جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 8

Relations

Outline

- Relations on Sets
- Properties of Relations
- Equivalence Relations

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- Relations on Sets
- Properties of Relations
- Equivalence Relations

Relations on Sets

- Definition of Relations
- Relations and Functions
- The Inverse of a Relation
- Direct Graphs on a Relation
- N-ary Relations
- Relational Databases

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Relations on Sets - Definition

- Let A and B be sets. A binary relation R from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written $x\ R\ y$, if and only if, (x, y) is in R.

- Symbolically:

$$x\ R\ y \leftrightarrow (x, y) \in R$$

and

$$x \not R\ y \leftrightarrow (x, y) \notin R$$

Relations on Sets - Example

- Suppose $A = \{Jerusalem, Beirut, Amman, Damascus\}$ and $B = \{Palestine, Lebanon, Jordan, Syria\}$.
- Define a relation R from A to B as follows: For all $(a, b) \in A \times B$,

$$a\ R\ b \leftrightarrow a\ is\ the\ capital\ of\ b$$

$$A\times B = \{(Jerusalem, Palestine), (Jerusalem, Lebanon), (Jerusalem, Jordan), (Jerusalem, Syria), (Beirut, Palestine), (Beirut, Lebanon), (Beirut, Jordan), (Beirut, Syria), (Amman, Palestine), (Amman, Lebanon), (Amman, Jordan), (Amman, Syria), (Damascus, Palestine), (Damascus, Lebanon), (Damascus, Jordan), (Damascus, Syria)\}$$

- R is the subset of $A\times B$ ($R \subseteq A\times B$) that satisfies the definition of the relation
  $R = \{(Jerusalem, Palestine), (Beirut, Lebanon), (Amman, Jordan), (Damascus, Syria)\}$

Relations on Sets - Example

- Suppose $A = \{Jerusalem, Beirut, Amman, Damascus\}$ and $B = \{Palestine, Lebanon, Jordan, Syria\}$.
- Define a relation R from A to B as follows: For all $(a, b) \in A \times B$,
$$a \; R \; b \;\leftrightarrow a \text{ is the capital of } b$$

We can say

$Jerusalem\ R\ Palestine$	since Jerusalem is the capital of Palestine
$Beirut\ R\ Lebanon$	since Beirut is the capital of Lebanon
$Amman\ R\ Jordan$	since Amman is the capital of Jordan
$Damascus\ R\ Syria$	since Damascus is the capital of Syria

Relations on Sets – Example

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$. Let us define a relation R from A to B where for all $(a, b) \in A\ x\ B$, $a\ R\ b \leftrightarrow a < b$.

- $0\ R\ 1$ since $0 < 1$
- $0\ R\ 2$ since $0 < 2$
- $0\ R\ 3$ since $0 < 3$
- $1\ R\ 2$ since $1 < 2$
- $1\ R\ 3$ since $1 < 3$
- $2\ R\ 3$ since $2 < 3$

- $0\ \not R\ 0$ since $0 \nless 0$
- $1\ \not R\ 1$ since $1 \nless 1$
- $2\ \not R\ 1$ since $2 \nless 1$
- $2\ \not R\ 2$ since $2 \nless 2$

Relations on Sets – Example

- If we expand the definition of the previous relation to become:

Define a relation L from $\mathbb{R}$ to $\mathbb{R}$ as follows: For all real numbers x and y, $x\ L\ y \leftrightarrow x < y$.

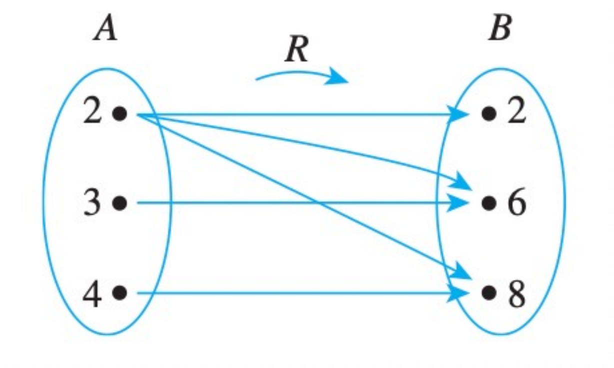
a. Is $57\ L\ 53$? No, because $57 \nless 53$

