BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 8

Relations

Outline

- Relations on Sets
- Properties of Relations
- Equivalence Relations

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- Relations on Sets
- Properties of Relations
- Equivalence Relations

Properties of Relations

- Reflexivity, Symmetry, and Transitivity
- The Transitive Closure of a Relation
- Properties of Relations on Infinite Sets

Properties of Relations

- Reflexivity, Symmetry, and Transitivity
- The Transitive Closure of a Relation
- Properties of Relations on Infinite Sets

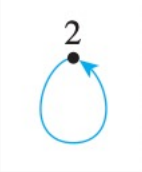
Reflexivity – Definition

- Let R be a binary relation on a set A. R is reflexive if, and only if, for all $x \in A, x\ R\ x$.

- Symbolically,

$$R\ is\ reflexive\ \leftrightarrow for\ all\ x\ in\ A, (x, x) \in R$$

- Informally: each element is related to itself.
- In directed graphs: each point of the graph has an arrow looping around from it back to itself.



Symmetry – Definition

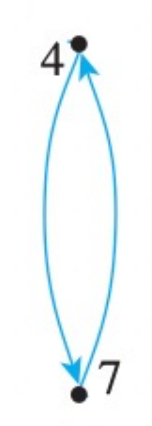
- R is symmetric if, and only if, for all $x \in A$, if $x\ R\ y$ then $y\ R\ x$.

- Symbolically,

$$R \text{ is symmetric } \leftrightarrow for\ all\ x\ and\ y\ in\ A, if\ (x, y) \in R\ then\ (y, x) \in R$$

- Informally: if any one element is related to any other element, then the second element is related to the first.

- In directed graphs: in each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.



Transitivity - Definition

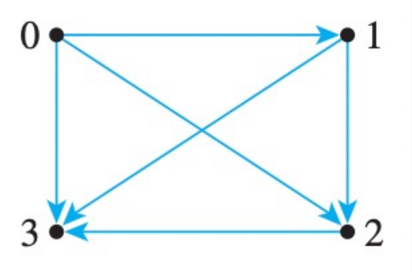
- R is transitive if, and only if, for all $x, y, z \in A$, if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

- Symbolically,

  $$R\ is\ transitive\ \leftrightarrow for\ all\ x, y, and\ z\ in\ A, if\ (x, y) \in R\ and\ (y, z) \in R\ then\ (x, z) \in R$$

- Informally: if any one element is related to a second and that second element is related to a third, then the first element is related to the third.

- In directed graphs: in each case where there is an arrow going from one point to a second and from a second point to a third, there is an arrow going from the first point to the third.



Negation of Reflexivity, Symmetry, and Transitivity

- Since the definition of reflexivity, symmetry, and transitivity are universal statements, their negation are existential statements as follows:

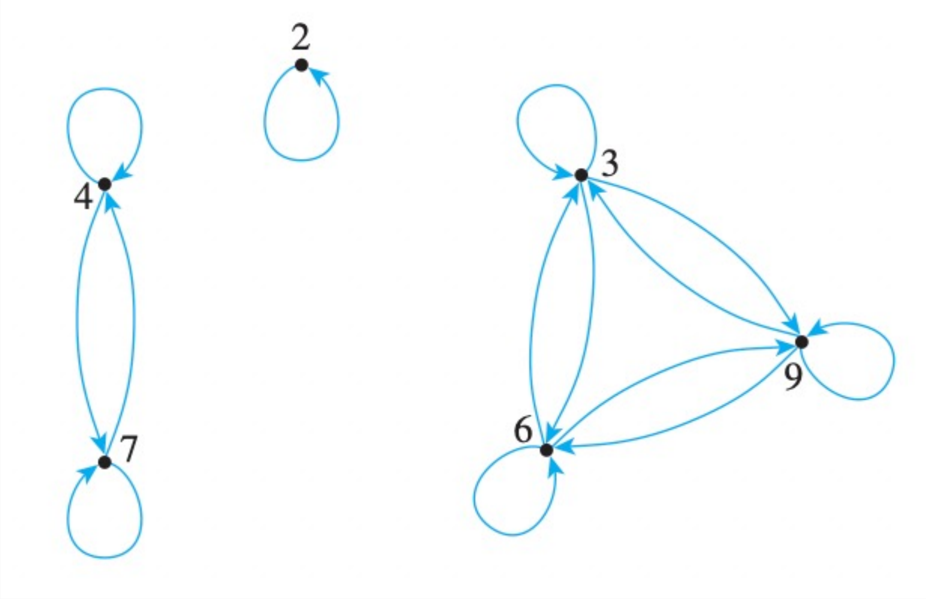
- R is not reflexive if, and only if, there is an element x in A such that $(x, x) \notin R$.
- R is not symmetric if, and only if, there are elements x and y in A such that $(x, y) \in R$, but $(y, x) \notin R$.
- R is not transitive if, and only if, there are elements x, y, and z in A such that $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$.

