جامعة بيرزيت

BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING AND TECHNOLOGY

COMPUTER SCIENCE DEPARTMENT

COMP233

Discrete Mathematics

CHAPTER 9

Counting and Probability

Counting and Probability

9.1 Introduction to Counting and Probability

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: The Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

Counting and Probability

What have we covered so far?

	Order Matters	Order Doesn't Matter
Repetition Not Allowed	Section 9.2 r-permutation $P(n, k)$	Section 9.5 r-combination $\binom{n}{k}$
Repetition Allowed	Section 9.2 multiplication rule n^k	Section 9.6 ???

r-Combinations with Repetition Allowed

- Introduction
- Examples
  - *Counting cans of soda*
  - *Counting triples*
  - *Counting iterations of a loop*
  - *Counting the number of solutions of an equation*
- Extra Example
  - *Counting the number of words*
- Summary

r-Combinations with Repetition Allowed

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r-Combinations with Repetition Allowed

- An ***r*-combination with repetition allowed**, or **multiset of size *r***, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed.

- If $X = \{x_1, x_2, \dots, x_n\}$, we write an *r*-combination with repetition allowed, or multiset of size *r*, as

$$[x_{i_1}, x_{i_2}, \dots, x_{i_r}]$$

  where each x_{i_j} is in X and some of the x_{i_j} may equal each other.

r-Combinations with Repetition Allowed

- Write a complete list to find the number of 3-combinations with repetition allowed, or multisets of size 3, that can be selected from {1, 2, 3, 4}.

[1, 1, 1]; [1,1,2]; [1,1,3]; [1, 1, 4]	all combinations with 1, 1
[1, 2, 2]; [1,2,3]; [1,2,4]	all additional combinations with 1, 2
[1, 3, 3]; [1,3,4]; [1,4,4]	all additional combinations with 1, 3 or 1, 4
[2, 2, 2]; [2,2,3]; [2,2,4]	all additional combinations with 2, 2
[2, 3, 3]; [2,3,4]; [2,4,4]	all additional combinations with 2, 3 or 2, 4
[3, 3, 3]; [3,3,4]; [3,4,4]	all additional combinations with 3, 3 or 3, 4
[4, 4, 4]	the only additional combination with 4, 4

- Thus there are twenty 3-combinations with repetition allowed.

r-Combinations with Repetition Allowed

- How can we count r-combinations with repetition allowed without listing all the items?

- Imagine representing the strings in the last example as such:

Category 1		Category 2		Category 3		Category 4	Result of the Selection
	|	×	|		|	× ×	1 from category 2 2 from category 4
×	|		|	×	|	×	1 each from categories 1, 3, and 4
× × ×	|		|		|		3 from category 1

- Three vertical bars are used to separate the four categories, and three crosses are used to indicate how many items from each category are chosen.

r-Combinations with Repetition Allowed

- With this scheme, any r-combination can be represented as a strings of six symbols consisting of three |'s and three ×'s, such as:

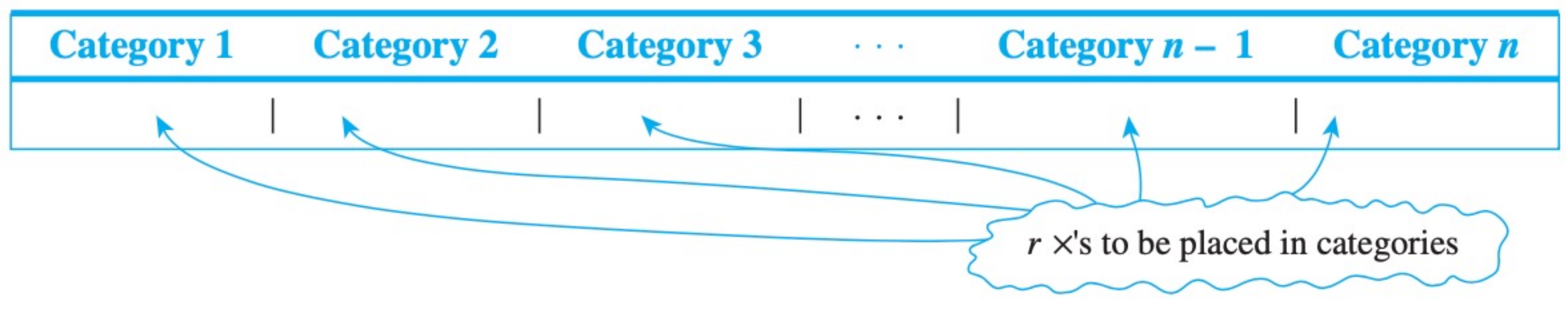
$$\times\times|\ |\times|\ \equiv [1,1,3]$$
$$\times|\times|\ |\times\ \equiv [1,2,4]$$
$$|\ |\ |\times\times\times\ \equiv [4,4,4]$$

