جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 9

Counting and Probability

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Possibility Trees and the Multiplication Rule

- Possibility Trees
- The Multiplication Rule
- Permutations

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Motivation

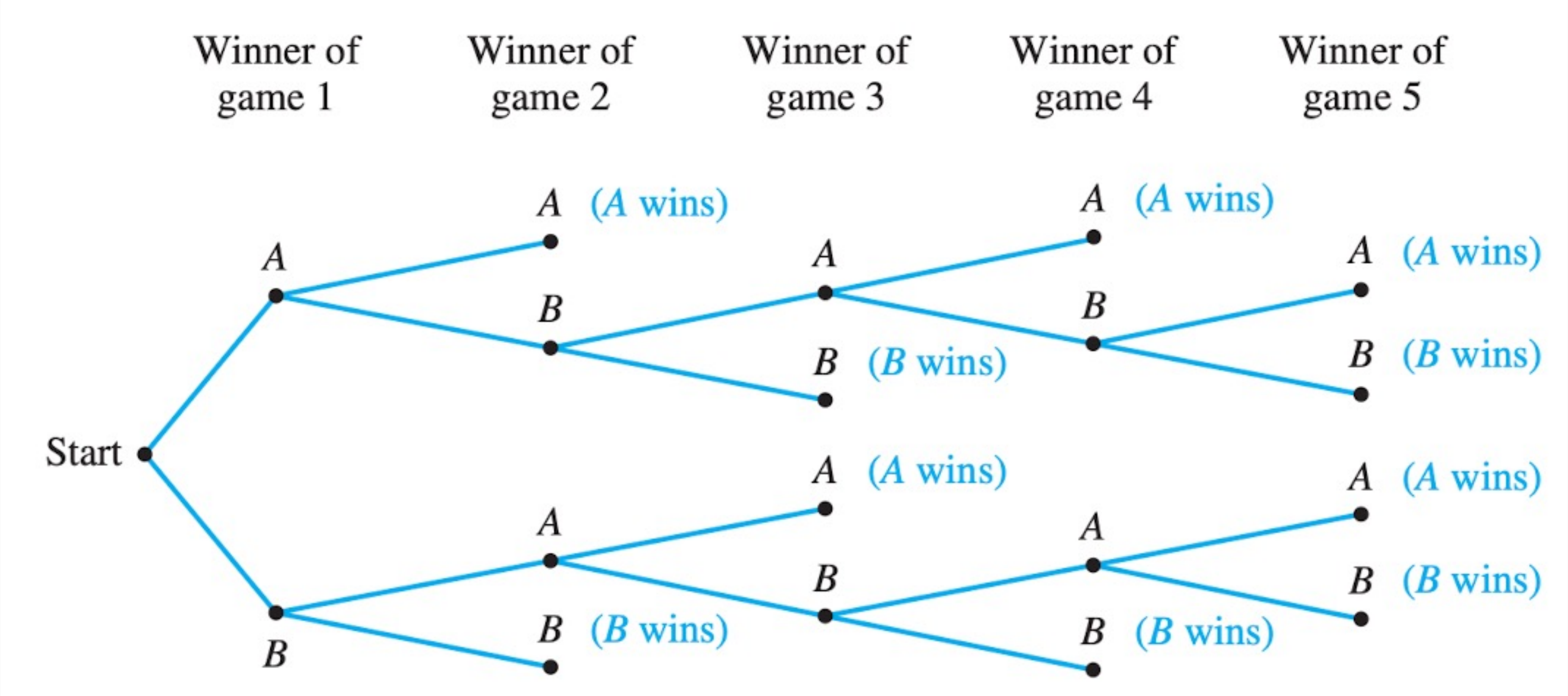
- A tree structure is a useful tool for keeping systematic track of all possibilities in situations in which events happen in order.
- If you want to draw a tree to describe tossing two coins and observing the outcome, it will look like this:

Possibility Trees

- Teams A and B are to play each other repeatedly until one wins two games in a row or a total of three games.
- How many ways can the tournament be played?

- We can use possibility trees to represent the possible ways for the tournament to be played by the distinct paths from root to leaf
  - *The label on each branching point indicates the winner of the game. The notations in parentheses indicate the winner of the tournament.*

Possibility Trees



Possibility Trees

- The fact that there are ten paths from the root of the tree to its leaves shows that there are ten possible ways for the tournament to be played.

