جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 8

Relations

Outline

- Relations on Sets
- Properties of Relations
- Equivalence Relations

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- Relations on Sets
- Properties of Relations
- Equivalence Relations

Equivalence Relations

- Partition Sets
- Equivalence Relations
  - *Equivalence Relation on a Set of Subsets*
  - *Equivalence Relation of Digital Logic Circuits*
  - *Equivalence Relation on a Set of Identifiers*
- Equivalence Classes
  - *Congruence Modulo n*

Equivalence Relations

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Partitions of Sets

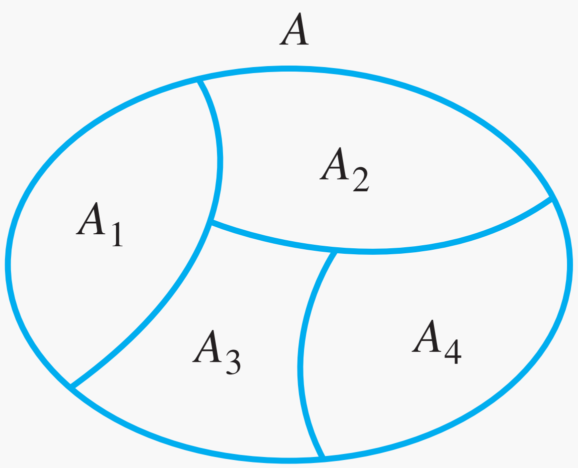
- A **partition of a set** is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset.

- For example, if we divide all the students in this class to groups, where no student is in two groups at the same time, then these groups will be a partition of the class.

Partition of Sets – Formal Definition

- A finite or infinite collection of non-empty sets $\{A_1, A_2, A_3, \dots\}$ is a partition of a set A if, and only if,

1. The set A is the union of all the $A_i s$
2. The sets $A_1, A_2, A_3, \dots$ are mutually disjoint



Partition of Sets - Example

Let $\mathbb{Z}$ be the set of all integers, and let

$T_0 = \{n \in \mathbb{Z} \mid n = 3k,\ for\ some\ integer\ k\}$,

$T_1 = \{n \in \mathbb{Z} \mid n = 3k + 1,\ for\ some\ integer\ k\}$,

$T_2 = \{n \in \mathbb{Z} \mid n = 3k + 2,\ for\ some\ integer\ k\}$.

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

Yes. By the quotient-remainder theorem, every integer n can be represented in exactly one of the three forms

n=3k   or   n=3k+1   or   n=3k+2

It also implies that every integer is in one of the sets T_0, T_1, T_2 , so $\mathbb{Z} = T_0 \cup T_1 \cup T_2$.

Relations Induced by Partitions

- A relation induced by a partition is a relation between two elements in the same partition.

- Given a partition of a set A, the relation induced by the partition, R, is defined on A as follows:
- For all $x, y \in A$,

  $x\ R\ y \leftrightarrow$ there is a subset A_i of the partition such that both x and y are in A_i.

Relations Induced by Partitions – Example

- Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:
$$\{0, 3, 4\}, \{1\}, \{2\}$$
- Find the relation R induced by this partition.

Since $\{0, 3, 4\}$ is a subset of the partition,

$0\ R\ 3$ because both 0 and 3 are in $\{0, 3, 4\}$,

$3\ R\ 0$ because both 3 and 0 are in $\{0, 3, 4\}$,

$0\ R\ 4$ because both 0 and 4 are in $\{0, 3, 4\}$,

$4\ R\ 0$ because both 4 and 0 are in $\{0, 3, 4\}$,

$3\ R\ 4$ because both 3 and 4 are in $\{0, 3, 4\}$,

$4\ R\ 3$ because both 4 and 3 are in $\{0, 3, 4\}$.

Relations Induced by Partitions – Example

- Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:
$$\{0, 3, 4\}, \{1\}, \{2\}$$
- Find the relation R induced by this partition.

Also,

$0\ R\ 0$ because both 0 and 0 are in $\{0, 3, 4\}$

$3\ R\ 3$ because both 3 and 3 are in $\{0, 3, 4\}$,

$4\ R\ 4$ because both 4 and 4 are in $\{0, 3, 4\}$.

