Key

Faculty of Information Technology

Computer Systems Engineering

Second Exam *[Time Allowed: 1.5 hours]*

ENCS431 (DSP) **Fall Semester 9/12/2012**

Name: **ID:** **section:**

Note: ***this exam consists of four questions, every candidate has to answer question 1, and any other two questions only.***

Question 1: [40 marks] (*sampling and reconstruction*)

a) In the context of discretize continuous-time signals, what do we mean be aliasing? Support your answer with an example. [6]

(or distortion on the signal)

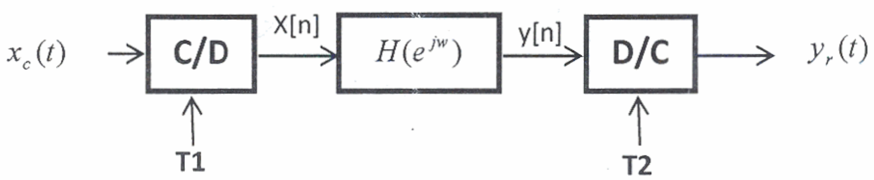
Aliasing is an overlap between shifted copies of spectra of the discretized signal because sampling freq. is less than the max. freq. in the sampled signal.

b) Explain how can we avoid aliasing? [6]

sampling freq. must be greater or equal to ~~the~~ double of of largest freq. in the sampled ~~the~~ signal.

$\Omega_s \geq 2\Omega_m$, when Ω_m is max. freq. in sampled sig. and Ω_s is the sampling freq.

Consider the system below



c) Let the continuous–time input signal $x_c(t) = \cos(150\pi t) + \cos(400\pi t) + \cos(600\pi t)$, what is the range of sampling period T1 that will avoid aliasing in the C/D converter? [8]

$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$. Ω_N is max. freq. in $x_c(t)$ which is $\Omega_N = 600\pi$.

$\frac{\cancel{2}\pi}{T} \geq 600\pi \Rightarrow T \leq \frac{1}{600}$

$\Omega_s \geq 1200\pi$

d) Now, assume the sampling period T1=0.001 sec and the discrete system is an ideal low-pass filter with frequency response as follow:

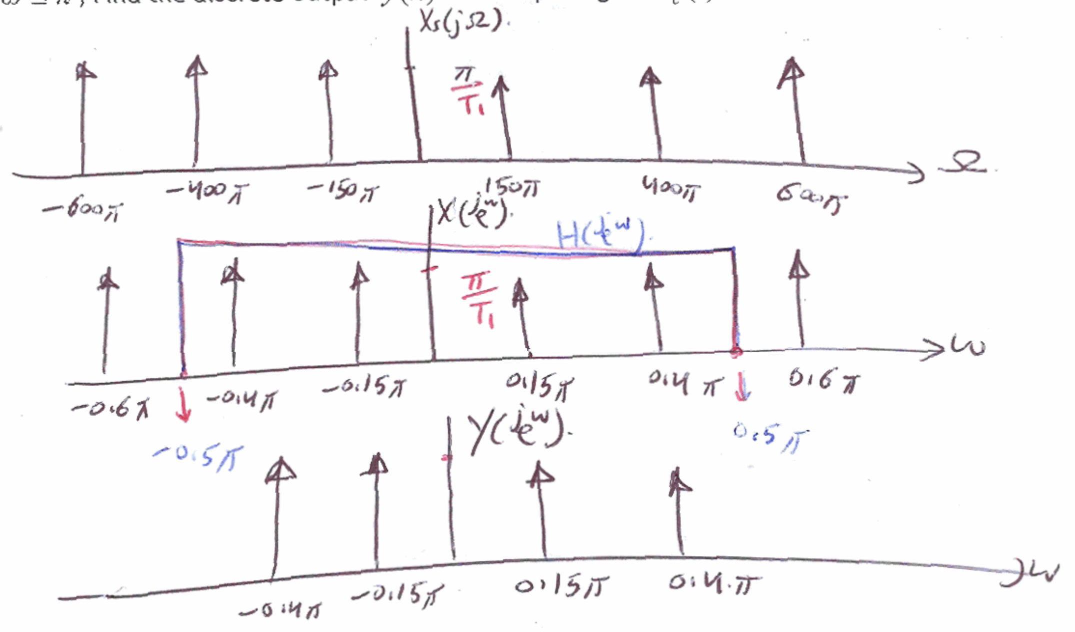
$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq 0.5\pi \\ 0 & \text{otherwise} \end{cases}$$

$\Omega_s = \frac{2\pi}{T} = \frac{2\pi}{0.001} = 2000\pi$.