- The number of distinct strings that can be created equals the number of ways to select three positions out of six because once three positions have been chosen for the ×'s, the |'s are placed in the remaining three positions, which equals

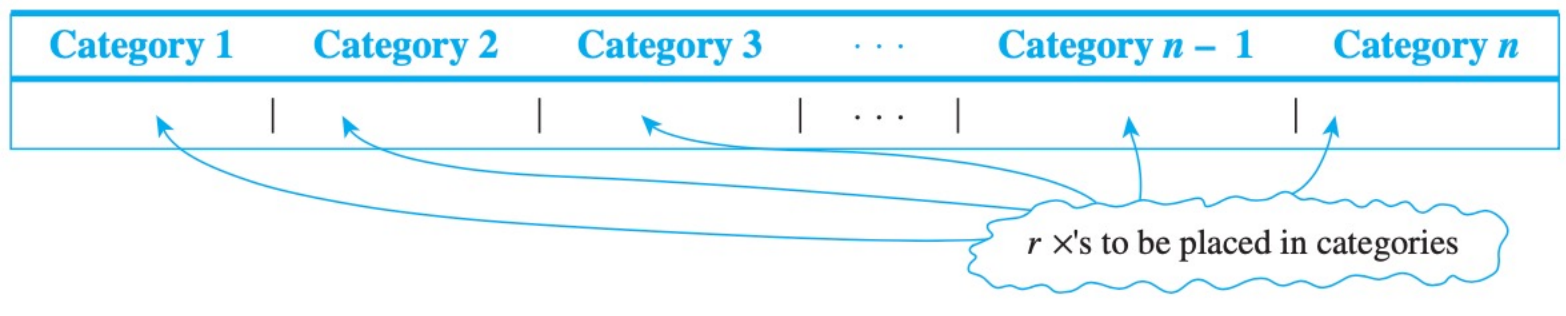
$$\binom{6}{3} = \frac{6!}{3!\,(6-3)!} = \frac{6\cdot 5\cdot 4\cdot 3!}{3\cdot 2\cdot 1\cdot 3!} = 20$$

r-Combinations with Repetition Allowed

- We can generalize this example:
- To count the number of r -combinations with repetition allowed, or multisets of size r , that can be selected from a set of n elements, think of the elements of the set as categories.
- Then each r -combination with repetition allowed can be represented as a string of $n - 1$ vertical bars (to separate the n categories) and r crosses (to represent the r elements to be chosen).
- The number of ×'s in each category represents the number of times the element represented by that category is repeated.



r-Combinations with Repetition Allowed



- The number of strings of $n - 1$ vertical bars and r crosses is the number of ways to choose r positions, into which to place the r crosses, out of a total of $r + (n - 1)$ positions, leaving the remaining positions for the vertical bars.

- We've already shown in the previous section that this number is

$$\binom{r+n-1}{r}$$

r-Combinations with Repetition Allowed

- The number of *r*-combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}$$

- This equals the number of ways *r* objects can be selected from *n* categories of objects with repetition allowed.

r-Combinations with Repetition Allowed

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Cans of Soft Drinks Example

- A person giving a party wants to set out 15 assorted cans of soft drinks for his guests. He shops at a store that sells five different types of soft drinks.
- How many different selections of cans of 15 soft drinks can he make?

Think of the five different types of soft drinks as the n categories and the 15 cans of soft drinks to be chosen as the *r* objects (so $n = 5$ and $r = 15$).

$$\binom{15+5-1}{15} = \binom{19}{15} = \frac{19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{15! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 3{,}876$$

Cans of Soft Drinks Example – cont.

- A person giving a party wants to set out 15 assorted cans of soft drinks for his guests. He shops at a store that sells five different types of soft drinks.

- If root beer is one of the types of soft drink, how many different selections include at least six cans of root beer?

If at least six cans of root beer are included, we can imagine choosing six such cans first and then choosing 9 additional cans.

$$\binom{9+5-1}{9} = \binom{14}{9} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9!}{9! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 715$$

Cans of Soft Drinks Example – cont.

