BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233
Discrete Mathematics

CHAPTER 9

Counting and Probability

Counting and Probability

9.1 Introduction to Counting and Probability

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: The Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

Counting and Probability

Motivation

- Suppose three members of a group of four are to be chosen to work as a team on a special project. How many distinct three-person teams can be selected?

- For example, if group contained Ann, Bob, Cyd, and Dan, we could form the following three-person teams:
  - *Ann, Bob, and Cyd*
  - *Ann, Bob, and Dan*
  - *Ann, Cyd, and Dan*
  - *Bob, Cyd, and Dan*
- This is called a 3-combination of the set of people we have.

Counting Subsets of a Set: Combinations

- Combinations
- Calculating combinations
- Neither, Both, At Least, At Most
- Permutations of a Set with Repeated Elements

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r-Combinations

- Let n and r be nonnegative integers with $r \leq n$. An ***r-combination*** of a set of n elements is a subset of r of the n elements. The symbol

$$\binom{n}{r}$$

which is read "n choose r", denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

- Alternative notations include $C(n, r)$, ${}_nC_r$, $C_{n,r}$, and ${}^nC_r$

Permutations vs. Combinations

- An r-permutation, $P(n, r)$ is an ordered selection of r elements from a set of n elements.
  In an ordered selection, it is not only what elements are chosen but also the order in which they are chosen that matters.
  Two ordered selections are said to be the same if the elements chosen are the same and also if the elements are chosen in the same order.
- An r-combination, $\binom{n}{r}$ is an unordered selection of r elements from a set of n elements.
  In an unordered selection, on the other hand, it is only the identity of the chosen elements that matters.
  Two unordered selections are said to be the same if they consist of the same elements, regardless of the order in which the elements are chosen.

Unordered Selections

- How many unordered selections of two elements can be made from the set $\{0,1,2,3\}$?

$\{0,1\},\{0,2\},\{0,3\}$	all the subsets containing the element 0
$\{1,2\},\{1,3\}$	all the subsets containing the element 1 not already listed
$\{2,3\}$	all the subsets containing the element 2 not already listed

Since this listing exhausts all possibilities, there are six subsets in all. Thus $\binom{4}{2} = 6$,

which is the number of unordered selections of two elements from a set of four.

Calculating Combinations

- When the values of n and r are small, it is reasonable to calculate values of r using the method of complete enumeration (listing all possibilities).

- But when n and r are large, it is not feasible to compute these numbers by listing and counting all possibilities.

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Calculating Combinations

- To derive the formula of calculating combinations, let's try this exercise.

