Prove or give a counterexample.

- Let $x_1 = 2$ and $x_2 = -2$, then by definition of f:

$$f(x_1) = f(2) = 2^2 = 4$$

and

$$f(x_2) = f(-2) = -2^2 = 4$$

Hence, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$, and so f is not one-to-one.

Outline

- One-to-one Functions
  - *Application: Hash Functions*
- Onto Functions
  - *Application: Exponential and Logarithmic Functions*
- One-to-one Correspondence Functions
- Inverse Functions

Hash Functions

- Hash functions are functions defined from larger to smaller sets of integers, frequently using the mod function.
- It can be used to map data of arbitrary sized to data of fixed-size values.

- Hash functions and their associated hash tables are used in data storage and retrieval applications to access data in a small and nearly constant time per retrieval.
- They require an amount of storage space only fractionally greater than the total space required for the data or records themselves.

Hash Functions – cont.

- Suppose there are no more than seven student records. Define a function $Hash$ from the set of all ID numbers to the set $\{0, 1, 2, 3, 4, 5, 6\}$ as follows:

$$Hash(n) = n \ mod \ 7 \text{ for all ID numbers } n.$$

- For example, if we have the ID number $328 - 34 - 3419$:
- Since $\frac{328343419}{7} = 46906202.71 \ldots$, then

$$Hash(328 - 34 - 3419) = 328343419 - (7 \cdot 46906202) = 5$$

0	356-63-3102
1	
2	513-40-8716
3	223-79-9061
4	
5	328-34-3419
6	

Hash Functions – cont.

- The problem with this approach is that $Hash$ may not be one-to one; $Hash$ might assign the same position in the table to records with different ID numbers.
- Such an assignment is called a ***collision***.
- When collisions occur, various collision resolution methods are used.

- One of the simplest is the following: if, when the record with ID number n is to be placed, position $Hash(n)$ is already occupied, start from that position and search downward to place the record in the first empty position that occurs, going back up to the beginning of the table if necessary.

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Onto Functions – Definition

- Let f be a function from a set X to a set Y. f is onto (or surjective) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = f(x)$.

Symbolically:

$$f: X \to Y \text{ is onto} \leftrightarrow \forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

Which also means:

$$f: X \to Y \text{ is not onto} \leftrightarrow \exists y \in Y \text{ such that } \forall x \in X, f(x) \neq y.$$