Example

- Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows:

$$\text{For all } x, y \in A, x\ R\ y \leftrightarrow 3 \mid (x - y)$$

- Draw the directed graph of R.



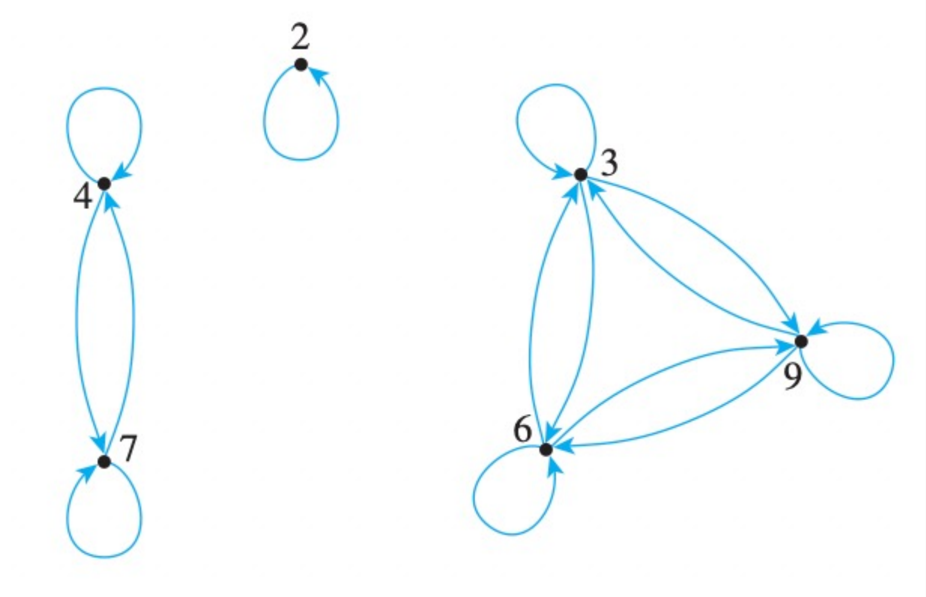
Example – cont.

- Is R reflexive?

Yes, because $(2, 2) \in R$, $(3, 3) \in R$, $(4, 4) \in R$, $(6, 6) \in R$, $(7, 7) \in R$, and $(9, 9) \in R$.

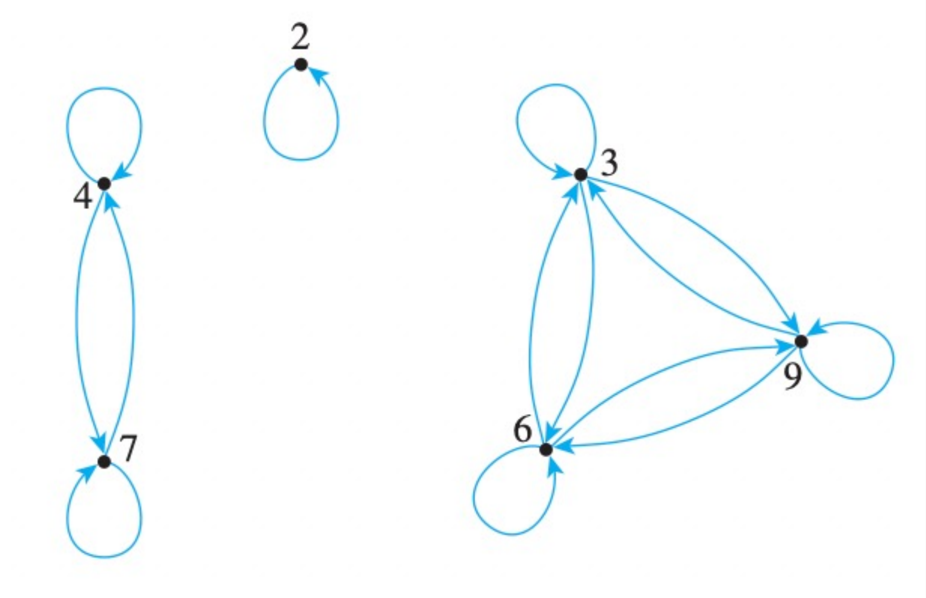
In other words,

$\forall x \in A, (x, x) \in R$.