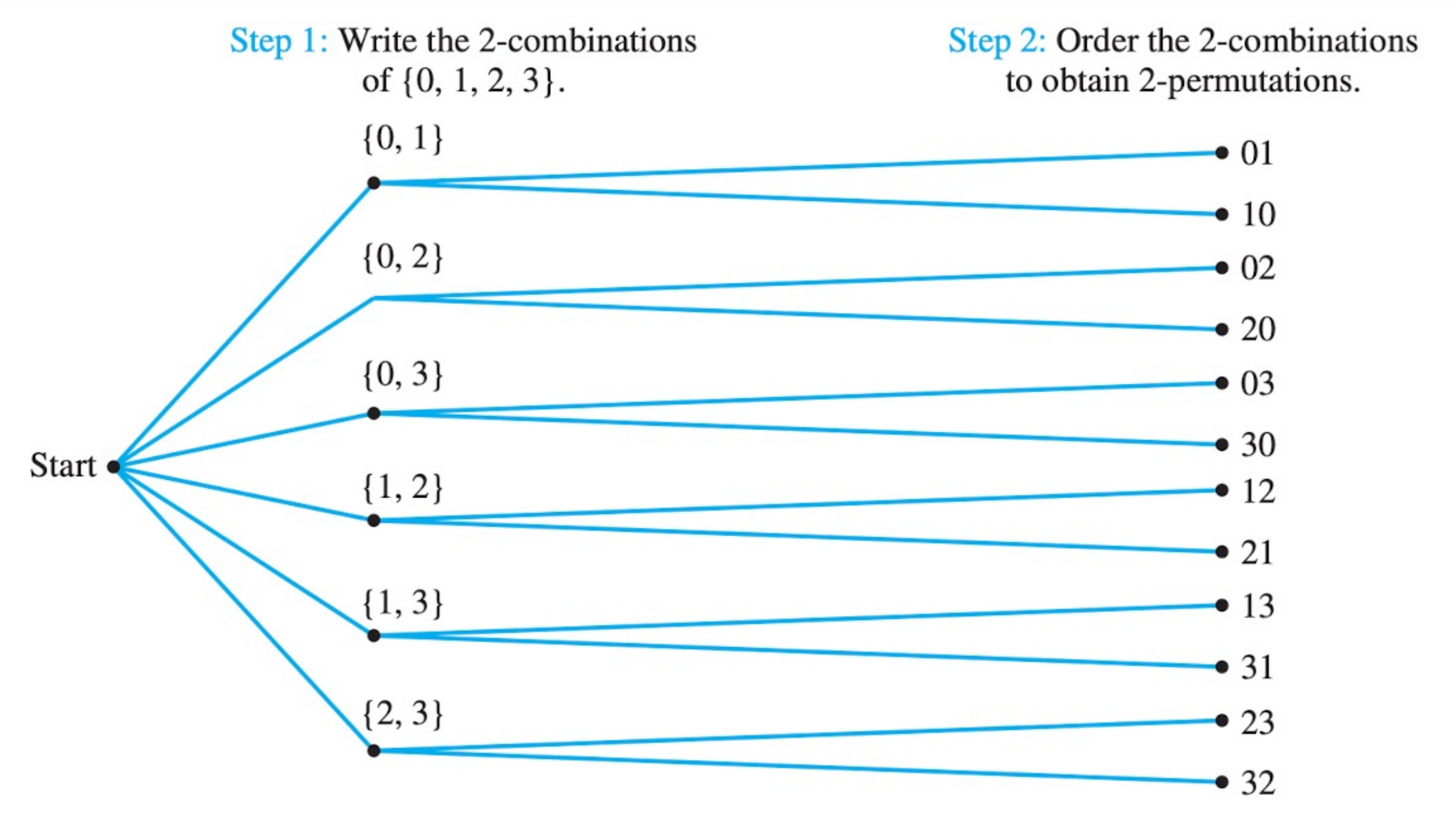
- Write all 2-combinations and 2-permutations of the set $\{0,1,2,3\}$. Find an equation relating the number of 2-permutations, $P(4,2)$, and the number of 2-combinations, $\binom{4}{2}$, and solve the equation for $\binom{4}{2}$.

Calculating Combinations

- The number of 2-permutations of the set $\{0,1,2,3\}$ is $P(4,2)$, which equals
$$\frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot 2!}{2!} = 4 \cdot 3 = 12$$

- Now the act of constructing a 2-permutation of $\{0,1,2,3\}$ can be thought of as a two-step process:
  - *Step 1: Choose a subset of two elements from* $\{0,1,2,3\}$*.*
  - *Step 2: Choose an ordering for the two-element subset.*

Calculating Combinations



Calculating Combinations

- The number of ways to perform step 1 is $\binom{4}{2}$, the same as the number of subsets of size 2 that can be chosen from $\{0,1,2,3\}$.
- The number of ways to perform step 2 is 2!, the number of ways to order the elements in a subset of size 2.
- Because the number of ways of performing the whole process is the number of 2-permutations of the set $\{0,1,2,3\}$, which equals $P(4,2)$, it follows from the product rule that:

$$P(4,2) = \binom{4}{2} \cdot 2!$$

Calculating Combinations

- Solving the equation for $\binom{4}{2}$ results in

$$\binom{4}{2} = \frac{P(4,2)}{2!}$$

- And we know that

$$P(4,2) = \frac{4!}{(4-2)!}$$

- Hence, substituting yields

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{4 \cdot 3 \cdot 2!}{2! \cdot 2!} = \frac{4 \cdot 3}{2} = \frac{12}{2} = 6$$

Calculating Combinations

- The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n,r)}{r!}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

when n and r are nonnegative integers with $r \leq n$.

n choose 0

- If r is zero and n is any nonnegative integer, then $\binom{n}{0}$ is the number of subsets of size zero of a set with n elements.
- We know that there is only one set that does not have any elements. Consequently, $\binom{n}{0} = 1$

- If we want to apply the previous formula,

$$\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{0!\,n!} = \frac{1}{0!} = \frac{1}{1} = 1$$

Calculating the Number of Teams

- Suppose five members of a group of twelve are to be chosen to work as a team on a special project. How many distinct five-person teams can be selected?