	A wins	B wins
2 games	A-A	B-B
3 games	B–A–A	A–B–B
4 games	A–B–A–A	B–A–B–B
5 games	A–B–A–B–A B–A–B–A–A	A–B–A–B–B B–A–B–A–B

- In five cases A wins, and in the other five B wins.
- The least number of games that must be played to determine a winner is two, and the most that will need to be played is five.

Possibility Trees

- Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?
- Since all the possible ways of playing the tournament are assumed to be equally likely, and the listing shows that five games are needed in four different cases (A–B-A–B–A, A–B–A–B–B, B–A–B–A–B, and B–A–B–A–A), the probability that five games are needed is $4/10 = 2/5 = 40\%$.

Possibility Trees and the Multiplication Rule

- Possibility Trees
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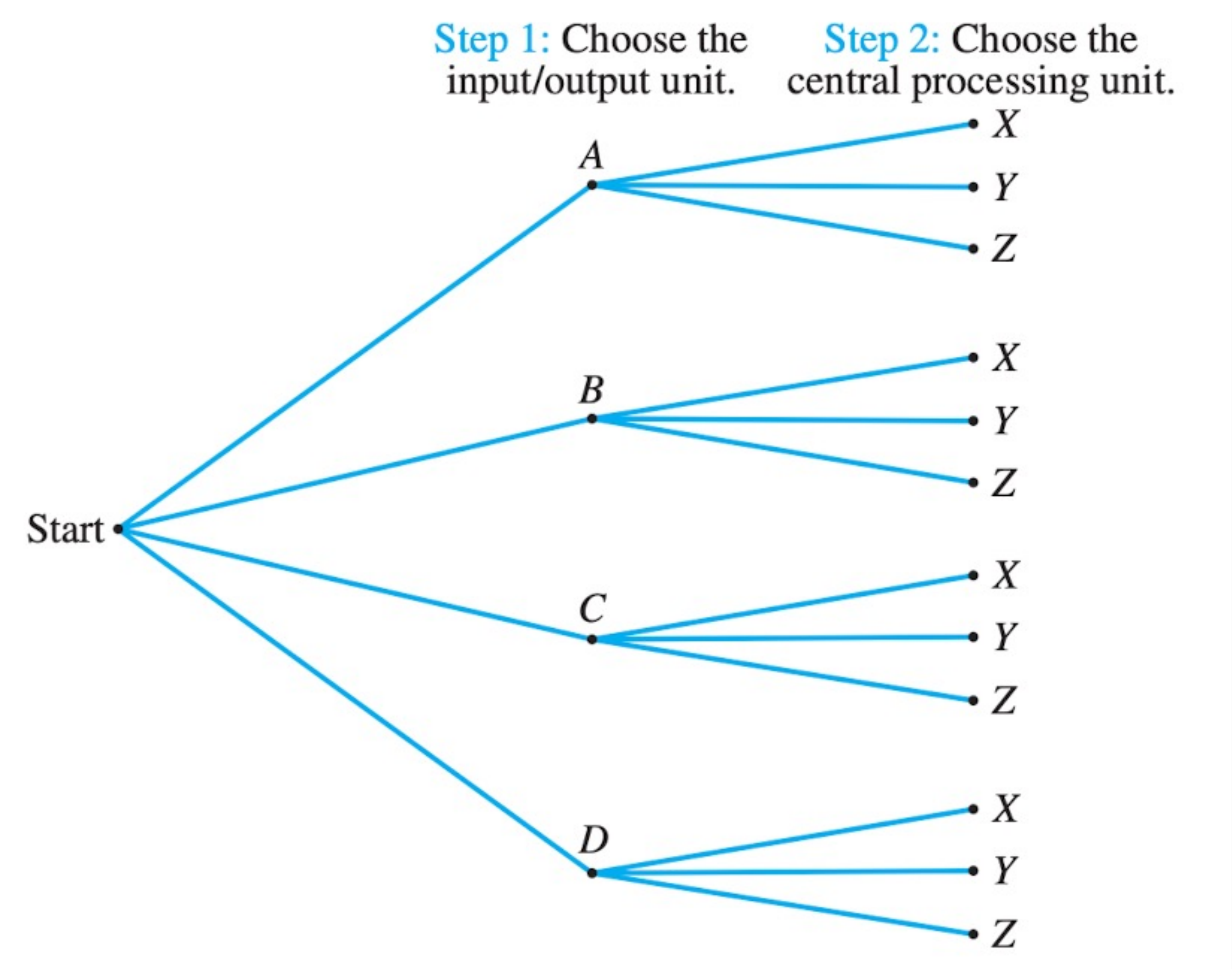
Example

- Suppose a computer installation has four input/output units (A, B, C, and D) and three central processing units (X , Y, and Z). Any input/output unit can be paired with any central processing unit.
- How many ways are there to pair an input/output unit with a central processing unit?
- To answer this question, imagine the pairing of the two types of units as a two-step operation:
  - *<u>Step 1:</u> Choose the input/output unit.*
  - *<u>Step 2:</u> Choose the central processing unit.*

Example – cont.