b. Is $(-17)\ L\ (-14)$? Yes, because $-17 < -14$

c. Is $143\ L\ 143$? No, because $143 \nless 143$

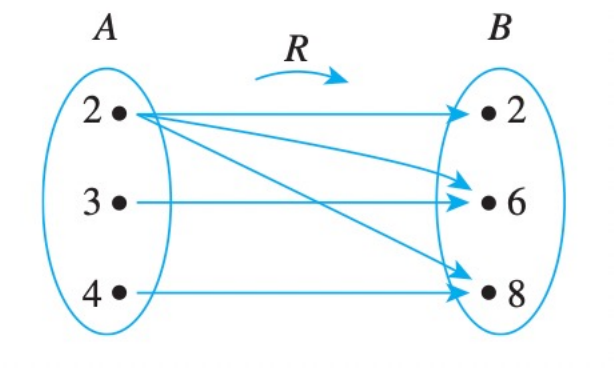
d. Is $(-35)\ L\ 1$? Yes, because $-35 < 1$

Arrow Diagrams of Relations



- From this diagram, we can notice that we do not have the same constraints in relationships that we had in functions.
- This means that
  - *there could be items in set* A *of the relations that are not part of the relations*
  - *one item in set* A *may be related to multiple items in set* B.

Arrow Diagrams of Relations



- The relations in this diagram are
  - *2 R 2*
  - *2 R 6*
  - *2 R 8*
  - *3 R 6*
  - *4 R 8*

$$R = \{(2,2),(2,6),(2,8),(3,6),(4,8)\}$$

The Congruence Modulo 2 Relation

- Define a relation E from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$,

$$m\ E\ n \leftrightarrow m - n \text{ is even.}$$

1. Is $4\ E\ 0$? Yes, because $4 - 0 = 4$, and 4 is even
2. Is $2\ E\ 6$? Yes, because $2 - 6 = -4$, and -4 is even
3. Is $3\ E\ (-3)$? Yes, because $3 - (-3) = 6$, and 6 is even
4. Is $5\ E\ 2$? No, because $5 - 2 = 3$, and 3 is not even

The Congruence Modulo 2 Relation

- Define a relation E from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$,

$$m\,E\,n \leftrightarrow m - n \text{ is even.}$$

List five integers that are related by E to 1.

1. 1, because $1 – 1 = 0$, and 0 is even
2. 3, because $3 – 1 = -2$, and 2 is even
3. 5, because $5 – 1 = 4$, and 4 is even
4. -1 because $(-1) – 1 = -2$, and -2 is even
5. -3 because $(-3) – 1 = -4$, and -4 is even

The Congruence Modulo 2 Relation

- Define a relation E from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$,

$$m\,E\,n \leftrightarrow m - n \text{ is even.}$$

Prove that if n is any odd integer, then $n\,E\,1$.

Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k.

Now by definition of E, $n\,E\,1$ if, and only if, $n - 1$ is even.

But by substitution, $n - 1 = (2k + 1) - 1 = 2k$, and since k is an integer, $2k$ is even.

Hence $n\,E\,1$, as was to be shown.

The Congruence Modulo 2 Relation

- It can be shown that integers m and n are related by E if, and only if,

$m \, mod \, 2 \; = \; n \, mod \, 2$

That is, both are even, or both are odd.

When this occurs m and n are said to be congruent modulo 2.

- Exercise for home:

Prove that for all integers m and n, $m \; - \; n$ is even if, and only if, both m and n are even or both m and n are odd.