- The number of distinct five-person teams is the same as the number of subsets of 12 size 5 (or *5-combinations*) that can be chosen from the set of twelve. This number is $\binom{12}{5}$.

$$\binom{12}{5} = \frac{12!}{5!\,(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5! \cdot 7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5!} = 792$$

Counting Subsets of a Set: Combinations

- Combinations
- Calculating combinations
- Neither, Both, At Least, At Most
- Permutations of a Set with Repeated Elements

Teams That Contain Both or Neither

- Suppose two members of the group of twelve insist on working as a pair—any team must contain either both or neither. How many five-person teams can be formed?
- Call the two members of the group that insist on working as a pair A and B. Then any team formed must contain both A and B or neither A nor B.
- Because a team that contains both A and B contains exactly three other people from the remaining ten in the group, there are as many such teams as there are subsets of three people that can be chosen from the remaining ten.

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\,7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3! \cdot 7!} = 120$$

Teams That Contain Both or Neither

- Suppose two members of the group of twelve insist on working as a pair—any team must contain either both or neither. How many five-person teams can be formed?
- Call the two members of the group that insist on working as a pair A and B. Then any team formed must contain both A and B or neither A nor B.
- Because a team that contains neither A nor B contains exactly five people from the remaining ten, there are as many such teams as there are subsets of five people that can be chosen from the remaining ten.

$$\binom{10}{5} = \frac{10!}{5!\,(10-5)!} = \frac{10!}{5!\,5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 5!} = 252$$

Teams That Contain Both or Neither

- Suppose two members of the group of twelve insist on working as a pair—any team must contain either both or neither. How many five-person teams can be formed?
- Call the two members of the group that insist on working as a pair A and B. Then any team formed must contain both A and B or neither A nor B.

- Because the set of teams that contain both A and B is disjoint from the set of teams that contain neither A nor B, by the addition rule,

The number of teams containing both A and B or neither A nor B =

the number of teams containing both A and B + the number of teams containing or neither A nor B

$$= 120 + 252 = 372$$

Teams That Do Not Contain Both

- Suppose two members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?
- Call the two people who refuse to work together C and D. If we want to use the addition rule, then any team formed must either contain C but not D, or contain D but not C, or not contain C nor D.
- Because any team that contains C but not D contains exactly four other people from the remaining ten in the group.

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{4! \cdot 6!} = 210$$

Teams That Do Not Contain Both

- Suppose two members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?
- Call the two people who refuse to work together C and D. If we want to use the addition rule, then any team formed must either contain C but not D, or contain D but not C, or not contain C nor D.
- Similarly, any team that contains D but not C contains exactly four other people from the remaining ten in the group.

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{4! \cdot 6!} = 210$$

Teams That Do Not Contain Both

- Suppose two members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?
- Call the two people who refuse to work together C and D. If we want to use the addition rule, then any team formed must either contain C but not D, or contain D but not C, or not contain C nor D.
- Because a team that contains neither C nor D contains exactly five people from the remaining ten, there are as many such teams as there are subsets of five people that can be chosen from the remaining ten.

$$\binom{10}{5} = \frac{10!}{5!\,(10-5)!} = \frac{10!}{5!\,5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 5!} = 252$$

Teams That Do Not Contain Both

- Suppose two members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?
- Call the two people who refuse to work together C and D. If we want to use the addition rule, then any team formed must either contain C but not D, or contain D but not C, or not contain C nor D.
- Since all those teams are disjoint, then the total number of teams that either contain C but not D, or contain D but not C, or not contain C nor D is the sum of the numbers of the three previous teams.

$$210 + 210 + 252 = 672$$

Teams That Do Not Contain Both

- Suppose two members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?
- Call the two people who refuse to work together C and D. Alternatively, the set of all five-person teams that don't contain both C and D equals the set difference between the set of all five-person teams and the set of all five-person teams that contain both C and D.
- The total number of five-person teams is $\binom{12}{5} = 792$.
- We have established that the number of teams that contain both C and $D = 120$.
- The number of teams that don't contain C and D together $= 792 - 120 = 672$.

Terminology

- The phrase at least n means "n or more."
- The phrase at most n means "n or fewer."

- For instance, if a set consists of three elements and you are to choose at least two, you will choose two or three.
- if you are to choose at most two, you will choose none, or one, or two.

Teams with Members of Two Types

- Suppose the group of twelve consists of five men and seven women.
- How many five-person teams can be chosen that consist of three men and two women?