Relations Induced by Partitions – Example

- Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:
$$\{0, 3, 4\}, \{1\}, \{2\}$$
- Find the relation R induced by this partition.

Since $\{1\}$ is a subset of the partition,

$1 \; R \; 1$ because both 1 and 1 are in $\{1\}$,

and since $\{2\}$ is a subset of the partition,

$2 \; R \; 2$ because both 2 and 2 are in $\{2\}$.

Hence
$$R = \{(0,0), (0,3), (0,4), (1,1), (2,2), (3,0), (3,3), (3,4), (4,0), (4,3), (4,4)\}$$

Properties of Relations Induced by Partitions

- Let A be a set with a partition and let R be the relation induced by the partition. Then R is ***reflexive***, ***symmetric***, and ***transitive***.

Proof that R is reflexive:

- Suppose $x \in A$. Since $\{A_1, A_2, \ldots, A_n\}$ is a partition of A, it follows that $x \in A_i$ for some i.
- But then the statement

  there is a set A_i of the partition such that $x \in A_i$ and $x \in A_i$ is true.

- Thus, by definition of R, $x \, R \, x$.

Properties of Relations Induced by Partitions

- Let A be a set with a partition and let R be the relation induced by the partition. Then R is ***reflexive***, ***symmetric***, and ***transitive***.

Proof that R is symmetric:

- Suppose x and y are elements of A such that $x\ R\ y$.
- Then there is a subset A_i of the partition such that $x \in A_i$ and $y \in A_i$ by definition of R.
- It follows that

  there is a subset A_i of the partition such that $y \in A_i$ and $x \in A_i$

- Hence, by definition of R, $y\ R\ x$.

Properties of Relations Induced by Partitions

- Let A be a set with a partition and let R be the relation induced by the partition. Then R is ***reflexive***, ***symmetric***, and ***transitive***.

Proof that R is transitive:

- Suppose x, y, and z are in A and $x\ R\ y$ and $y\ R\ z$. By definition of R, there are subsets A_i and A_j of the partition such that x and y are in A_i and y and z are in A_j .
- Suppose $A_i \neq A_j$. Then $A_i \cap A_j = \emptyset$ since $\{A_1, A_2, \ldots, A_n\}$ is a partition of A.
- But y is in A_i and y is also in A_j. Hence $A_i \cap A_j \neq \emptyset$. Thus, $A_i = A_j$ (by contradiction).
- It follows that x, y, and z are all in A_i , and so in particular, x and z are in A_i .
- Thus, by definition of R, $x\ R\ z$.

Equivalence Relations

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Equivalence Relations

- A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an equivalence relation.

- <u>Definition:</u>

Let A be a set and R a relation on A. R is an equivalence relation if, and only if, R is reflexive, symmetric, and transitive.

- Thus, according to the theorem that we have just proved, the relation induced by a partition is an equivalence relation.

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An Equivalence Relation on a Set of Subsets

- Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

- Define a relation R on X as follows: For all A and B in X,

$$A\ R\ B \Leftrightarrow \textit{the least element of } A \textit{ equals the least element of } B$$

- Prove that R is an equivalence relation on X.

- R is reflexive:

Suppose A is a nonempty subset of $\{1, 2, 3\}$. We must show that $A\ R\ A$.

It is true to say that the least element of A equals the least element of A.

Thus, by definition of R, $A\ R\ A$.

An Equivalence Relation on a Set of Subsets

- Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then
$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$
- Define a relation R on X as follows: For all A and B in X,
$$A\ R\ B \iff \textit{the least element of } A \textit{ equals the least element of } B$$
- Prove that R is an equivalence relation on X.

- R is symmetric:

Suppose A and B are nonempty subsets of $\{1, 2, 3\}$ and $A\ R\ B$. We must show that $B\ R\ A$.

Since $A\ R\ B$, the least element of A equals the least element of B. But this implies that the least element of B equals the least element of A, and so, by definition of R, $B\ R\ A$.