Example – cont.

- Is R symmetric?



Yes, because both $(3, 9) \in R$ and $(9, 3) \in R$,
and both $(3, 6) \in R$ and $(6, 3) \in R$,
and both $(6, 9) \in R$ and $(9, 6) \in R$,
and both $(4, 7) \in R$ and $(7, 4) \in R$

In other words,

$\forall x, y \in A,\ if\ (x, y) \in R\ then\ (y, x) \in R.$

Example – cont.

- Is R transitive?

Yes, because $(3,6) \in R$, $(6,9) \in R$, and $(3,9) \in R$.
And $(3,9) \in R$, $(9,6) \in R$, and $(3,6) \in R$
And $(6,3) \in R$, $(3,9) \in R$, and $(6,9) \in R$
And $(6,9) \in R$, $(9,3) \in R$, and $(6,3) \in R$
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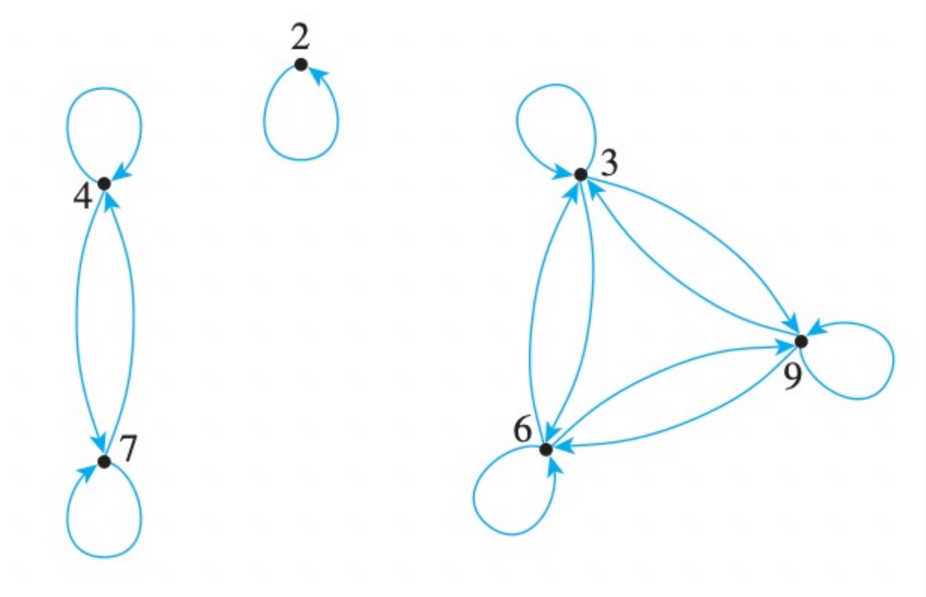
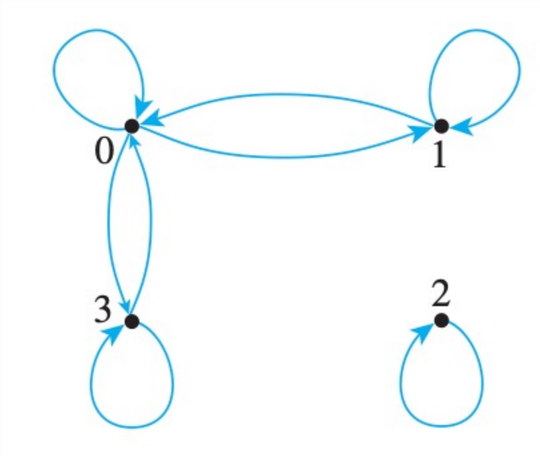


In other words,
$\forall x, y, z \in R, if\ (x, y) \in R\ and\ (y, z) \in R\ then\ (x, z) \in R$

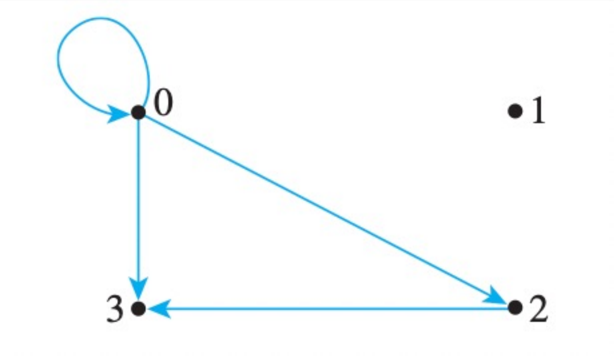
Example

- Is the relation represented in the following graph:
  - *Reflexive?*
    - Yes
  - *Symmetric?*
    - Yes
  - *Transitive?*
    - No, because $(1,0) \in R$ and $(0,3) \in R$ but $(1,3) \notin R$.