- Think of forming a team as a two-step process:
  - *Step 1: Choose the men.*
  - *Step 2: Choose the women.*

Teams with Members of Two Types

- There are $\binom{5}{3}$ ways to choose 3 men out of the 5 men in the group.

$$\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5 \cdot 4 \cdot 3!}{3!\,2!} = 5 \cdot 2 = 10$$

- There are $\binom{7}{2}$ ways to choose 2 men out of the 7 men in the group.

$$\binom{7}{2} = \frac{7!}{2!\,(7-2)!} = \frac{7 \cdot 6 \cdot 5!}{2!\,5!} = 7 \cdot 3 = 21$$

- By the product rule:

The number of teams containing 3 men and 2 women $= \binom{5}{3}\binom{7}{2} = 10 \cdot 21 = 210$

Teams with Members of Two Types

- Suppose the group of twelve consists of five men and seven women.
- How many five-person teams contain at least one man?

- This question can also be answered either by the addition rule or by the difference rule. The solution by the difference rule is shorter.

The number of teams contain at least one man
= the total number of teams – the number of teams containing no men

$$= \binom{12}{5} - \binom{7}{5} = 792 - 21 = 771$$

Teams with Members of Two Types

- Alternatively, if we use the addition rule, we observe that:

$\binom{5}{1}\binom{7}{4}$ teams with one man and four women $= 175$

$\binom{5}{2}\binom{7}{3}$ teams with two men and three women $= 350$

$\binom{5}{3}\binom{7}{2}$ teams with three men and two women $= 210$

$\binom{5}{4}\binom{7}{1}$ teams with four men and one woman $= 21$

$\binom{5}{5}\binom{7}{0}$ teams with five men and no women $= 1$

The total number of teams $= 175 + 350 + 210 + 21 + 1 = 771$

Teams with Members of Two Types

- Suppose the group of twelve consists of five men and seven women.
- How many five-person teams contain at most one man?

- The set of teams containing at most one man can be partitioned into the set that does not contain any men and the set that contains exactly one man.
- Hence, by the addition rule,

The number of teams containing at most one man
= the number of teams containing one man + the number of teams containing no men

$$= \binom{5}{1}\binom{7}{4} + \binom{5}{0}\binom{7}{5} = 21 + 175 = 196$$

Counting Subsets of a Set: Combinations

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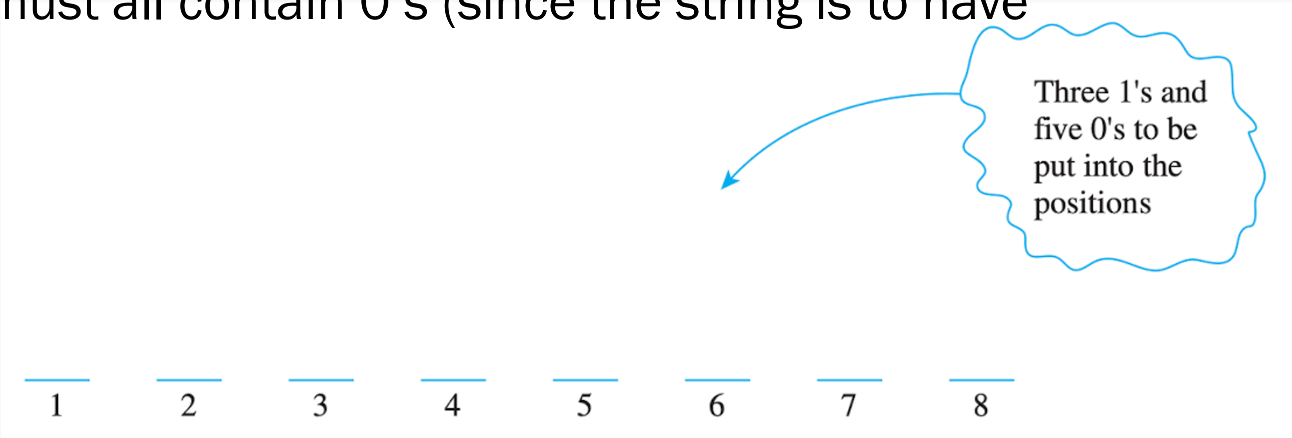
Permutations of a Set with Repeated Elements