- The topmost path from root to leaf indicates that input/output unit A is to be paired with central processing unit X. The next lower branch indicates that input/output unit A is to be paired with central processing unit Y. And so forth.
- Thus the total number of ways to pair the two types of units is the same as the number of branches of the tree, which is

$$3 + 3 + 3 + 3 = 4 \cdot 3 = 12$$

The Multiplication Rule

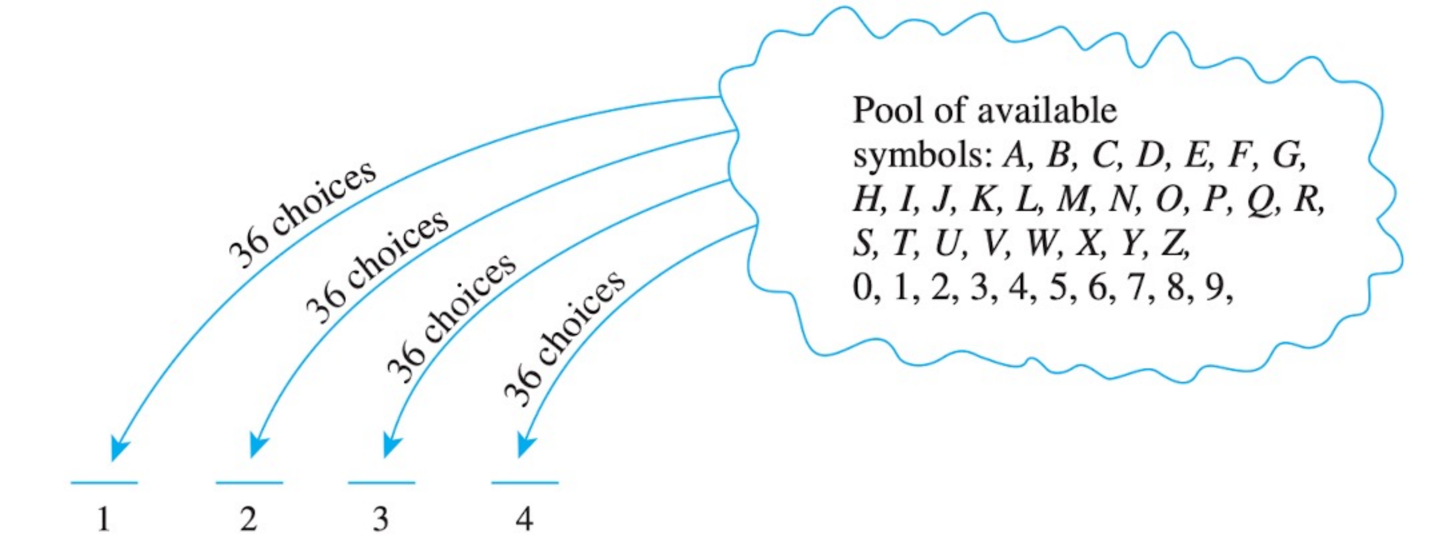
- If an operation consists of k steps and
  - *the first step can be performed in n_1 ways,*
  - *the second step can be performed in n_2 ways, __regardless of how the first step was performed__,*

    ...
  - *the k^{th} step can be performed in n_k ways, __regardless of how the preceding steps were performed,__*

  then the entire operation can be performed in $n_1 \cdot n_2 \cdot \cdots \cdot n_k$ ways.

Example – Number of Passwords

- If you are generating a random password that consists of any four symbols chosen from the 26 letters in the English alphabet (all uppercase) and the ten digits, with repetition allowed. How many different passwords are possible?