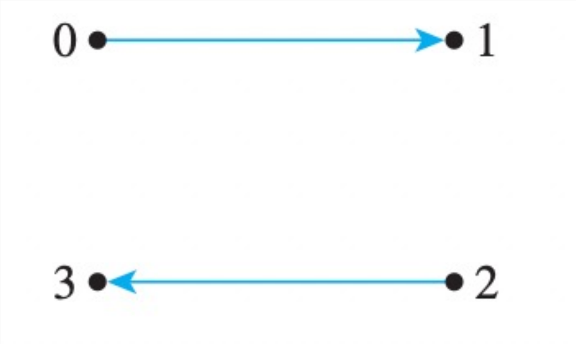
Example

- Is the relation represented in the following graph:
  - *Reflexive?*
    - No, because $(1, 1) \notin R$
  - *Symmetric?*
    - No, because $(0, 2) \in R$ but $(2, 0) \notin R$.
  - *Transitive?*
    - Yes.



Example

- Is the relation represented in the following graph:
  - *Reflexive?*
    - No, because $(0,0) \notin R$
  - *Symmetric?*
    - No, because $(0,1) \in R$ but $(1,0) \notin R$.
  - *Transitive?*
    - Yes.



Properties of Relations

- Reflexivity, Symmetry, and Transitivity
- The Transitive Closure of a Relation
- Properties of Relations on Infinite Sets

The Transitive Closure of a Relation

- Let A be a set and R a binary relation on A. The transitive closure of R is the binary relation R^t on A that satisfies the following three properties:
  - R^t *is transitive*
  - $R \subseteq R^t$
  - *If* S *is any other transitive relation that contains* R*, then* $R^t \subseteq S$

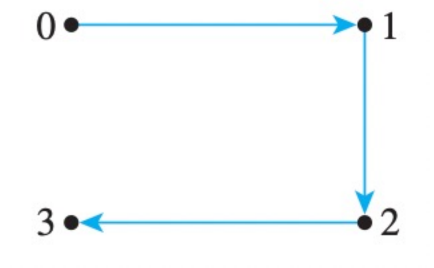
- In other words, R^t is the relation that is obtained by adding ***<u>the least number</u>*** of ordered pairs to ensure transitivity.

Transitive Closure – example

- Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as follows:

$$R = \{(0, 1), (1, 2), (2, 3)\}$$

- Find the transitive closure of R.



The directed graph of R looks like this:

All the ordered pairs in R are in R^t.

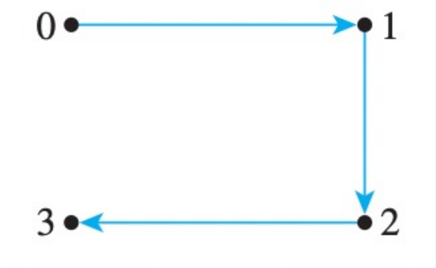
Since there are arrows going from 0 to 1 and from 1 to 2, R^t must have an arrow going from 0 to 2. Hence $(0, 2) \in R^t$

This results in both $(0, 2) \in R^t$ and $(2, 3) \in R^t$, which means we must also add $(0, 3)$ to R^t.

Transitive Closure – example

- Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as follows:

$$R \ = \{(0,1), (1,2), (2,3)\}$$

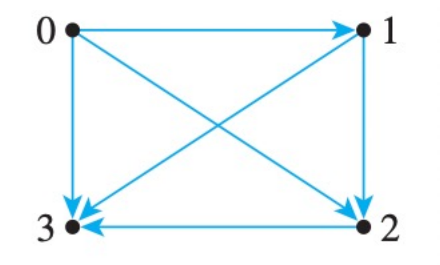


- Find the transitive closure of R.

Also, since $(1,2) \in R^t$ and $(2,3) \in R^t$, then we need to also add $(1,3)$ to R^t.

Therefore,

$$R^t = \{(0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$$



Properties of Relations

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Proving Properties of Relations on Infinite Sets

- Proving that relations are reflexive, symmetric, or transitive are proofs of universal statements.

- This means, we need to use methods of direct proof to prove them.
- Or we need a counterexample to disprove them.