An Equivalence Relation on a Set of Subsets

- Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then
$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$
- Define a relation R on X as follows: For all A and B in X,
$$A\ R\ B \Leftrightarrow \textit{the least element of } A \textit{ equals the least element of } B$$
- Prove that R is an equivalence relation on X.

- R is transitive:

Suppose A, B, and C are nonempty subsets of $\{1, 2, 3\}$, $A\ R\ B$, and $B\ R\ C$. We must show that $A\ R\ C$.

Since $A\ R\ B$, the least element of A equals the least element of B, and since $B\ R\ C$, the least element of B equals the least element of C.

Thus, the least element of A equals the least element of C, and so, by definition of R, $A\ R\ C$.

An Equivalence Relation on a Set of Subsets

- Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then
$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$
- Define a relation R on X as follows: For all A and B in X,
$$A\ R\ B \iff the\ least\ element\ of\ A\ equals\ the\ least\ element\ of\ B$$
- Prove that R is an equivalence relation on X.

- **Since $\boldsymbol{R}$ is reflexive, symmetric, and transitive, then $\boldsymbol{R}$ is an equivalence relation on $\boldsymbol{X}$.**

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Equivalence of Digital Logic Circuits

- Let S be the set of all digital logic circuits with a fixed number n of inputs. Define a relation E on S as follows:

For all circuits C_1 and C_2 in S,

$$C_1 \ E \ C_2 \ \leftrightarrow \ C_1 \text{ has the same input/output table as } C_2.$$

- If $C_1 \ E \ C_2$, then circuit C_1 is said to be equivalent to circuit C_2.
- Prove that E is an equivalence relation on S.

Equivalence of Digital Logic Circuits

For all circuits C_1 and C_2 in S,

$$C_1 \; E \; C_2 \;\leftrightarrow\; C_1 \text{ has the same input/output table as } C_2.$$

- E is reflexive:

Suppose C is a digital logic circuit in S. We must show that $C \; E \; C$.

Certainly, C has the same input/output table as itself.

Thus, by definition of E, $C \; E \; C$ as was to be shown.

Equivalence of Digital Logic Circuits

For all circuits C_1 and C_2 in S,

$$C_1 \; E \; C_2 \;\leftrightarrow\; C_1 \text{ has the same input/output table as } C_2.$$

- E is symmetric:

Suppose C_1 and C_2 are digital logic circuits in S such that $C_1 \; E \; C_2$. We must show that $C_2 \; E \; C_1$.

By definition of E, since $C_1 \; E \; C_2$, then C_1 has the same input/output table as C_2.

It follows that C_2 has the same input/output table as C_1.

Hence, by definition of E, $C_2 \; E \; C_1$ as was to be shown.

Equivalence of Digital Logic Circuits

For all circuits C_1 and C_2 in S,

$$C_1\ E\ C_2 \leftrightarrow C_1 \text{ has the same input/output table as } C_2.$$

- E is transitive:

Suppose C_1, C_2, and C_3 are digital logic circuits in S such that $C_1\ E\ C_2$ and $C_2\ E\ C_3$. We must show that $C_1\ E\ C_3$.

By definition of E, since $C_1\ E\ C_2$ and $C_2\ E\ C_3$, then

C_1 has the same input/output table as C_2

and C_2 has the same input/output table as C_3.

It follows that C_1 has the same input/output table as C_3

Hence, by definition of E, $C_1\ E\ C_3$ as was to be shown.

Equivalence of Digital Logic Circuits

For all circuits C_1 and C_2 in S,

$$C_1 \; E \; C_2 \;\leftrightarrow\; C_1 \text{ has the same input/output table as } C_2.$$

- **Since $\boldsymbol{E}$ is reflexive, symmetric, and transitive, then $\boldsymbol{E}$ is an equivalence relation on $\boldsymbol{S}$.**

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A Relation on a Set of Identifiers

- Let L be the set of all allowable identifiers in a certain computer language, and define a relation R on L as follows: For all strings s and t in L,

  $$s\ R\ t \leftrightarrow \text{the first eight characters of } s \text{ equal the first eight characters of } t.$$

- Prove that R is an equivalence relation on L.