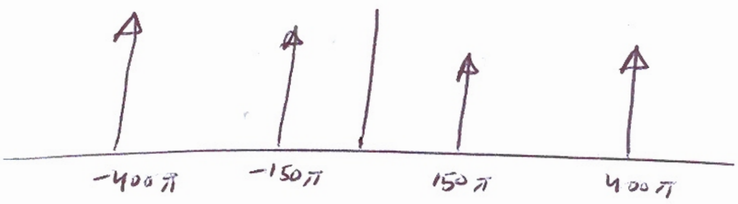
Since $\Omega_s = 2000\pi > 2\Omega_N$ (1200π)

$\Rightarrow$ No aliasing

For $-\pi \leq \omega \leq \pi$, Find the discrete output $y(n)$ if the input signal $x_c(t)$ is the same as specified in part **(c)** above? [10]

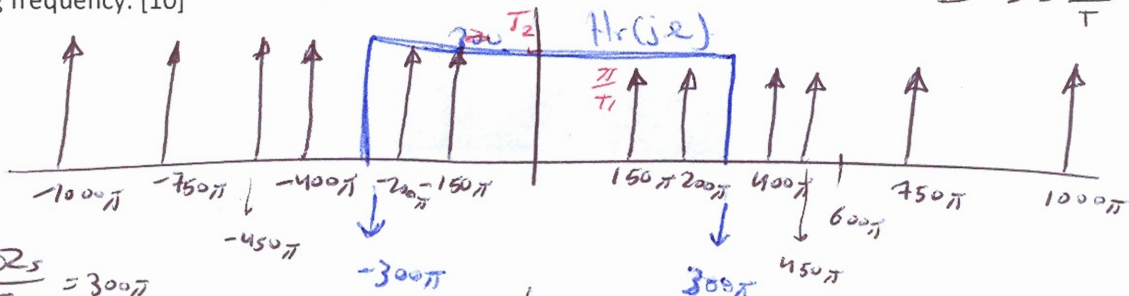


So, output of L.P.F is $y(n) = \cos(0.15\pi n) + \cos(0.4\pi n)$.

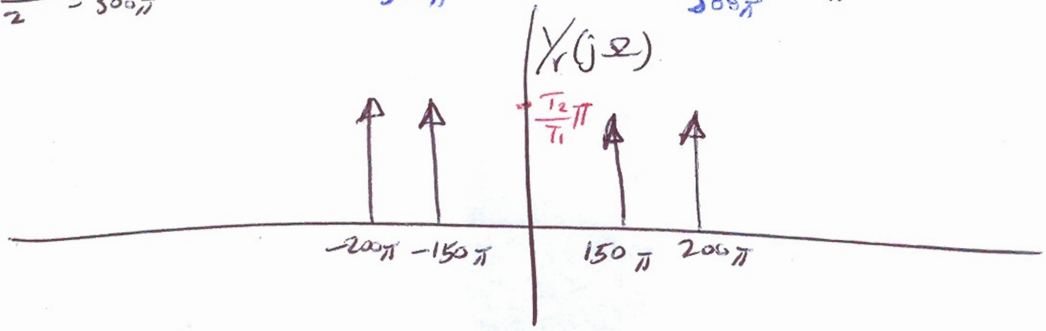


e) If the sampling period of **D/C** converter T2 = (1/300) sec, Find the reconstructed continuous-time signal $y_r(t)$ if the reconstruction filter in the **D/C** converter, $H_r(j\Omega)$, has a gain of (1/T2) and cut-off frequency, Ω_c half of the sampling frequency. [10]

$$\Omega_s = \frac{2\pi}{T} = 600\pi$$



$$\Omega_c = \frac{\Omega_s}{2} = 300\pi$$



$$y_r(t) = \frac{T_2}{T_1}\cos(150\pi t) + \frac{T_2}{T_1}\cos(200\pi t).$$

Question 2: [30 marks, 6 pts each] (*Discrete Fourier Transform-DFT*)

Suppose we have two four-point sequences x(n) and h(n) as follows:

x(n) = {1, 0, 0, 1}, and h(n) = {1, 1, 0, 0} ,for n=0,1,2,3 respectively.

a) Calculate the 4-point DFT X(k).