Properties of Equality

- Define a relation R on $\mathbb{R}$ as follows:

  For all real numbers x and y, $x \; R \; y \; \leftrightarrow \; x \; = \; y$.

- Is R reflexive?

  Yes, because for all $x \in \mathbb{R}$, $x \; R \; x$.

  Which is the same as saying:

  for all $x \in \mathbb{R}$, $x = x$.

Properties of Equality

- Define a relation R on $\mathbb{R}$ as follows:

  For all real numbers x and y, $x\ R\ y \leftrightarrow x = y$.

- Is R symmetric?

  Yes, because for all $x, y \in \mathbb{R}$, $if\ x\ R\ y\ \ then\ \ y\ R\ x$.

  Which is the same as saying:

  for all $x, y \in \mathbb{R}$, $if\ x = y\ \ then\ \ y = x$.

Properties of Equality

- Define a relation R on $\mathbb{R}$ as follows:

  For all real numbers x and y, $x\ R\ y \leftrightarrow x = y$.

- Is R transitive?

  Yes, because for all $x, y, z \in \mathbb{R}$, $if\ x\ R\ y\ and\ y\ R\ z\ then\ x\ R\ z$.

  Which is the same as saying:

  for all $x, y, z \in \mathbb{R}$, $if\ x = y\ and\ y = z\ then\ x = z$.

Properties of "Less Than"

- Define a relation R on $\mathbb{R}$ as follows:

  For all real numbers x and y, $x\ R\ y \leftrightarrow x < y$.

- Is R reflexive?

  No, because for all $x \in \mathbb{R}$, $x \nless x$.

  Counterexample:

  $0 \in \mathbb{R}$, and $0 \nless 0$

Properties of “Less Than”

- Define a relation R on $\mathbb{R}$ as follows:

  For all real numbers x and y, $x\ R\ y \ \leftrightarrow\ x\ <\ y$.

- Is R symmetric?

  No, because for all $x, y \in \mathbb{R},\ if\ x < y,\ \ y \nless x$.

  Counterexample:

  $0, 1 \in \mathbb{R}$, and $0 < 1$ but $1 \nless 0$.

Properties of “Less Than”

- Define a relation R on $\mathbb{R}$ as follows:

  For all real numbers x and y, $x\ R\ y \leftrightarrow x < y$.

- Is R transitive?

  Yes, because for all $x, y, z \in \mathbb{R}$, $if\ x < y\ and\ y < z\ then\ x < z$.

  by the transitive law of order for real number.

Properties of Congruence Modulo 3

- Define a relation T on $\mathbb{Z}$ (the set of all integers) as follows: For all integers m and n,
$$m\,T\,n \leftrightarrow 3 \mid (m - n)$$
- This relation is called congruence modulo 3.

- Is T reflexive?

Suppose m is a particular but arbitrarily chosen integer. We must show that $m\,T\,m$.

Now $m - m = 0$.

But $3 \mid 0$ since $0 = 3 \cdot 0$.

Hence $3 \mid (m - m)$.

Thus, by definition of T, mTm, as was to be shown.

Properties of Congruence Modulo 3

- Define a relation T on $\mathbb{Z}$ (the set of all integers) as follows: For all integers m and n,
$$m\,T\,n \leftrightarrow 3 \mid (m - n)$$
- Is T symmetric?

Suppose m and n are particular but arbitrarily chosen integers that satisfy the condition $m\,T\,n$. We must show that $n\,T\,m$.

By definition of T , since $m\,T\,n$ then $3 \mid (m - n)$.

By definition of “divides,” this means that $m - n = 3k$, for some integer k.

Multiplying both sides by -1 gives $n - m = 3(-k)$.

Since $-k$ is an integer, this equation shows that $3 \mid (n - m)$.

Hence, by definition of T, $n\,T\,m$ as was to be shown.

Properties of Congruence Modulo 3

- Define a relation T on $\mathbb{Z}$ (the set of all integers) as follows: For all integers m and n,

$$m\,T\,n \leftrightarrow 3 \mid (m - n)$$

- Is T transitive?

Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m\,T\,n$ and $n\,T\,p$. We must show that $m\,T\,p$.

By definition of T , since $m\,T\,n$ and $n\,T\,p$, then $3 \mid (m - n)$ and $3 \mid (n - p)$.

By definition of "divides," this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s.

Adding the two equations gives $(m - n) + (n - p) = 3r + 3s$,

and simplifying gives that $m - p = 3(r + s)$.

Since $r + s$ is an integer, this equation shows that $3 \mid (m - p)$. Hence, by definition of T , $m\,T\,p$ as was to be shown.