جامعة بيرزيت
BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 9

Counting and Probability

Counting and Probability

Counting and Probability

Motivation

- In the last section we discussed counting problems that can be solved using possibility trees.
- In this section we look at counting problems that can be solved by counting the number of elements in the union of two sets, the difference of two sets, or the intersection of two sets.

Counting Elements of Disjoint Sets: The Addition Rule

- The Addition Rule
- The Difference Rule
- The Inclusion/Exclusion Rule

Counting Elements of Disjoint Sets: The Addition Rule

- The Addition Rule
- The Difference Rule
- The Inclusion/Exclusion Rule

The Addition Rule

- This rule states that the number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

- Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \dots, A_k$.

  Then

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k)$$

- You can prove this rule using mathematical induction. (do it at home)

The Addition Rule

- If we know that in the COMP233 class, there are the following students:
  - *25 Computer Science students*
  - *23 Computer Engineering students*

  and we know that there are no other students from a different discipline.
  Can we determine how many students there are in the COMP233 class?

- The total number of students is the number of Computer Science students and Computer Engineering students: $25 + 23 = 48$ students.

Counting Passwords with Three or Fewer Letters

- A computer access password consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. How many different passwords are possible?

- The set of all passwords can be partitioned into subsets consisting of those of length 1, those of length 2, and those of length 3.
- These subsets are mutually disjoint.
- The total number of passwords that are possible is the summation of the total number of 1-letter passwords, 2-letter passwords, and 3-letter passwords.

Counting Passwords with Three or Fewer Letters

- A computer access password consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. How many different passwords are possible?

number of passwords of length $1 = 26$

number of passwords of length $2 = 26 \times 26 = 26^2$

number of passwords of length $3 = 26 \times 26 \times 26 = 26^3$

Hence, the total number of passwords = $26 + 26^2 + 26^3 = 18{,}278$

Counting the Number of Integers Divisible by 5

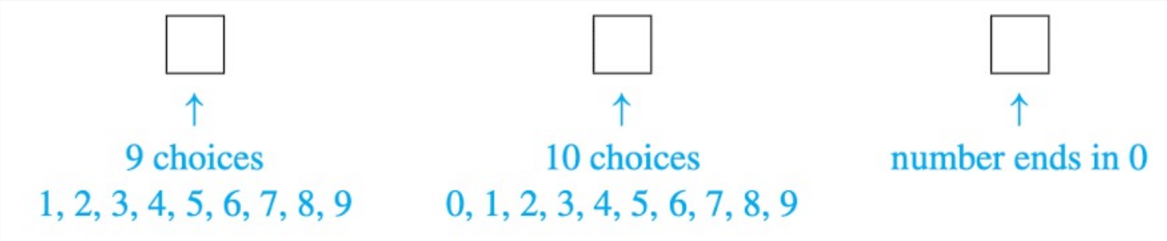
- How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

- We have already covered a solution for this question in section 9.1.
- A different approach to solving it using the addition rule is:

  Integers that are divisible by 5 end either in 5 or in 0. Thus, the set of all three-digit integers that are divisible by 5 can be split into two mutually disjoint subsets A_1 and A_2, where A_1 is the set of all three-digit numbers that end in 0, and A_2 is the set of all three-digit numbers that end in 5.

Counting the Number of Integers Divisible by 5

- How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?
- The number of possible integers that end in 0 are:



$\mathrm{N}(A_1) = 9 \times 10 \times 1 = 90$

- Similar reasoning (using 5 instead of 0) shows that $\mathrm{N}(A_2) = 90$ also.
- The total number of three-digit integers that are divisible by $5 = N(A_1) + N(A_2) = 90 + 90 = 180$.

Counting Elements of Disjoint Sets: The Addition Rule

- The Addition Rule
- The Difference Rule
- The Inclusion/Exclusion Rule

The Difference Rule

- An important consequence of the addition rule is the fact that if the number of elements in a set A and the number in a subset B of A are both known, then the number of elements that are in A and not in B can be computed.

- If A is a finite set and B is a subset of A, then
$$N(A - B) = N(A) - N(B)$$

The Difference Rule

- If we know that in the COMP233 class, there are 48 total students. 25 of them are Computer Science students, and the rest are Computer Engineering students.

  Can we determine how many Computer Engineering students there are in the COMP233 class?

- The number of Computer Engineering students is the total number of students in the class minus the number of Computer Science students: $48 - 25 = 23$ students.

Counting Passwords with Repeated Symbols

- The passwords discussed in Section 9.2 are made from exactly four symbols chosen from the 26 letters of the alphabet and the ten digits, with repetitions allowed.

- How many passwords contain repeated symbols?
- According to our previous solution, there are $36^4 = 1{,}679{,}616$ passwords when repetition is allowed, and there are $36 \times 35 \times 34 \times 33 = 1{,}413{,}720$ passwords when repetition is not allowed.
- Thus, by the difference rule, there are $1{,}679{,}616 - 1{,}413{,}720 = 265{,}896$ passwords that contain at least one repeated symbol.

Counting Passwords with Repeated Symbols

- The passwords discussed in Section 9.2 are made from exactly four symbols chosen from the 26 letters of the alphabet and the ten digits, with repetitions allowed.

- If all passwords are equally likely, what is the probability that a randomly chosen password contains a repeated symbol?

We have already shown that there are $1{,}679{,}616$ passwords in total, and we just showed that $265{,}896$ of these contain at least one repeated symbol.

Thus, by the equally likely probability formula, the probability that a randomly chosen password contains a repeated symbol is $\frac{265{,}896}{1{,}679{,}616} \cong 0.158 = 15.8\%$

Counting Passwords with Repeated Symbols

- Alternatively, if we consider S to be the set of all passwords, and A is the set of all passwords with no repeating symbols, then $S - A$ is the set of all passwords with repeating symbols.