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \;;\; 0 \le k \le N-1$$

$$X(0) = \sum_{n=0}^{3} x(n)\, W_4^{0} = \sum_{n=0}^{3} x(n) = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)\, W_4^{n} = (1)\,W_4^0 + (1)\,W_4^3 = 1 + e^{-j\frac{2\pi}{4}\cdot 3} = 1+j$$

$$X(2) = \sum_{n=0}^{3} x(n)\, W_4^{2n} = (1)\,W_4^0 + (1)\,W_4^6 = 1 + e^{-j\frac{2\pi}{4}\cdot 6} = 1-1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)\, W_4^{3n} = (1)\,W_4^0 + (1)\,W_4^9 = 1 + e^{-j\frac{2\pi}{4}\cdot 9} = 1-j$$

So, $X(k) = \{2,\ 1+j,\ 0,\ 1-j\}$ for $k = 0, 1, 2, 3$.

b) Calculate the 4-point DFT H(k).

$$H(0) = \sum_{n=0}^{3} h(n)\, W_4^{0} = \sum_{n=0}^{3} h(n) = 2$$

$$H(1) = \sum_{n=0}^{3} h(n)\, W_4^{n} = (1)\,W_4^0 + W_4^1 = 1 + e^{-j\frac{2\pi}{4}(1)} = 1-j$$

$$H(2) = \sum_{n=0}^{3} h(n)\, W_4^{2n} = (1)\,W^0 + (1)\,W_4^2 = 1 + e^{-j\pi} = 0$$

$$H(3) = \sum_{n=0}^{3} h(n)\, W_4^{3n} = (1)\,W_4^0 + (1)\,W_4^3 = 1 + e^{-j\frac{2\pi}{4}(3)} = 1+j$$

So, $H(k) = \{2,\ 1-j,\ 0,\ 1+j\}$ for $k = 0, 1, 2, 3$.

c) Prove the circular convolution property of the DFT, i.e. circular convolution in time-domain corresponds to multiplication in frequency-domain, and vice versa.

$$x_1(n) \, \textcircled{N} \, x_2(n) \xleftrightarrow{DFT} X_1(k) \, X_2(k)$$

let $y(n) = x_1(n) \, \textcircled{N} \, x_2(n)$

$$Y(k) = \sum_{n=0}^{N-1} y(n) W_N^{kn} = \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} x_1(m) \, x_2[((n-m))_N] \right] W_N^{kn}$$

by Interchanging order of sums and substitude $n-m = L + Nr$, where L and r are Integers with $0 \leq L \leq N-1$.

$$Y(k) = \sum_{m=0}^{N-1} x_1(m) \left[\sum_{n=0}^{N-1} x_2[((n-m))_N \, W_N^{kn} \right]$$

$$= \sum_{m=0}^{N-1} x_1(m) \left[\sum_{L=0}^{N-1} x_2(L) \, W_N^{k(L+m+Nr)} \right] = \left(\sum_{m=0}^{N-1} x_1(m) W_N^{km} \right) X_2(k)$$

$$= X_1(k) \, X_2(k)$$

d) Calculate y(n) = x(n) (4) h(n) by doing the circular convolution directly.

using Tabular method!

n	0	1	2	3
x(n)	1	0	0	1
h(n)	1	1	0	0
	1	0	0	1
	1	1	0	0
	0	0	0	0
	0	0	0	0
	2	1	0	1

$y(n) = \{2, 1, 0, 1\}$, $\quad n = 0, 1, 2, 3$.

e) Calculate y(n) of Part (c) by multiplying the DFTs of x(n) and h(n) and performing an inverse DFT.