Example – Number of Passwords

- If you are generating a random password that consists of any four symbols chosen from the 26 letters in the English alphabet (all uppercase) and the ten digits, with repetition allowed. How many different passwords are possible?
  - *Step 1: Choose the first symbol.*
  - *Step 2: Choose the second symbol.*
  - *Step 3: Choose the third symbol.*
  - *Step 4: Choose the fourth symbol.*
- There is a fixed number of ways to perform each step, namely 36, ***regardless of how preceding steps were performed***. And so, by the multiplication rule, there are

$$36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1{,}679{,}616 \text{ passwords in all.}$$

Example – Number of Passwords

- What if repeating is not allowed in your random password generator?
- This means that whenever you chose a symbol in any step, the pool of possible symbols for the following step will be reduced by one.
  - *There are 36 ways to choose the first symbol,*
  - *35 ways to choose the second (since the first symbol cannot be used again),*
  - *34 ways to choose the third (since the first two symbols cannot be reused),*
  - *and 33 ways to choose the fourth (since the first three symbols cannot be reused).*
- Thus, the multiplication rule can be applied to conclude that there are

$36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720$ password with no repeating symbols

Example – Number of Passwords

- If all passwords are equally likely, what is the probability that a password chosen at random contains no repeated symbol?

$$P(password\ with\ no\ repeating\ symbols) = \frac{N(passwords\ with\ no\ repeating\ symbols)}{N(total\ possible\ passwords)}$$

$$= \frac{1{,}413{,}720}{1{,}679{,}616} \cong 0.8417$$

The Number of Elements in a Cartesian Product

- Suppose A_1, A_2, A_3, and A_4 are sets with n_1, n_2, n_3, and n_4 elements, respectively. Show that the set $A_1 \times A_2 \times A_3 \times A_4$ has $n_1 \cdot n_2 \cdot n_3 \cdot n_4$ elements.
- Each element in $A_1 \times A_2 \times A_3 \times A_4$ is an ordered 4-tuple of the form (a_1, a_2, a_3, a_4), where $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$, and $a_4 \in A_4$.
- Imagine the process of constructing these ordered tuples as a four-step operation:
  - *Step 1: Choose the first element of the 4-tuple.*
  - *Step 2: Choose the second element of the 4-tuple.*
  - *Step 3: Choose the third element of the 4-tuple.*
  - *Step 4: Choose the fourth element of the 4-tuple.*
- There are n_1 ways to perform step 1, n_2 ways to perform step 2, n_3 ways to perform step 3, and n_4 ways to perform step 4.
- Hence, by the multiplication rule, there are $n_1 \cdot n_2 \cdot n_3 \cdot n_4$ ways to perform the entire operation. Therefore, there are $n_1 \cdot n_2 \cdot n_3 \cdot n_4$ distinct 4-tuples in $A_1 \times A_2 \times A_3 \times A_4$

Counting the Number of Iterations of a Nested Loop

- Consider the following nested for-loop

for $i := 1$ **to** 4
    for $j := 1$ **to** 3
        [*Statements in body of inner loop.*
        None contain branching statements
        that lead out of the inner loop.]
    next j
next i

- How many times will the inner loop be iterated when the algorithm is implemented and run?

Counting the Number of Iterations of a Nested Loop

- The outer loop is iterated four times, and during each iteration of the outer loop, there are three iterations of the inner loop.

i	1 →			2 →			3 →			4 →		
j	1	2	3	1	2	3	1	2	3	1	2	3

$3 + 3 + 3 + 3 = 12$

- Hence by the multiplication rule, the total number of iterations of the inner loop is $4 \cdot 3 = 12$

Counting Example

- Three officers — a president, a treasurer, and a secretary — are to be chosen from among four people: Ann, Bob, Cyd, and Dan.
- Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

  - *There are three choices for president (all except Ann),*
  - *three choices for treasurer (all except the one chosen as president),*
  - *and two choices for secretary (Cyd or Dan).*

Therefore, by the multiplication rule, there are $3 \cdot 3 \cdot 2 = 18$ *choices in all.*