$P(S-A) = \frac{N(S-A)}{N(S)}$ by definition of probability in equally likely case.

$= \frac{N(S)-N(A)}{N(S)}$ by the difference rules

$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$ by the laws of fractions

$= 1 - P(A)$ by definition of probability in the equally likely case.

$\cong 1 - 0.842$ from the solution we reached in Section 9.2

$= 0.158 = 15.8\%$

Probability of the Complement of an Event

- From the previous example, we can infer a more general property of probabilities:

The probability of the complement of an event is obtained by subtracting the probability of the event from the number 1.

- If S is a finite sample space and A is an event in S, then

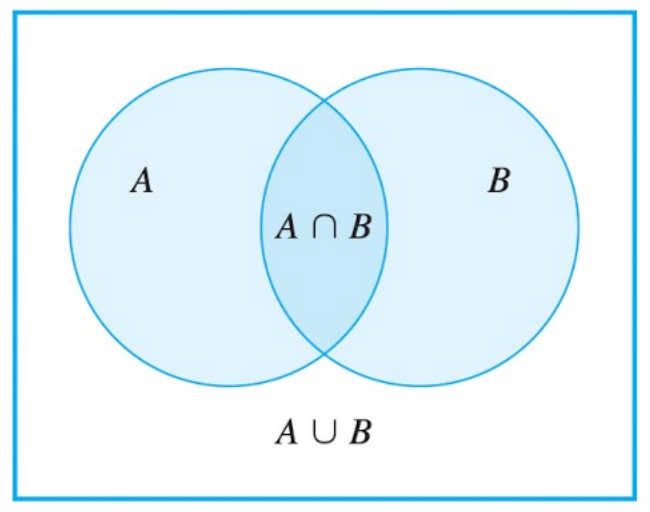
$$P(A^c) = 1 - P(A)$$

Counting Elements of Disjoint Sets: The Addition Rule

- The Addition Rule
- The Difference Rule
- The Inclusion/Exclusion Rule

The Inclusion/Exclusion Rule

- The addition rule says how many elements are in a union of sets if the sets are mutually disjoint.
- Now consider the question of how to determine the number of elements in a union of sets when some of the sets overlap.



The Inclusion/Exclusion Rule

- If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

Counting Elements of a General Union

- How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

- Let A = the set of all integers from 1 through 1,000 that are multiples of 3.
- Let B = the set of all integers from 1 through 1,000 that are multiples of 5.
- Then $A \cup B$ = the set of all integers from 1 through 1,000 that are multiples of 3 or multiples of 5
- And $A \cap B$ = the set of all integers from 1 through 1,000 that are multiples of both 3 and 5 = the set of all integers from 1 through 1,000 that are multiples of 15.

Counting Elements of a General Union

- Because every third integer from 3 through 999 is a multiple of 3, each can be represented in the form $3k$, for some integer k from 1 through 333.
  Hence there are 333 multiples of 3 from 1 through 1,000, and so $N(A) = 333$.

- Similarly, each multiple of 5 from 1 through 1,000 has the form $5k$, for some integer k from 1 through 200.
  Thus, there are 200 multiples of 5 from 1 through 1,000 and $N(B) = 200$.

- Finally, each multiple of 15 from 1 through 1,000 has the form $15k$, for some integer k from 1 through 66 (since $990 = 66 \cdot 15$).
  Hence there are 66 multiples of 15 from 1 through 1,000, and $N(A \cap B) = 66$.

Counting Elements of a General Union

- It follows by the inclusion/exclusion rule that

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$
$$= 333 + 200 - 66$$
$$= 467$$

- Thus, 467 integers from 1 through 1,000 are multiples of 3 or multiples of 5.

Counting Elements of a General Union

- How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

- There are 1,000 integers from 1 through 1,000, and we have shown that 467 of these are multiples of 3 or multiples of 5.
- Thus, by the set difference rule, there are $1{,}000 - 467 = 533$ numbers that are neither multiples of 3 nor multiples of 5.

Counting the Number of Elements in an Intersection

- A professor in a discrete mathematics class passes out a form asking students to check all the mathematics and computer science courses they have recently taken. The finding is that out of a total of 50 students in the class,
  - *30 took precalculus;*
  - *18 took calculus;*
  - *26 took Java;*
  - *9 took both precalculus and calculus;*
  - *16 took both precalculus and Java;*
  - *8 took both calculus and Java;*
  - *47 took at least one of the three courses.*

Counting the Number of Elements in an Intersection

- How many students did not take any of the three courses?

By the difference rule, the number of students who did not take any of the three courses equals the number in the class minus the number who took at least one course.

Thus, the number of students who did not take any of the three courses is

$$50 - 47 = 3$$

Counting the Number of Elements in an Intersection

- How many students took all three courses?

Let

P = the set of students who took precalculus
C = the set of students who took calculus
J = the set of students who took Java

Then, by the inclusion/exclusion rule,

$$N(P \cup C \cup J) = N(P) + N(C) + N(J) - N(P \cap C) - N(P \cap J) - N(C \cap J) + N(P \cap C \cap J)$$

Substituting known values, we get

$$47 = 30 + 26 + 18 - 9 - 16 - 8 + N(P \cap C \cap J)$$

Solving for $N(P \cap C \cap J)$ gives

$$N(P \cap C \cap J) = 6$$

Hence, there are six students who took all three courses.

Counting the Number of Elements in an Intersection

- How many students took precalculus and calculus but not Java?

Since $N(P \cap C \cap J) = 6$, put the number 6 inside the innermost region. Then work outward to find the numbers of students represented by the other regions of the diagram.