R is reflexive:

- Let $s \in L$. We must show that $s\ R\ s$.
- Clearly s has the same first eight characters as itself. Thus, by definition of R, $s\ R\ s$ as was to be shown.

A Relation on a Set of Identifiers

R is symmetric:

- Let s and t be in L and suppose that $s\ R\ t$. We must show that $t\ R\ s$.
- By definition of R, since $s\ R\ t$, the first eight characters of s equal the first eight characters of t.
- But then the first eight characters of t equal the first eight characters of s.
- And so, by definition of R, $t\ R\ s$ as was to be shown.

A Relation on a Set of Identifiers

R is transitive:

- Let s, t, and u be in L and suppose that $s\ R\ t$ and $t\ R\ u$. We must show that $s\ R\ u$.
- By definition of R, since $s\ R\ t$ and $t\ R\ u$,

  the first eight characters of s equal the first eight characters of t,

  and the first eight characters of t equal the first eight characters of u.
- Hence the first eight characters of s equal the first eight characters of u.
- Thus, by definition of R, $s\ R\ u$ as was to be shown.

A Relation on a Set of Identifiers

- Let L be the set of all allowable identifiers in a certain computer language, and define a relation R on L as follows: For all strings s and t in L,

  $s\ R\ t \leftrightarrow$ the first eight characters of s equal the first eight characters of t.

- Prove that R is an equivalence relation on L.

- **Since $\boldsymbol{R}$ is reflexive, symmetric, and transitive, then $\boldsymbol{R}$ is an equivalence relation on $\boldsymbol{L}$.**

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Equivalence Classes of an Equivalence Relation

- Suppose A is a set and R is an equivalence relation on A. For each element a in A, the equivalence class of a, denoted $[a]$ and called the class of a for short, is the set of all elements x in A such that x is related to a by R.
- In symbols:

$$[a] = \{x \in A \mid x\ R\ a\}$$

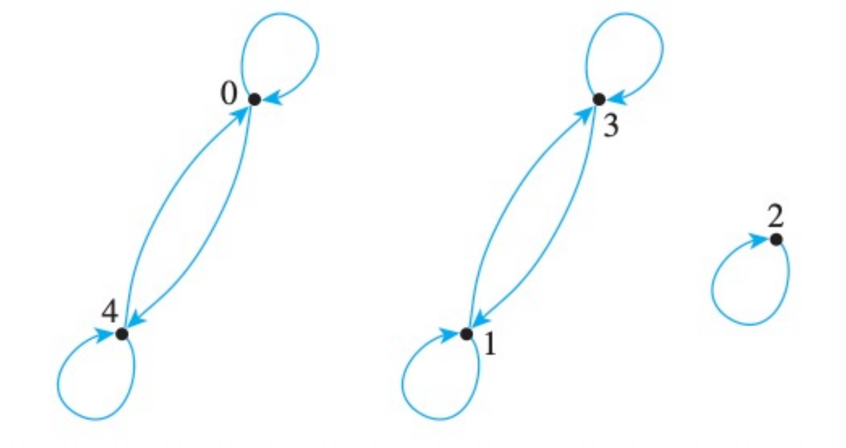
- When several equivalence relations on a set are under discussion, the notation $[a]_R$ is often used to denote the equivalence class of a under R.
- The procedural version of this definition is

$$\text{for all } x \in A,\ x \in [a] \leftrightarrow x\ R\ a$$

Equivalence Classes of a Relation – Example

- Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:
$$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$$

- Draw the directed graph of R.



- Does this graph show that R is an equivalence relation?