Onto Functions in Arrow Diagrams

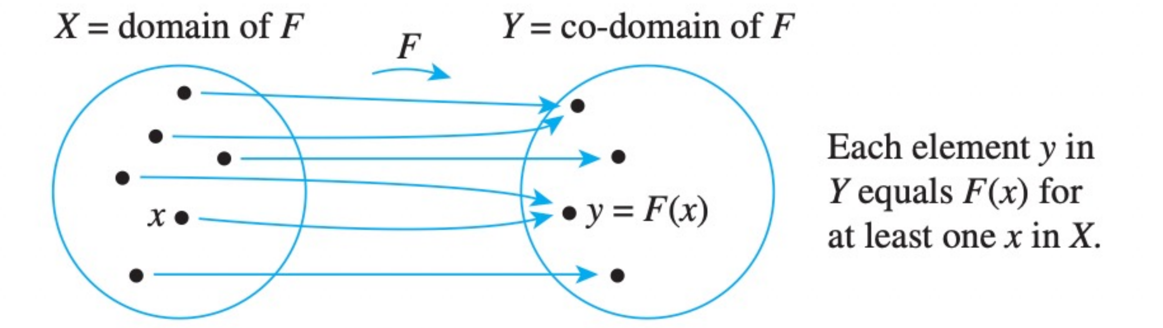


Figure 7.2.3(a) A Function That Is Onto

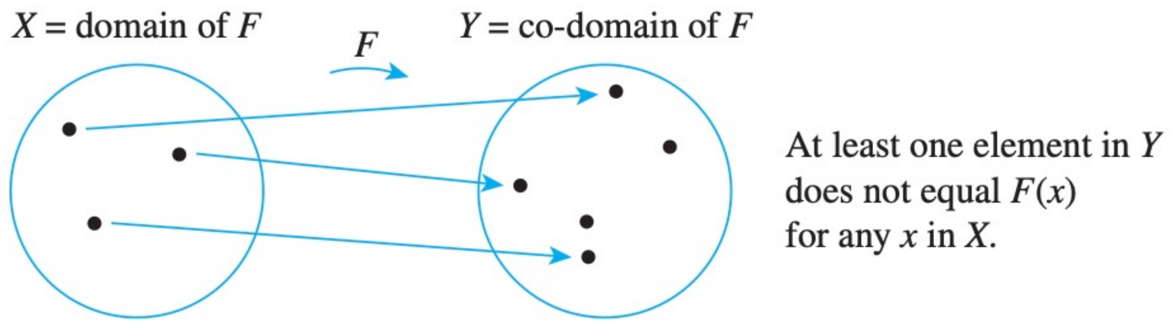
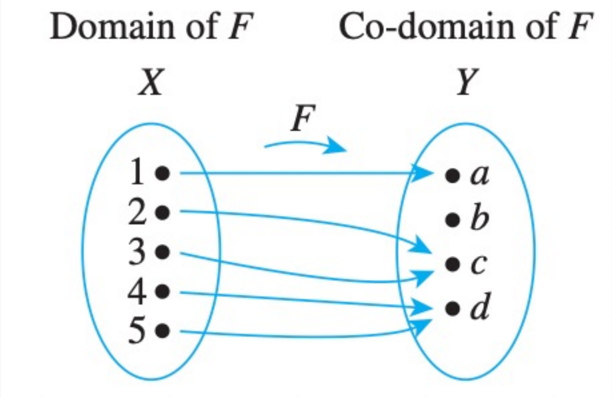


Figure 7.2.3(b) A Function That Is Not Onto

Onto Functions – examples

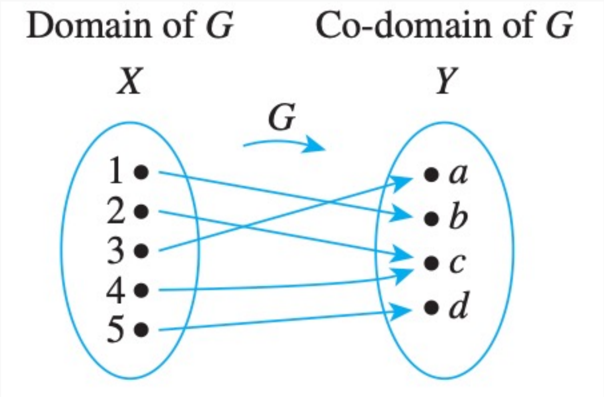
- Is the following function onto?



- No, because $b \neq F(x)$ for any x in X.

Onto Functions – examples

- Is the following function onto?



- Yes, because each element of Y equals $G(x)$ for some x in X.

Onto Functions – examples

- Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $h: X \to Y$ as follows:

$$h(1) = c,\ h(2) = a,\ h(3) = c,\ h(4) = b$$

  And define $k: X \to Y$ as follows:

$$k(1) = c,\ k(2) = b,\ k(3) = b,\ k(4) = c$$

- Is either of these functions onto?

➢ h is onto because each of the three element of the co-domain of h is the image of some element in the domain of h.

➢ k is not onto because $a \neq k(x)$ for any x in $\{1, 2, 3, 4\}$.

Proving that Functions are Onto

- To prove that a function is onto, you will ordinarily use the method of generalizing from the generic particular:
  - *Suppose y is any element of Y*
  - *Show that there is an element of X with $f(x) = y$*

- To show that a function is not onto, you need a counterexample:
  - *Find an elements y of Y such that $y \neq f(x)$ for any x in X.*

Proof on Infinite Sets

- Define $f: \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$.
  Is f onto? Prove or give a counterexample.