$$Y(k) = X(k)H(k)$$

$$= \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix} \cdot * \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$W_4^{-1} = e^{+j\frac{2\pi}{4}(1)} = e^{+j\frac{\pi}{2}} = +j$$

$$W_4^{-2} =$$

$$W_4^{-3} = e^{+j\frac{2\pi}{4}(3)} = e^{+j\frac{3\pi}{2}} = +j$$

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} \cancel{X}(k)\, W_N^{-kn}$$

$$y(0) = \frac{1}{4}\sum_{k=0}^{N-1} Y(k)\, \cancel{W_4^{0}} = \frac{1}{4}\sum_{k=0}^{3} Y(k) = 8/4 = 2$$

$$y(1) = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, W_4^{-k} = \frac{1}{4}\left\{(4)W_4^{0} + (2)W_4^{-1} + (2)W_4^{-3}\right\} = \frac{1}{4}\left[4 + 2j + 2j\right] = 1$$

$$y(2) = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, W_4^{-2k} = \frac{1}{4}\left\{4W_4^{0} + (2)W_4^{-2} + (2)W_4^{-6}\right\} = 0$$

$$y(3) = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, W_4^{-3k} = \frac{1}{4}\left\{(4)W_4^{0} + (2)W_4^{-3} + (2)W_4^{-9}\right\} = 1$$

so, $y(n) = \{2, 1, 0, 1\}$ for $n = 0, 1, 2, 3$.

Question 3: [30 marks] (*Fast Fourier Transform –FFT*)

The N-point DFT, X(k), of discrete sequence x(n) is given by

$$X(k)=\sum_{n=0}^{N-1} x(n)W_N^{nk}, k=0,1,..N-1, W_N^{nk}=e^{-j2\pi kn/N}$$

a) Derive an expression for N/2-point DFT X(k). [8]

$$X(k)=\sum_{n=0}^{N-1} x(n)\,W_N^{nk} = \underbrace{\sum_{n\ \text{even}} x(n)\,W_N^{nk}}_{n=2r} + \underbrace{\sum_{n\ \text{odd}} x(n)\,W_N^{nk}}_{n=2r+1}, \quad r=0,1,..\tfrac{N}{2}-1$$

$$X(k)=\sum_{r=0}^{\frac{N}{2}-1} x(2r)\,W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\,W_N^{(2r+1)k}$$

$$W_N^{(2r+1)k} = W_N^{k}\,W_N^{rk}\,W_N^{2}$$

$$W_N^{2} = e^{-j\frac{2\pi}{N}\cdot 2} = e^{-j2\pi/\frac{N}{2}} = W_{\frac{N}{2}}$$

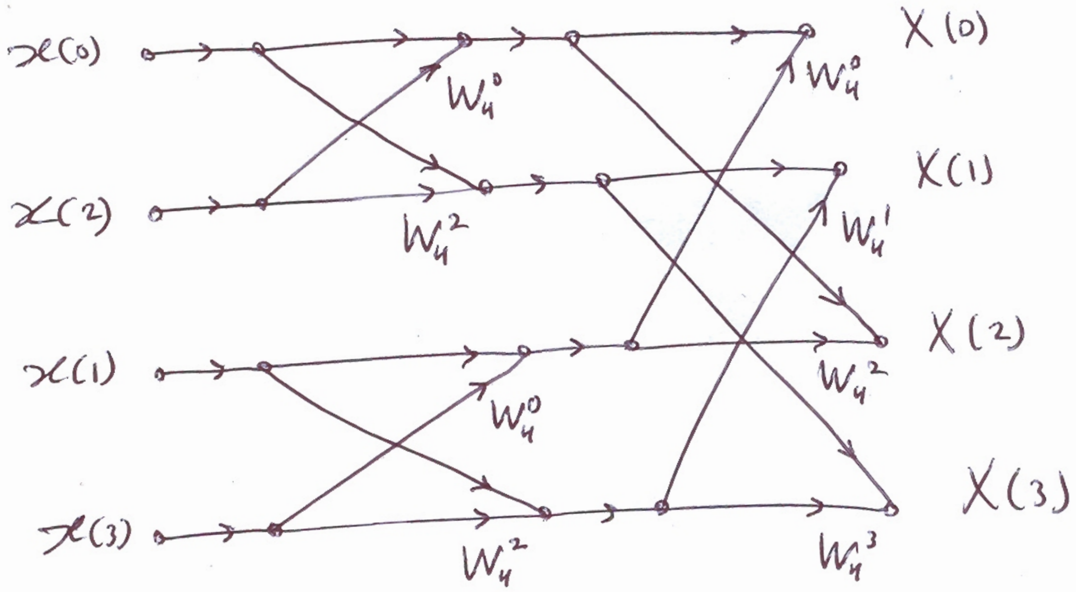
so, $$X(k)=\underbrace{\sum_{r=0}^{\frac{N}{2}-1} x(2r)\,W_{\frac{N}{2}}^{rk}}_{\frac{N}{2}\text{-point DFT}} + W_N^{k}\underbrace{\sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\,W_{\frac{N}{2}}^{rk}}_{\frac{N}{2}\text{-point DFT.}}$$

b) Explain the advantage of computing X(k) by this method compared to the N-point DFT method. [6]