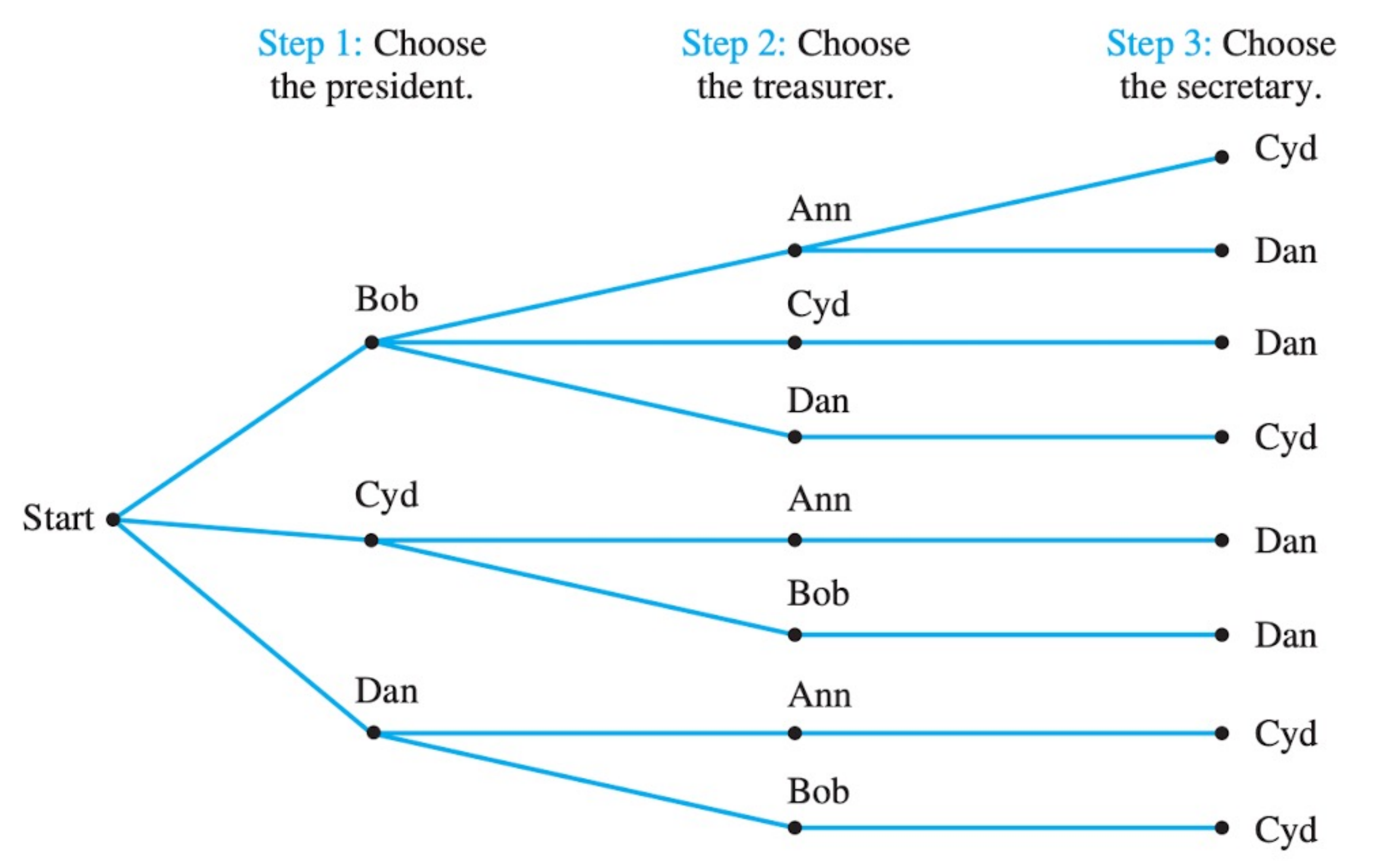
WRONG

Counting Example

- Three officers — a president, a treasurer, and a secretary — are to be chosen from among four people: Ann, Bob, Cyd, and Dan.
- Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

  - *The number of ways to choose the secretary varies depending on who is chosen for president and treasurer.*
  - *For instance, if Bob is chosen for president and Ann for treasurer, then there are two choices for secretary: Cyd and Dan.*
  - *But if Bob is chosen for president and Cyd for treasurer, then there is just one choice for secretary: Dan.*