- A person giving a party wants to set out 15 assorted cans of soft drinks for his guests. He shops at a store that sells five different types of soft drinks.

- If the store has only five cans of root beer but at least 15 cans of each other type of soft drink, how many different selections are there?

If the store has only five cans of root beer, then the number of different selections of 15 cans of soft drinks of the five types is the same as the number of different selections that contain five or fewer cans of root beer.

Cans of Soft Drinks Example – cont.

- A person giving a party wants to set out 15 assorted cans of soft drinks for his guests. He shops at a store that sells five different types of soft drinks.
- If the store has only five cans of root beer but at least 15 cans of each other type of soft drink, how many different selections are there?

We know from the last section that at most 5 cans, means 5 or fewer, and at least 6 cans means 6 or more.

We also know that these two sets are disjoint. Therefore, we can apply the difference rule.

Cans of Soft Drinks Example – cont.

- A person giving a party wants to set out 15 assorted cans of soft drinks for his guests. He shops at a store that sells five different types of soft drinks.

- If the store has only five cans of root beer but at least 15 cans of each other type of soft drink, how many different selections are there?

If T is the set of all selections, and $R_{\leq 5}$ is the set of selections containing 5 or fewer root beer cans, and $R_{\geq 6}$ is the set of selections containing 6 or more root beer cans:

$$N(R_{\leq 5}) = N(T) - N(R_{\geq 6}) = 3{,}876 - 715 = 3{,}161.$$

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Counting Triples (i, j, k) Example

- If n is a positive integer, how many triples of integers from 1 through n can be formed in which the elements of the triple are written in increasing order but are not necessarily distinct?

- In other words, how many triples of integers (i, j, k) are there with $1 \leq i \leq j \leq k \leq n$?

Any triple of integers (i, j, k) with $1 \leq i \leq j \leq k \leq n$ can be represented as a string of $n - 1$ vertical bars and three crosses, with the positions of the crosses indicating which three integers from 1 to n are included in the triple.

Counting Triples (i, j, k) Example – cont.

- How many triples of integers (i, j, k) are there with $1 \leq i \leq j \leq k \leq n$?
- If $n = 5$, this table is an example of two possible triples that follow this rule:

Category					Result of the Selection
1	2	3	4	5	
		× ×		×	(3, 3, 5)
×	×		×		(1, 2, 4)

Counting Triples (i, j, k) Example – cont.

- How many triples of integers (i, j, k) are there with $1 \le i \le j \le k \le n$?

- The number of such triples is the same as the number of strings of $(n - 1)$ \|'s and 3 $\times$'s, which is

$$\binom{3 + (n-1)}{3} = \binom{n+2}{3} = \frac{(n+2)!}{3!\,(n+2-3)!} = \frac{(n+2)(n+1)n(n-1)!}{3!\,(n-1)!} = \frac{n(n+1)(n+2)}{6}$$

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Counting Iterations of a Loop Example

- Assuming n is a positive integer, how many times will the innermost loop be iterated when the algorithm segment below is implemented and run?

for $k := 1$ **to** n
    for $j := 1$ **to** k
        for $i := 1$ **to** j
            [*Statements in the body of the inner loop, none containing branching statements that lead outside the loop*]
        next i
    next j
next k

Counting Iterations of a Loop Example – cont.

- If we construct a trace table for the loop execution, we will get something similar to this:

k	1	2 →			3 →						$\cdots$	n →						
j	1	1	2 →		1	2 →		3 →			$\cdots$	1	2 →		$\cdots$	n →		
i	1	1	1	2	1	1	2	1	2	3	$\cdots$	1	1	2	$\cdots$	1	$\cdots$	n

- Observe that there is one iteration of the innermost loop for each column of this table, and there is one column of the table for each triple of integers (i, j, k) with $1 \le i \le j \le k \le n$

Counting Iterations of a Loop Example – cont.

- If we construct a trace table for the loop execution, we will get something similar to this:

k	1	2		→	3					→	⋯	n						→
j	1	1	2	→	1	2	→	3		→	⋯	1	2	→	⋯	n		→
i	1	1	1	2	1	1	2	1	2	3	⋯	1	1	2	⋯	1	⋯	n

- Observe that there is one iteration of the innermost loop for each column of this table, and there is one column of the table for each triple of integers (i, j, k) with $1 \le i \le j \le k \le n$, which makes it identical to the previous questions.

$$\frac{n(n+1)(n+2)}{6}$$

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The Number of Integral Solutions of an Equation Example

- How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if x_1, x_2, x_3, and x_4 are nonnegative integers?