A Relation on a Power Set

- Let $X = \{a, b, c\}$. Then $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
- Define a relation S from $\mathcal{P}(X)$ to $\mathbb{Z}$ as follows: For all sets A and B in $\mathcal{P}(X)$ (i.e., for all subsets A and B of X),

$$A\ S\ B \leftrightarrow A \text{ has at least as many elements as } B.$$

a. Is $\{a, b\}\ S\ \{b, c\}$? Yes, because both sets have two elements

b. Is $\{a\}\ S\ \emptyset$? Yes, because $\{a\}$ has one element, and $\emptyset$ has 0 elements, and $1 \geq 0$

c. Is $\{b, c\}\ S\ \{a, b, c\}$? No, because $\{b, c\}$ has two elements, and $\{a, b, c\}$ has 3 elements and $2 \ngeq 3$

d. Is $\{c\}\ S\ \{a\}$? Yes, because both sets have one element

Relations on Sets

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Relations and Functions

- Functions are a special type of relations.
- To define functions in terms of relations:

A function F from a set A to a set B is a relation from A to B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
   In other words: every x in the domain has a relation to some element in the co-domain
2. For all elements x in A and y and z in B, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.
   in other words: every x in the domain has a relation to some unique element in the co-domain.

If F is a function from A to B, we write

$$y = F(x) \leftrightarrow (x, y) \in F$$

Relations and Functions

- Let $A = \{2, 4, 6\}$, and $B = \{1, 3, 5\}$. The relation R is defined from A to B as:

$$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$$

Is R a function?

No, because $(4, 1) \in R$ and $(4, 3) \in R$, but $1 \neq 3$.

Relations and Functions

- Let $A = \{2, 4, 6\}$, and $B = \{1, 3, 5\}$. The relation S is defined from A to B as:

For all $(x, y) \in A \times B, (x, y) \in S \leftrightarrow y = x + 1$

Is S a function?

No, because the element 6 of the domain A does not have a relation to any element in the co-domain B.

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The Inverse of a Relation

- Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}$$

  Symbolically,

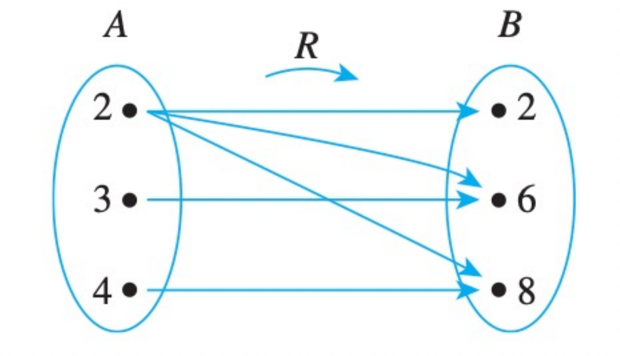
$$For\ all\ x \in X\ and\ y \in Y, (y, x) \in R^{-1} \leftrightarrow (x, y) \in R$$

- Unlike functions, every inverse of a relation is a relation.

The Inverse of a Relation – Example

- Let A = {2, 3, 4} and B = {2, 6, 8}, and let R be the “divides” relation from A to B:

$$For\ all\ (x, y) \in A{\times}B, \qquad x\ R\ y \ \leftrightarrow x|y$$



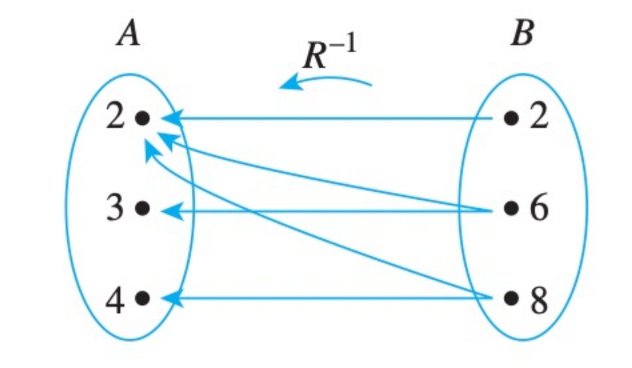
- State explicitly:

1. Which are the ordered pairs of R

$$R \ = \ \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$$

2. Which are the ordered pairs of R^{-1}

$$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$$



The Inverse of a Relation – Example

- Let A = {2, 3, 4} and B = {2, 6, 8}, and let R be the “divides” relation from A to B:

$$For\ all\ (x, y) \in A \times B, \qquad x\ R\ y\ \leftrightarrow x|y$$

Define R^{-1} in words:

$$For\ all\ (y, x) \in B \times A, \qquad y\ R^{-1}\ x\ \leftrightarrow y\ is\ a\ multiple\ of\ x$$

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Directed Graph on a Relation

- A binary relation ***on a set A*** is a binary relation from A to A.