Counting Example



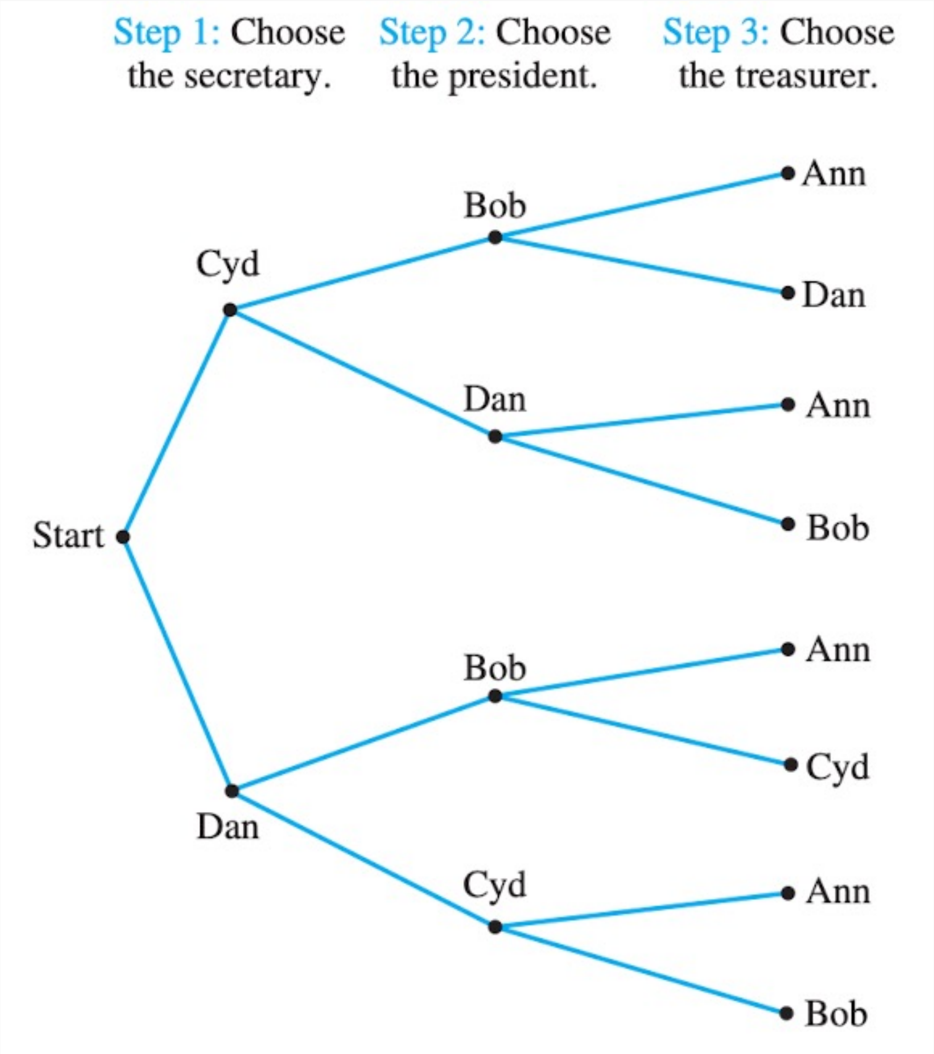
- From the tree it is easy to see that there are only eight ways to choose a president, treasurer, and secretary so as to satisfy the given conditions.
- This tree is not homogenous, and that's why the multiplication rule does not apply.

Counting Example

- The steps of solving the previous example can be reordered in a slightly different way so that the number of ways to perform each step is constant regardless of the way previous steps were performed.
  - *Step 1: Choose the secretary - two ways (either Cyd or Dan may be chosen)*
  - *Step 2: Choose the president - two ways (neither Ann nor the person chosen in step 1 may be chosen but either of the other two may)*
  - *Step 3: Choose the treasurer – two ways (either of the two people not chosen as secretary or president may be chosen as treasurer)*
- Thus, by the multiplication rule, the total number of ways to choose officers is

$$2 \cdot 2 \cdot 2 = 8$$

Counting Example



Possibility Trees and the Multiplication Rule

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Permutations

- A permutation of a set of objects is an ordering of the objects in a row.
- For example, the set of elements a, b, and c has six permutations.

abc acb cba bac bca cab

Permutations

- In general, given a set of n objects, how many permutations does the set have?
  - *Step 1: Choose an element to write first.*
  - *Step 2: Choose an element to write second.*
  - *. . . .*
  - *Step n: Choose an element to write* n^{th}.

Permutations

- The number of ways to perform each of these steps:
  - *Any element of the set can be chosen in step 1, so there are n ways to perform step 1.*
  - *Any element except that chosen in step 1 can be chosen in step 2, so there are* $n - 1$ *ways to perform step 2.*
  - *In general, the number of ways to perform each successive step is one less than the number of ways to perform the preceding step.*
  - *At the point when the* n^{th} *element is chosen, there is only one element left, so there is only one way to perform step* n*.*

- Hence, by the multiplication rule, there are

$$n(n - 1)(n - 2) \cdots 2 \cdot 1 = n!$$

Permutations

- Theorem:

  For any integer n with $n \geq 1$,

  the number of permutations of a set with n elements is $n!$.

Permutations of the Letters in a Word

- How many ways can the letters in the word COMPUTER be arranged in a row?

- All the eight letters in the word COMPUTER are ***distinct***, so the number of ways in which we can arrange the letters equals the number of permutations of a set of eight elements. This equals $8! = 40{,}320$.s

Permutations of the Letters in a Word

- How many ways can the letters in the word COMPUTER be arranged if the letters CO must remain next to each other (in order) as a unit?

- If the letter group CO is treated as a unit, then there are effectively only seven objects that are to be arranged in a row.