Think of the number 10 as divided into ten individual units and the variables x_1, x_2, x_3, and x_4 as four categories into which these units are placed.

The number of units in each category x_i indicates the value of x_i in a solution of the equation.

Categories x_1	x_2	x_3	x_4	Solution to the equation $x_1 + x_2 + x_3 + x_4 = 10$
× × |	× × × × × |	|	× × ×	$x_1 = 2,\quad x_2 = 5,\quad x_3 = 0,\quad$ and $\quad x_4 = 3$
× × × × |	× × × × × × |	|		$x_1 = 4,\quad x_2 = 6,\quad x_3 = 0,\quad$ and $\quad x_4 = 0$

The Number of Integral Solutions of an Equation Example

- How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if x_1, x_2, x_3, and x_4 are nonnegative integers?

Categories				Solution to the equation $x_1 + x_2 + x_3 + x_4 = 10$
x_1	x_2	x_3	x_4	
× × |	× × × × × |	|	× × ×	$x_1 = 2, \quad x_2 = 5, \quad x_3 = 0, \quad \text{and} \quad x_4 = 3$
× × × × |	× × × × × × |	|		$x_1 = 4, \quad x_2 = 6, \quad x_3 = 0, \quad \text{and} \quad x_4 = 0$

$$\binom{10+4-1}{10} = \binom{13}{10} = \frac{13!}{10! \cdot (13-10)!} = \frac{13 \cdot 12 \cdot 11 \cdot 10!}{10! \cdot 3 \cdot 2 \cdot 1} = 286$$

Additional Constraints on the Number of Solutions Example

- How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if each $x_i \geq 1$?

In this case imagine starting by putting one cross in each of the four categories. Then distribute the remaining six crosses among the categories.

For example, the string

$$\times\times\times|\,|\times\times|\times$$

indicates that there are three more crosses in category x_1 in addition to the one cross already there (so $x_1 = 4$), no more crosses in category x_2 in addition to the one already there (so $x_2 = 1$), two more crosses in category x_3 in addition to the one already there (so $x_3 = 3$), and one more cross in category x_4 in addition to the one already there (so $x_4 = 2$).

Additional Constraints on the Number of Solutions Example

- How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if each $x_i \geq 1$?

In this case imagine starting by putting one cross in each of the four categories. Then distribute the remaining six crosses among the categories.

$$\binom{6+4-1}{6} = \binom{9}{6} = \frac{9!}{6! \cdot (9-6)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 3 \cdot 2 \cdot 1} = 84$$

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Number of Words with Restrictions

- How many words made up of 3 vowels and 3 consonants can be generated, assuming repetition is not allowed?

To solve this, first we need to find out how many different combinations of 3 vowels and 3 consonants we can make.

Then we need to find out how many ways we can rearrange these letters.

Number of Words with Restrictions

- How many words made up of 3 vowels and 3 consonants can be generated, assuming repetition is not allowed?

Selecting 3 vowels from the 5 available vowels is $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3!}{3!2!} = 10$

Selecting 3 consonants from the available 21 consonants is $\binom{21}{3} = \frac{21!}{3!(21-3)!} = \frac{21 \cdot 20 \cdot 19!}{3!19!} = 70$

Applying the multiplication rule, we get $10 \times 70 = 700$ possible combination of letters.

Number of Words with Restrictions

- How many words made up of 3 vowels and 3 consonants can be generated, assuming repetition is not allowed?

Arranging a group of 6 letters will result in

$6! = 720$ possible different combinations for each set of 6 letters

The total possible number of words we can generate is $700 \times 720 = 504{,}000$ words.

What if repetition is allowed?

- If repetition is allowed, you will have to look at several possible cases, where you have 3 repeated vowels and 3 repeated consonants, 3 repeated vowels and 2 repeated consonants, 3 repeated vowels and no repeated consonants, and so on.

- This will produce 9 different mutually disjoint cases that you have to solve one by one.
- Once these are solved, you can apply the addition rule to get the total number of possible words.

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Summary

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Repetition Not Allowed	Section 9.2 r-permutation $P(n,k)$	Section 9.5 r-combination $\binom{n}{k}$
Repetition Allowed	Section 9.2 multiplication rule n^k	Section 9.6 r-combination with repetition $\binom{k+n-1}{k}$