- N-point DFT Computation requires N^2 MADS.
- $\frac{N}{2}$-point DFT requires $2\left(\frac{N}{2}\right)^2+N = \frac{N^2}{2}+N$ MADS

Which is less than N^2.

c) Let N=4, draw the complete flow-graph of decimation in time FFT. The graph should have all the necessary butterflies and 2-point DFTs. Also their associated input and output labels should be properly labeled. [8]



d) If the input signal x(n) = {1, 2, 4, 8}, n=0,1,2,and 3. Compute the outputs of the FFT flow-graph shown in part (c) for k=0 and k=3. [8]

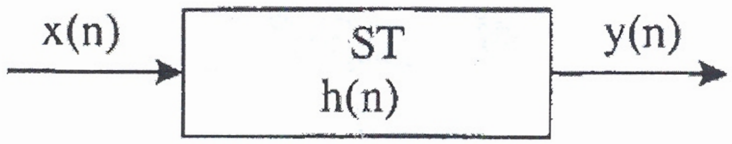
from the flow-graph above:

$$X(0) = x(0) + W_4^0 x(2) + \left(x(1) + W_4^0 x(3)\right) W_4^0 = 1 + 2 + 4 + 8 = 15$$

$$X(3) = x(0) + W_4^2 x(2) + \left[x(1) + W_4^2 x(3) \right] W_4^3 =$$

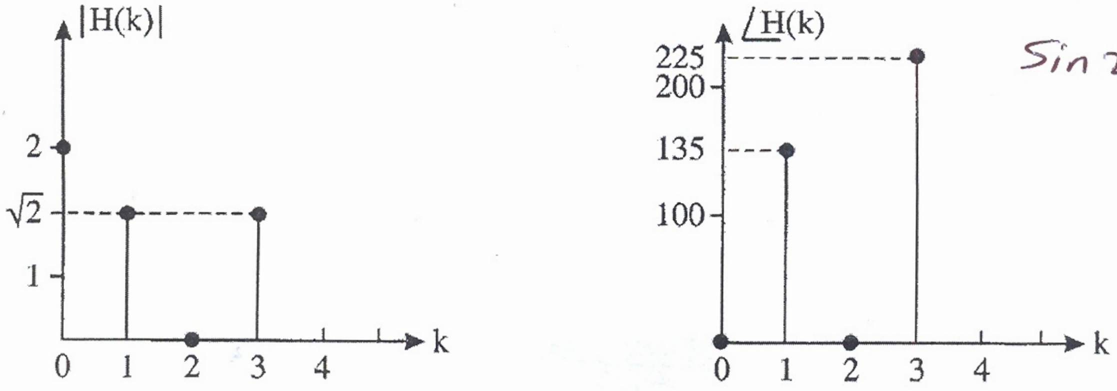
Question 4: [30 marks] (*Discrete Fourier Transform-DFT*)

Consider a discrete time system shown below, where x(n), h(n) and y(n) are input signal, impulse response and output signal of the system. The magnitude and phase of the complex DFT H(k) of h(n) are shown in fig (a) and fig (b) below, respectively.



$\cos 135 = -\frac{1}{\sqrt{2}}$
$\sin 135 = \frac{1}{\sqrt{2}}$

$\cos 225 = -\frac{1}{\sqrt{2}}$
$\sin 225 = -\frac{1}{\sqrt{2}}$.