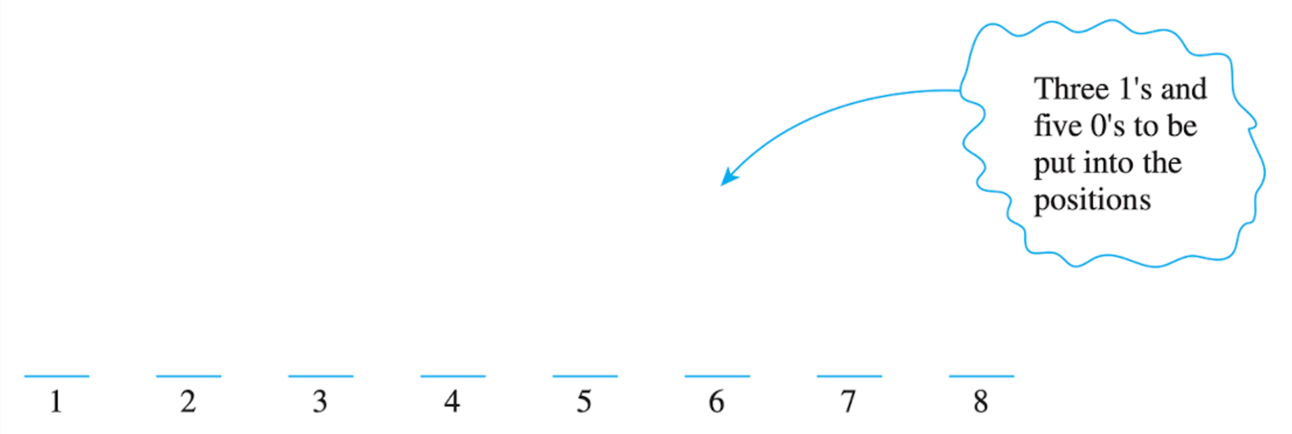
- How many eight-bit strings have exactly three 1's?
- To solve this problem, imagine eight empty positions into which the 0's and 1's of the bit string will be placed.
- In step 1, choose positions for the three 1's, and in step 2, put the 0's into place.
- Once a subset of three positions has been chosen from the eight to contain 1's, then the remaining five positions must all contain 0's (since the string is to have exactly three 1's).



Permutations of a Set with Repeated Elements

- How many eight-bit strings have exactly three 1's?
- It follows that the number of ways to construct an eight-bit string with exactly three 1's is the same as the number of subsets of three positions that can be chosen from the eight into which to place the 1's.

This equals $\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3! \cdot 5!} = 56$



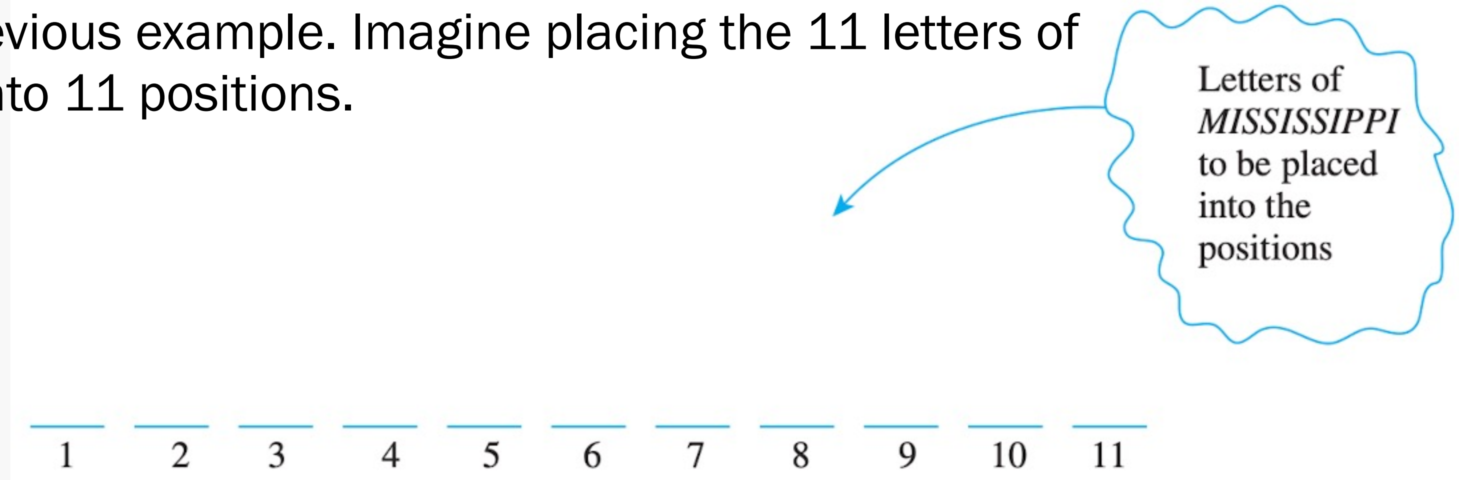
Permutations of a Set with Repeated Elements

- Consider various ways of ordering the letters in the word MISSISSIPPI:

  IIMSSPISSIP, ISSSPMIIPIS, PIMISSSSIIP, and so on.