- When a binary relation R is defined on a set A, the arrow diagram of the relation can be modified so that is becomes a directed graph.
- This means:
Instead of representing A as two separate sets of points represent A only once and draw an arrow diagram from each point of A to each related point.

- For all points x and y in A,

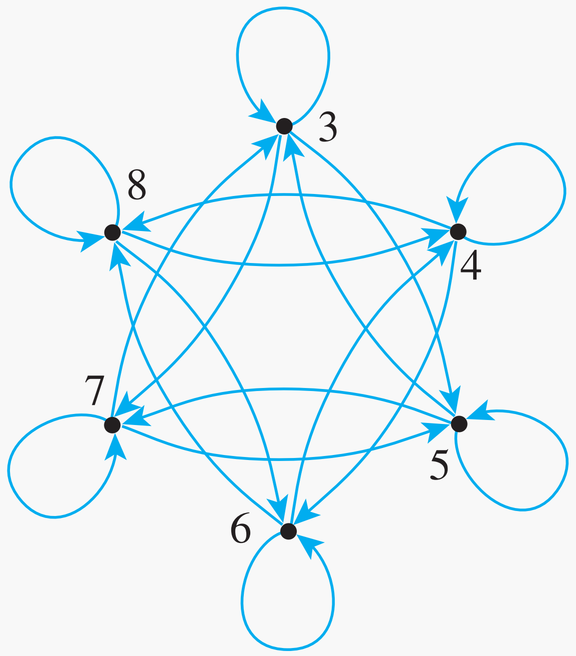
$$\text{there is an arrow from } x\ to\ y \leftrightarrow x\ R\ y \leftrightarrow (x, y) \in R$$

Directed Graph of a Relation – Example

- Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows:

$$\text{For all } x, y \in A, x\, R\, y \Leftrightarrow 2 \mid (x - y)$$

Draw the directed graph of R



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N-ary Relations – Definition

- Given sets $A_1, A_2, \dots, A_n$, an n-ary relation R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called binary, ternary, and quaternary relations, respectively.

- A binary relation is a subset of the Cartesian product of two sets.
- Similarly, an n-ary relation is a subset of the Cartesian product of n sets.

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Relational Databases

- Let A_1 be a set of positive integers,
  A_2 be a set of alphabetic character strings
  A_3 be a set of numeric character strings
  A_4 a set of alphabetic character strings.
- Define a quaternary relation R on $A_1 \times A_2 \times A_3 \times A_4$ as follows:

$(a_1, a_2, a_3, a_4) \in R \leftrightarrow$ a patient with patient ID number a_1, named a_2, was admitted on date a_3, with primary diagnosis a_4.

Relational Databases – cont.

- At some hospital, this relation might contain the following 4-tuples:

  (011985, John Schmidt, 020710, asthma)

  (574329, Tak Kurosawa, 0114910, pneumonia)

  (466581, Mary Lazars, 0103910, appendicitis)

  (008352, Joan Kaplan, 112409, gastritis)

  (011985, John Schmidt, 021710, pneumonia)

  (244388, Sarah Wu, 010310, broken leg)

  (778400, Jamal Baskers, 122709, appendicitis)

Relational Databases – cont.

- When talking about relational databases, tuples are normally thought of as being written in tables.
- The name of the table is the name of the relation
- Each row of the table corresponds to one tuple.
- The header for each column gives the descriptive attribute for the elements in the column

Patient			
Patient ID	Patient Name	Admission Date	Primary Diagnosis
011985	John Schmidt	020795	asthma
574329	Tak Kurosawa	011495	pneumonia
466581	Mary Lazars	010395	appendicitis
…	…	…	…