- Let $y \in \mathbb{R}$, and let $x = \frac{(y+1)}{4}$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers.

$$f(x) = f\left(\frac{y+1}{4}\right) \quad \textit{by substitution}$$

$$= 4 \cdot \left(\frac{y+1}{4}\right) - 1 \quad \textit{by definition of } f$$

$$= (y+1) - 1 = y \quad \textit{by basic algebra}$$

which is what was to be shown.

Proof on Infinite Sets

- Define $f: \mathbb{Z} \to \mathbb{Z}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{Z}$.
  Is f onto? Prove or give a counterexample.

- The co-domain of f is $\mathbb{Z}$ and $0 \in \mathbb{Z}$. But $f(x) \neq 0$ for any integer x. For if $f(x) = 0$, then:

$4x - 1 = 0$	*by definition of f*
$4x = 1$	*by adding 1 to both sides*
$x = \frac{1}{4}$	*by dividing both sides by 4*

But $\frac{1}{4}$ is not an integer. Hence, there is no integer x for which $f(x) = 0$, and so f is not onto.

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Exponential and Logarithmic Functions

- The exponential function with base b, denoted exp_b, is the function from $\mathbb{R}$ to $\mathbb{R}^+$ defined as follows:
- For all real numbers x,

$$exp_b(x) = b^x$$

where $b^0 = 1$ and $b^{-x} = \frac{1}{b^x}$

Laws of Exponents

- If b and c are any positive real numbers, and u and v are any real numbers, the following laws of exponents hold true:

$$b^u b^v = b^{u+v}$$

$$(b^u)^v = b^{uv}$$

$$\frac{b^u}{b^v} = b^{u-v}$$

$$(bc)^u = b^u c^u$$

Exponential and Logarithmic Functions

- It can be shown using calculus that both the exponential and logarithmic functions are one-to-one and onto.
- Therefore, by definition of one-to-one, the following properties hold true:

For any positive real number b with $b \neq 1$,

if $b^u = b^v$ then $u = v$ for all real numbers u and v,

and

if $\log_b u = \log_b v$ then $u = v$ for all positive real numbers u and v.

Properties of Logarithms

- For any positive real numbers b, c and x with $b \neq 1$ and $c \neq 1$:

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b(x^a) = a \log_b x$$

$$\log_c x = \frac{\log_b x}{\log_b c}$$

Properties of Logarithms – Proof

- Show that for any positive real numbers b, c and x with $b \neq 1$ and $c \neq 1$:

$$\log_c x = \frac{\log_b x}{\log_b c}$$

Suppose positive real numbers b, c, and x are given. Let

(1) $u = \log_b c$ (2)$v = \log_c x$ (3)$w = \log_b x$

Then, by definition of logarithm,

(1') $c = b^u$ (2') $x = c^v$ (3') $x = b^w$

Properties of Logarithms – Proof – cont.

- Substituting (1’) into (2’) and using one of the laws of exponents gives
$$x = c^v = (b^u)^v = b^{uv}$$
- But by (3), $x = b^w$ also. Hence
$$b^{uv} = b^w$$
- And so by the one-to-oneness of the exponential function,
$$uv = w$$
- Substituting from (1), (2), and (3) gives that
$$(\log_b c)(\log_c x) = \log_b x$$
- And dividing both sides by $\log_b c$ (which is nonzero because $c \neq 1$) results in
$$\log_c x = \frac{\log_b x}{\log_b c}$$

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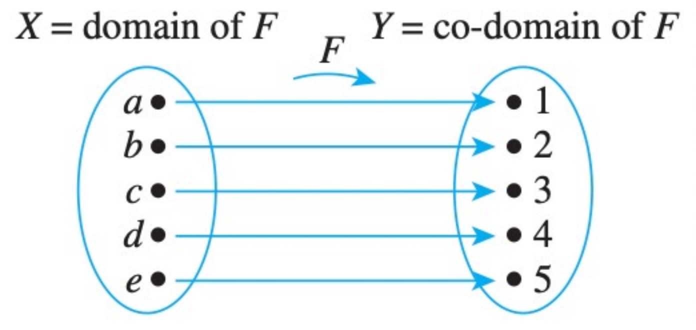
One-to-one Correspondence Functions - Definition

- A one-to-one correspondence function (or bijection) from a set X to a set Y is a function $f: X \to Y$ that is both one-to-one and onto.