Hence there are as many ways to write the letters as there are permutations of a set of seven elements, namely $7! = 5{,}040$.

Permutations of the Letters in a Word

- If letters of the word COMPUTER are randomly arranged in a row, what is the probability that the letters CO remain next to each other (in order) as a unit?
- When the letters are arranged randomly in a row, the total number of arrangements is 40,320, and the number of arrangements with the letters CO next to each other (in order) as a unit is 5,040.

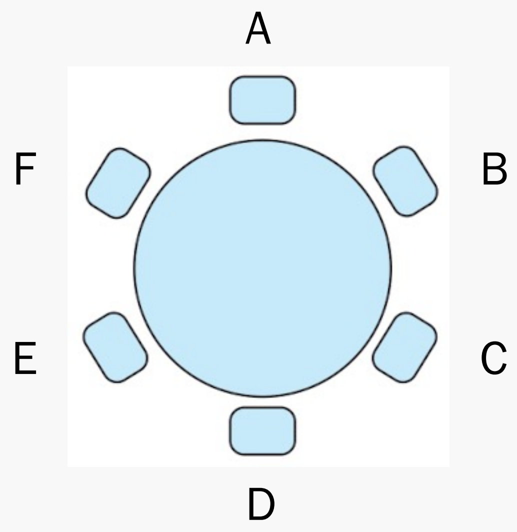
  Thus the probability is

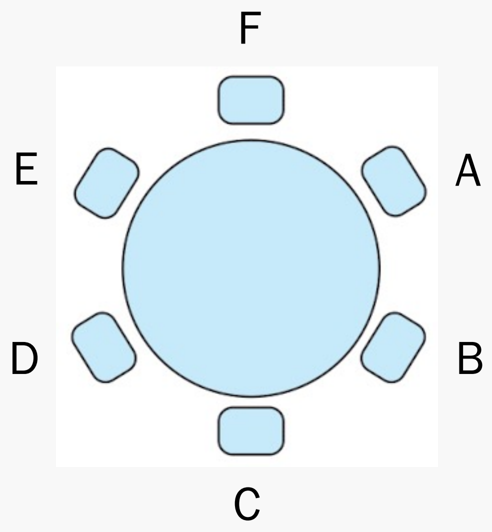
$$\frac{5,040}{40,320} = \frac{1}{8} = 12.5\%$$

Permutations of Objects Around a Circle

- At a meeting of diplomats, the six participants are to be seated around a circular table.
- Since the table has no ends to confer particular status, it doesn’t matter who sits in which chair. But it does matter how the diplomats are seated relative to each other.
- In other words, two seatings are considered the same if one is a rotation of the other.
- How many different ways can the diplomats be seated?

Permutations of Objects Around a Circle

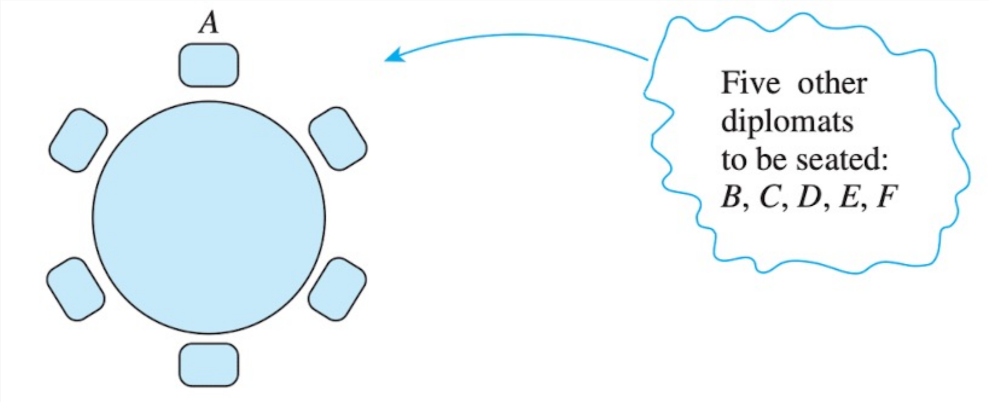




These seatings are considered the same.

Permutations of Objects Around a Circle

- Since only relative position matters, you can start with any diplomat (say A), place that diplomat anywhere (say in the top seat), and then consider all arrangements of the other diplomats around that one.



- B through F can be arranged in the seats around diplomat A in all possible orders. So, there are $5! = 120$ ways to seat the group.