- How many distinguishable orderings are there?

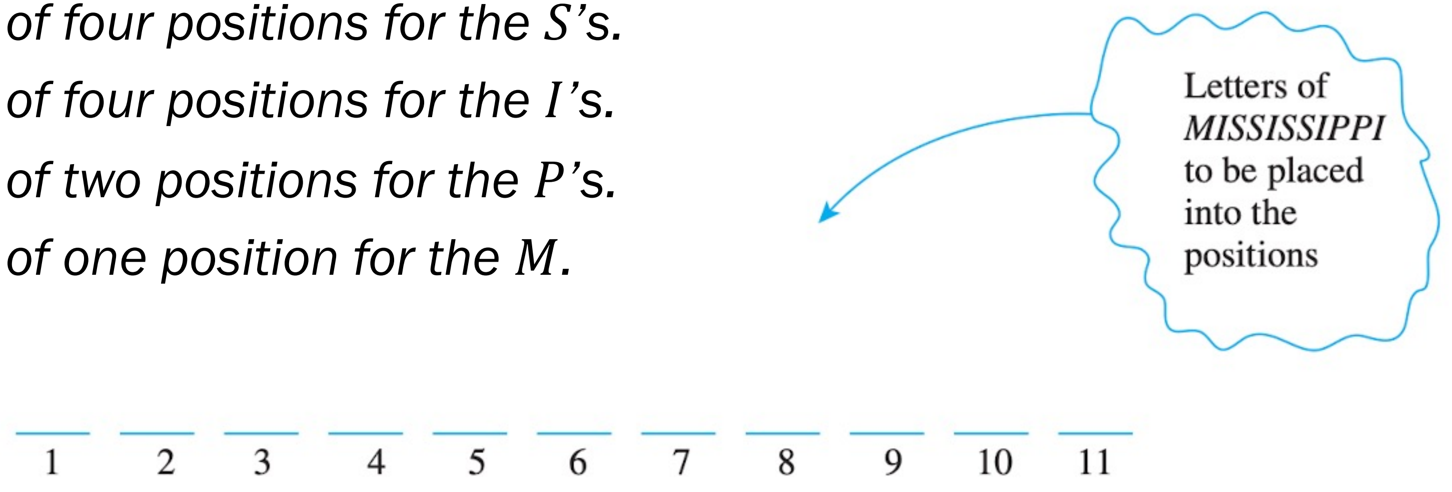
This example generalizes the previous example. Imagine placing the 11 letters of MISSISSIPPI one after another into 11 positions.



Permutations of a Set with Repeated Elements

- Because copies of the same letter cannot be distinguished from one another, once the positions for a certain letter are known, then all copies of the letter can go into the positions in any order. It follows that constructing an ordering for the letters can be thought of as a four-step process:

  - *Step 1: Choose a subset of four positions for the S's.*
  - *Step 2: Choose a subset of four positions for the I's.*
  - *Step 3: Choose a subset of two positions for the P's.*
  - *Step 4: Choose a subset of one position for the M.*



Permutations of a Set with Repeated Elements

- Since there are 11 positions, there are $\binom{11}{4}$ subsets of four positions for the S's.
- Once the four S's are in place, there are seven positions that remain empty, so there are $\binom{7}{4}$ subsets for four positions for the I's.
- After the I's are in place, there are three positions left empty, so there are $\binom{3}{2}$ subsets of two positions for the P's.
- That leaves just one position for the M, which is $\binom{1}{1}$.

Hence, by the multiplication rule,

$$\binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1} = \frac{11!}{4!\,7!}\cdot\frac{7!}{4!\,3!}\cdot\frac{3!}{3!\,1!}\cdot\frac{1!}{1!\,0!} = \frac{11!}{4!\,4!\,2!\,1!} = 34{,}650$$

Permutations with sets of Indistinguishable Objects

- Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other

n_2 are of type 2 and are indistinguishable from each other

…

n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$.

Then the number of distinguishable permutations of the n objects is:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\dots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!\,n_2!\,n_3!\dots n_k!}$$

Recommendation

- Refer to the book for advice about counting.

  Check the example on double counting in order to be able to avoid such mistakes.