One-to-one Correspondence Functions - Example

- Let T be the set of all finite strings of x's and y's. Define $f: T \to T$ by the rule:

  For all strings $s \in T$,

  $f(s) =$ the string obtained by writing the characters of s in reverse order.

- Is f a one-to-one correspondence function from T to itself?
  - *We need to show that f is one-to-one*
  - *And we need to show that f is onto*

One-to-one Correspondence Functions – Example – cont.

- To show that f is one-to-one, we must show that for some strings s_1 and s_2 in T, if we suppose that $f(s_1) = f(s_2)$, thens_1 and s_2.

  $f(s_1) = f(s_2)$ means that the string obtained by writing the characters of s_1 in reverse order equals the string obtained by writing the characters of s_2 in reverse order.

  But, if s_1 and s_2 are equal when written in reverse order, then they must be equal to start with.

  In other words, s_1 and s_2, which was to be shown.

One-to-one Correspondence Functions – Example – cont.

- To show f is onto, we need to show that for any string t in T, there is a string s in T that $f(s) = t$.

  Let $s = f(t)$.

  By definition of f, $s = f(t)$ is the string in T obtained by writing the characters of t in reverse order.

  But when the order of the characters of a string is reversed once and then reversed again, the original string is recovered.

  Thus

  $f(s) = f\big(f(t)\big) =$ the string obtained by writing the characters of t in reverse order and then writing those characters in reverse order again $= t$

  This is what was to be shown.

Outline

Inverse Functions – Definition

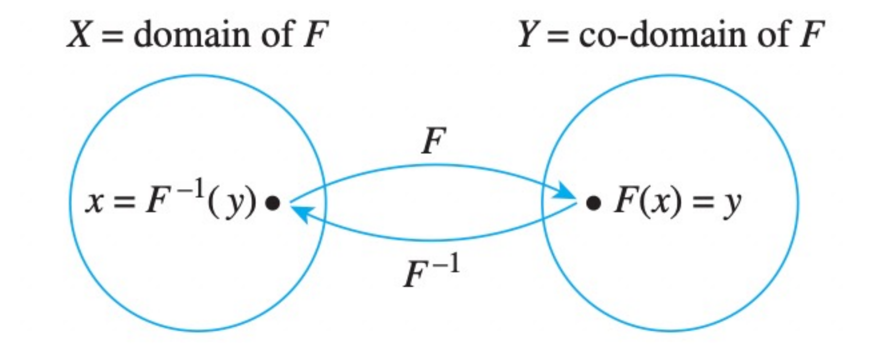
- Suppose $f: X \to Y$ is a one-to-one correspondence; that is, suppose f is one-to-one and onto. Then there is a function $f^{-1}: Y \to X$ that is defined as follows:

Given any element y in Y,

$$f^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } f(x) \text{ equals } y.$$

In other words,

$$f^{-1}(y) = x \leftrightarrow y = f(x)$$



The function f^{-1} is called the inverse function of f.

Finding the Inverse Function

- We have previously shown that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula

$$f(x) = 4x - 1 \text{ for all real numbers } x$$

is both one-to-one and onto. What is the inverse function of f?

By definition of f^{-1}: $f^{-1}(y) = the\ unique\ real\ number\ y\ such\ that\ f(x) = y$

But $f(x) = 4x - 1 = y$ by definition of f

$x = \frac{y+1}{4}$ by adding 1 and dividing both sides by 4

Hence $f^{-1}(y) = \frac{y+1}{4}$