(a) (b)

a) Determine the sequence h(n). [10]

~~H(k) = √2~~,

$H(0) = 2e^{j0} = 2$, $H(1) = \sqrt{2}e^{j135} = -1+j$, $H(2) = 0$, $H(3) = -1-j$

$h(n) = \frac{1}{4}\sum_{k=0}^{3} H(k) W_4^{-kn}$

$h(0) = \frac{1}{4}\sum_{k=0}^{3} H(k) W_4^{0} = 0$

$h(1) = \frac{1}{4}\sum_{k=0}^{3} H(k) W_4^{-k} = \frac{1}{4}\left[2W_4^0 + (-1+j)W_4^{-1} + (-1-j)W_4^{-3}\right]$

$= \frac{1}{4}\{2 - j - 1 + \cancel{j} - 1\} = 0$

$h(2) = \frac{1}{4}\sum_{k=0}^{3} H(k) W_4^{-2k} = \frac{1}{4}\left\{2W_4^0 + (-1+j)W_4^{-2} + (-1-j)W_4^{-6}\right\}$

$= 1$

$h(3) = \frac{1}{4}\sum_{k=0}^{3} H(k) W_4^{-3k} = \frac{1}{4}\left[2W_4^0 + (-1+j)W_4^{-3} + (-1-j)W_4^{-9}\right]$

$= 1$

b) If X(k) is the DFT of x(n), write down expressions of DFTs of the following signals: [10, 5 each]

(i) x[((n-2))N]

Using circular shift property of DFT, circular shift in the time domain is a multiplication of complex exponential in the freq. domain. i.e.

$$x[((n-m))_N] \xleftrightarrow{DFT} W_N^{-m} X(k)$$

so, $$DFT\left\{x[((n-2))_N]\right\} = W_N^{-2} X(k).$$

$$= e^{-j\frac{2\pi}{N}(2)} X(k).$$

(ii) 2x(n) + x[((n+1))N]

Using linearity of DFT and circular shift property:

we get.

$$DFT\left[2x(n) + x[((n+1))_N]\right] = 2X(k) + W_N X(k).$$

$$= (2+W_N) X(k).$$

c) Explain with an example how linear convolution can be used to compute the circular convolution, and how circular convolution can be used to compute the linear convolution of two discrete signals x1(n) and x2(n). [10]

Let $x_3(n) = x_1(n) \; N \; x_2(n)$.

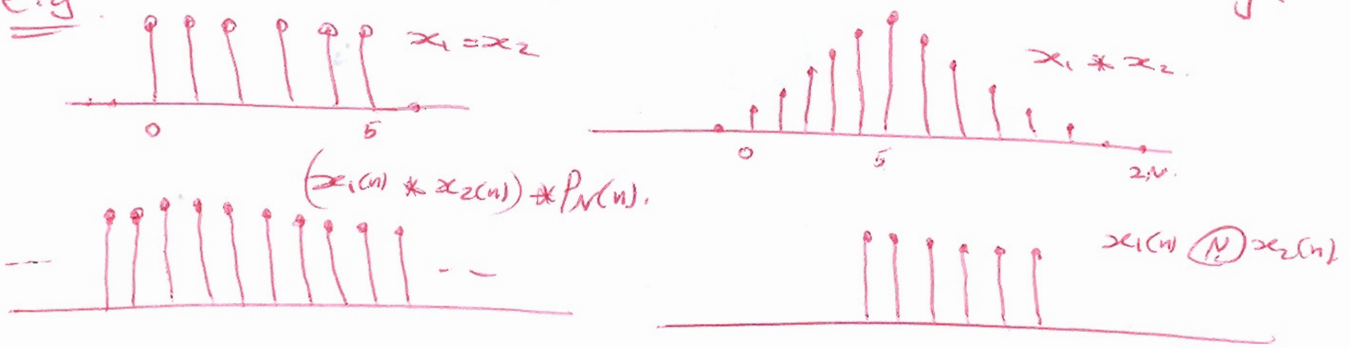
$$= \left[\sum_{m=0}^{N-1} x_1(m)\, x_2[((n-m))_N]\right] R_N(n)$$

$$x_3(n) = \left[x_1(n) * x_2(((n))_N)\right] R_N(n).$$

So, Circular convolution is linear conv. of $x_1(n)$ and periodic seq. $x_2(n)$ with period N. (i.e. length of circ. conv.).

or it can be thought as linear convolution convolved with impulse train, and extract one period, so it is lin. conv. + aliasing.

e.g



To get linear conv. from circular conv., we need to do circular conv. on 2N point. i.e. $x_1(n) * x_2(n) = x_1(n) \; 2N \; x_2(n)$

$x_1(n) \; 2N \; x_2(n) = x_1(n) * x_2(n)$.

-Good Luck-