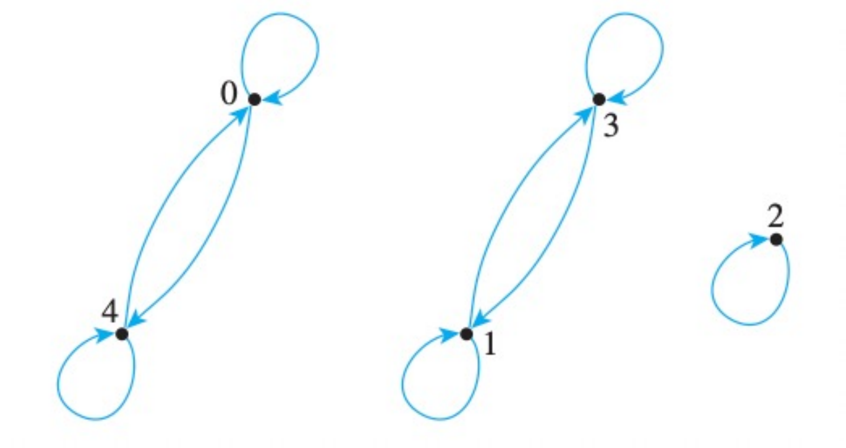
  Yes, it does.

Equivalence Classes of a Relation – Example

- Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

  $$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$$

- Find the distinct equivalence classes of R.



First find the equivalence class of every element of A.

$[0] = \{x \in A \mid x\ R\ 0\} = \{0, 4\}$

$[1] = \{x \in A \mid x\ R\ 1\} = \{1, 3\}$

$[2] = \{x \in A \mid x\ R\ 2\} = \{2\}$

$[3] = \{x \in A \mid x\ R\ 3\} = \{1, 3\}$

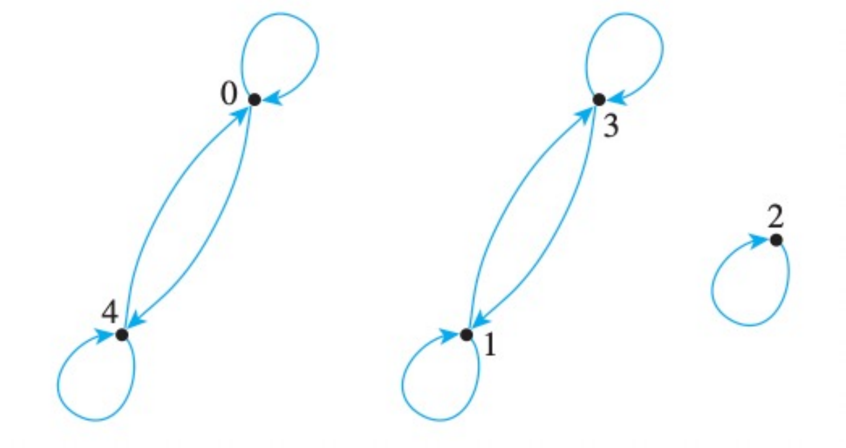
$[4] = \{x \in A \mid x\ R\ 4\} = \{0, 4\}$

Equivalence Classes of a Relation – Example

- Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

$$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$$

- Find the distinct equivalence classes of R.



Note that $[0] = [4]$ and $[1] = [3]$.

Thus, the distinct equivalence classes of the relation are

$\{0, 4\}$, $\{1, 3\}$, and $\{2\}$.

Lemmas of Equivalence Classes

- Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a\ R\ b$, then $[a] = [b]$.

- If A is a set, R is an equivalence relation on A, and a and b are elements of A, then either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

- If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

Class Representative

- Suppose R is an equivalence relation on a set A and S is an equivalence class of R.
- A representative of the class S is any element a such that $[a] = S$.

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Congruence Modulo 3 – Example

- Let R be the relation of congruence modulo 3 on the set $\mathbb{Z}$ of all integers.
- That is, for all integers m and n,

$$m\,R\,n \Leftrightarrow 3 \mid (m - n) \leftrightarrow m \equiv n \ (mod\ 3)$$

- Describe the distinct equivalence classes of R.
- For each integer a,

$$[a] = \{x \in Z \mid x\,R\,a\}$$
$$= \{x \in Z \mid 3 \mid (x - a)\}$$
$$= \{x \in Z \mid x - a = 3k, for\ some\ integer\ k\}$$

