When the input to a causal LTI system is

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3} 2^n u[-n-1]$$

The z-transform of the output is

$$Y(z) = \frac{1+z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-2z^{-1})}$$

(a) Find z-transform of x[n]?
(b) What is the region of convergence (ROC) of Y(z)?
(c) Find the system function of this system H(z)? Plot its zero-pole diagram and indicate ROC?
(d) Is the system stable?

(a) $X(z) = \dfrac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \dfrac{4/3}{1-2z^{-1}} = \dfrac{-\frac{1}{3}+\frac{2}{3}z^{-1}+\frac{4}{3}-\frac{2}{3}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \dfrac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$, $\frac{1}{2}<|z|<2$

(b) $X(z)$ has two poles at $z=\frac{1}{2}$ and at $z=2$

$Y(z)$ has three poles at $z=1$, $z=-\frac{1}{2}$, and $z=2$, therefore the two poles $z=1$ and $z=-\frac{1}{2}$ comes from the system Function $H(z)$.

Since $H(z)$ is Causal $\Rightarrow$ it's Roc extends from outermost pole to ∞

$\Rightarrow$ Roc of $H(z)$ is $|z| > 1$

We know, RoC of $Y(z)$ is the common area of Roc of $X(z)$ and Roc of $H(z)$

Which is $1 < |z| < 2$.

(c) $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1+z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-2z^{-1})} \cdot \dfrac{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}{1} = \dfrac{(1+z^{-1})(1-\frac{1}{2}z^{-1})}{(1-z^{-1})(1+\frac{1}{2}z^{-1})}$

ROC
$|z| > 1$ (Causal system)
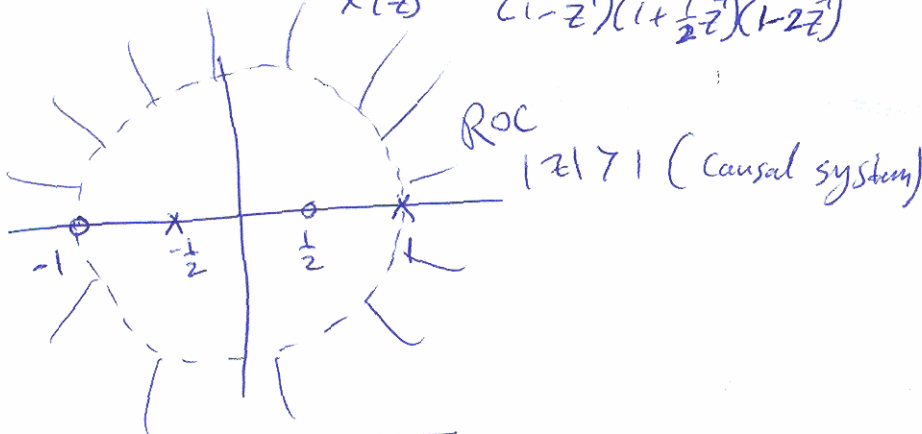
(d) System is unstable since its ROC doesn't include Unit circle.