Permutations of Selected Elements

- Given the set $\{a, b, c\}$, there are six ways to select two letters from the set and write them in order.

  ab ac ba bc ca cb

- Each such ordering of two elements of $\{a, b, c\}$ is called a 2-permutation of $\{a, b, c\}$.

Permutations of Selected Elements

- An r-permutation of a set of n elements is an ordered selection of r elements taken from the set of n elements.

  The number of r-permutations of a set of n elements is denoted $P(n,r)$.

- If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

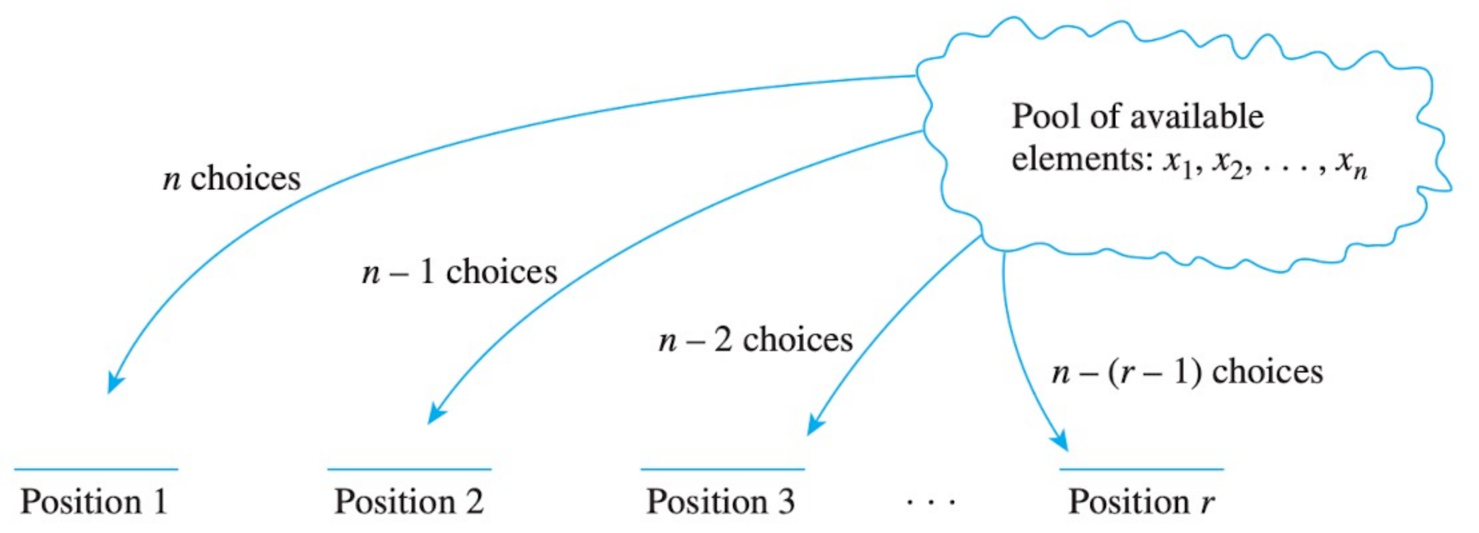
$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

  or, equivalently,

$$P(n,r) = \frac{n!}{(n-r)!}$$

- Alternative notations you might see are ${}_nP_r,\ P_{n,r},\ {}^nP_r$.

Permutations of Selected Elements



$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}$$
$$= n(n-1)(n-2)\cdots(n-r+1).$$

Evaluating r-Permutations

- Evaluate $P(5, 2)$.

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 5 \cdot 4 = 20$$

- How many 4-permutations are there of a set of seven objects?

$$P(7,4) = \frac{7!}{(7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$$

- How many 5-permutations are there of a set of five objects?

$$P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120$$

Permutations of Selected Letters of a Word

- How many different ways can three of the letters of the word BYTES be chosen and written in a row?

The answer equals the number of 3-permutations of a set of five elements. This equals

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 5 \cdot 4 \cdot 3 = 60$$

- How many different ways can this be done if the first letter must be B?

Since the first letter must be B, there are effectively only two letters to be chosen and placed in the other two positions out of the remaining 4.

$$P(4,2) = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 4 \cdot 3 = 12$$

Proving a Property of $P(n, r)$

- Prove that for all integers $n \geq 2$,
$$P(n,2) + P(n,1) = n^2$$

- Suppose n is an integer that is greater than or equal to 2
$$P(n,2) = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1)$$
and
$$P(n,1) = \frac{n(n-1)!}{(n-1)!} = n$$
hence
$$P(n,2) + P(n,1) = n(n-1) + n = n^2 - n + n = n^2$$