Congruence Modulo 3 – Example

- Therefore,

$$[a] = \{x \in Z \mid x = 3k + a, for\ some\ integer\ k\}$$

In particular,

$$\begin{aligned}[0] &= \{x \in Z \mid x = 3k + 0, for\ some\ integer\ k\} \\ &= \{x \in Z \mid x = 3k, for\ some\ integer\ k\} \\ &= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\}\end{aligned}$$

$$\begin{aligned}[1] &= \{x \in Z \mid x = 3k + 1, for\ some\ integer\ k\} \\ &= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\}\end{aligned}$$

$$\begin{aligned}[2] &= \{x \in Z \mid x = 3k + 2, for\ some\ integer\ k\} \\ &= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}\end{aligned}$$

Congruence Modulo 3 – Example

- Now since $3\ R\ 0$, then $[3] = [0]$.
- More generally, by the same reasoning, $[0] = [3] = [-3] = [6] = [-6] = \cdots$, and so on.
- Similarly, $[1] = [4] = [-2] = [7] = [-5] = \cdots$, and so on.
- And $[2] = [5] = [-1] = [8] = [-4] = \cdots$, and so on.

- Notice that every integer is in class $[0]$, $[1]$, or $[2]$. Hence the distinct equivalence classes are

$$\{x \in Z \mid x = 3k, for\ some\ integer\ k\},$$

$$\{x \in Z \mid x = 3k + 1, for\ some\ integer\ k\}, \text{ and}$$

$$\{x \in Z \mid x = 3k + 2, for\ some\ integer\ k\}.$$

Congruence Modulo 3 – Example

- the distinct equivalence classes are

$$\{x \in Z \mid x = 3k, for\ some\ integer\ k\},$$

$$\{x \in Z \mid x = 3k + 1, for\ some\ integer\ k\}, \text{ and}$$

$$\{x \in Z \mid x = 3k + 2, for\ some\ integer\ k\}.$$

- In words, the three classes of congruence modulo 3 are

(1) the set of all integers that are divisible by 3,

(2) the set of all integers that leave a remainder of 1 when divided by 3, and

(3) the set of all integers that leave a remainder of 2 when divided by 3.

Congruence Modulo n

- Let m and n be integers and let d be a positive integer.
- We say that m is congruent to n modulo d and write

$$m \equiv n \ (mod \ d) \text{ if, and only if, } d \mid (m - n).$$

- Symbolically:

$$m \equiv n \ (mod \ d) \leftrightarrow d \mid (m - n)$$

Evaluating Congruences

- Determine which of the following congruences are true and which are false.

a. $12 \equiv 7 \pmod{5}$

   – *True.* $12 - 7 = 5 = 5 \cdot 1$. *Hence* $5 \mid (12 - 7)$, *and so* $12 \equiv 7 \pmod{5}$.

b. $6 \equiv -8 \pmod{4}$

   – *False.* $6 - (-8) = 14$, *and* $4 \nmid 14$ *because* $14 \neq 4 \cdot k$ *for any integer* k. *Consequently,* $6 \not\equiv -8 \pmod{4}$.

c. $3 \equiv 3 \pmod{7}$

   – *True.* $3 - 3 = 0 = 7 \cdot 0$. *Hence* $7 \mid (3 - 3)$, *and so* $3 \equiv 3 \pmod{7}$.

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Rational Numbers As Equivalence Classes

- Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$$A = Z \times (Z - \{0\})$$

- Define a relation R on A as follows: For all $(a, b), (c, d) \in A$,

$$(a, b) R\ (c, d) \leftrightarrow ad = bc$$

- The fact is that R is an equivalence relation.

  Describe the distinct equivalence classes of R.

Rational Numbers As Equivalence Classes

- There is one equivalence class for each distinct rational number.
- Each equivalence class consists of all ordered pairs (a, b) that, if written as fractions $\frac{a}{b}$, would equal each other.
- The reason for this is that the condition for two rational numbers to be equal is the same as the condition for two ordered pairs to be related.
- For instance, the class of $(1, 2)$ is

$$[(1,2)] = \{(1,2), (-1,-2), (2,4), (-2,-4), (3,6), (-3,-6), \dots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth.