ENEE2302
Signals and Systems

Complex Number Review

Complex Numbers

Complex numbers arise as roots of polynomials.

Definition of imaginary # j and some resulting properties:

$$j = \sqrt{-1} \Rightarrow j^2 = -1$$
$$\Rightarrow (-j)(j) = 1$$
$$\Rightarrow (-j)(-j) = -1$$

Recall that the solution of differential equations involves finding roots of the "characteristic polynomial"

So…differential equations often involve complex numbers

Rectangular form of a complex number:

$$z = a + jb \qquad a = \mathrm{Re}\{z\}$$
$$b = \mathrm{Im}\{z\}$$

real numbers

The rules of addition and multiplication are straight-forward:

$$Add: \quad (a + jb) + (c + jd) = (a + c) + j(b + d)$$
$$Multiply: \quad (a + jb)(c + jd) = (ac - bd) + j(ad + bc)$$

Polar Form

$$\boxed{z = re^{j\theta}} \qquad r > 0$$

If r is negative then it is <u>NOT</u> in polar form!!!

Polar form… an alternate way to express a complex number…

<u>Polar Form</u>…

good for multiplication and division

Note: you may have learned polar form as $r\angle\theta$… we will **<u>NOT</u>** use that here!!

The advantage of the $re^{j\theta}$ is that when it is manipulated using rules of exponentials and it behaves properly according to the rules of complex #s:

$$(a^x)(a^y) = a^{x+y} \qquad a^x / a^y = a^{x-y}$$

Multiplying Using Polar Form

$$\left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

$$\Rightarrow |z_1 z_2| = |z_1||z_2|$$

$$\angle\{z_1 z_2\} = \angle\{z_1\} + \angle\{z_2\}$$

$$z^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta}$$

$$z^{1/n} = r^{1/n} e^{j\theta/n}$$

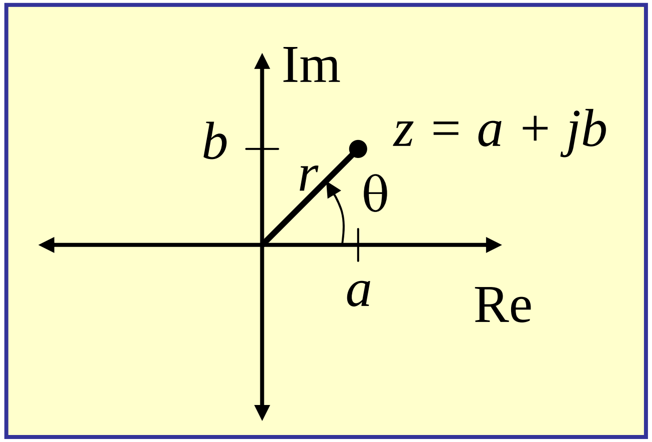
Dividing Using Polar Form

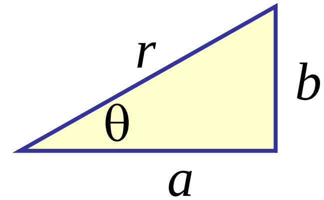
$$\frac{\left(r_1 e^{j\theta_1}\right)}{\left(r_2 e^{j\theta_2}\right)} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$$

$$\frac{1}{z_2} = \frac{1}{r_2 e^{j\theta_2}} = \frac{1}{r_2} e^{-j\theta_2}$$

We need to be able convert between Rectangular and Polar Forms… this is made easy and obvious by looking at the geometry (and trigonometry) of complex #s:

<u>Geometry of Complex Numbers</u>





Conversion Formulas

$$b = r\sin\theta$$

$$a = r\cos\theta$$

$$\Longleftrightarrow$$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Complex Exponentials vs. Sines and Cosines

Euler's Equations:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad \text{(A)}$$

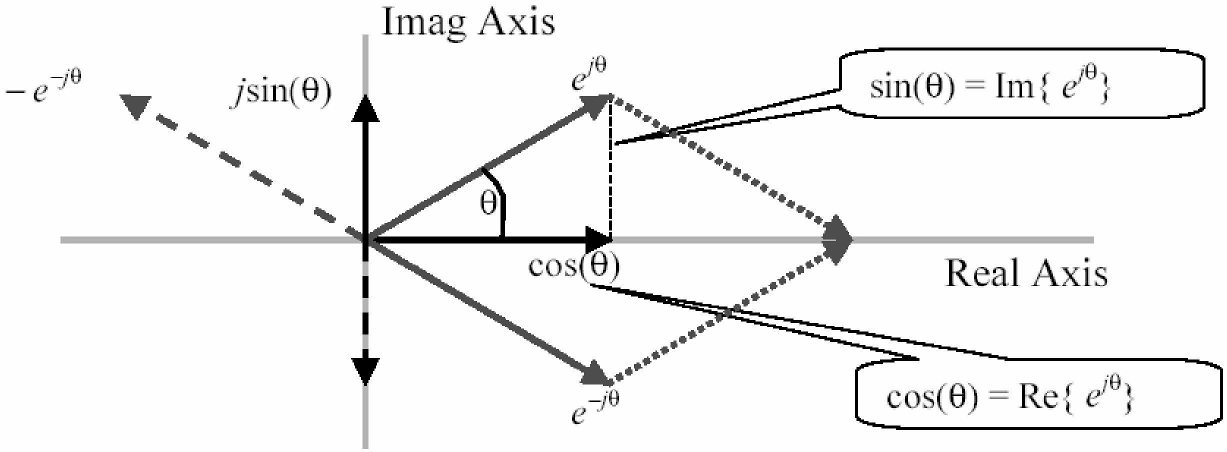
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta) \qquad \text{(B)}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \text{(C)}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad \text{(D)}$$

Note: Eq. C = (Eq. A+ Eq. B)/2 D = (A – B)/2

A = C + jD B = C – jD



Summary of Rectangular & Polar Forms

Rect Form:

$$z = a + jb$$

$$\mathrm{Re}\{z\} = a = r\cos\theta$$

$$\mathrm{Im}\{z\} = b = r\sin\theta$$

Polar Form:

$$z = re^{j\theta} \qquad r \geq 0 \qquad \theta \in (-\pi, \pi]$$

$$|z| = r = \sqrt{a^2 + b^2}$$

$$\angle z = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Warning: If you calculate the angle by first dividing *b*/*a* and then taking the inverse tangent… your *calculator* will give you the **wrong answer** whenever you have $a < 0$. In other words, for *z* values that lie in the II and III quadrants.

You can always fix this by either adding or subtracting π… choose add or subtract in order to give an angle that lies between –π and + π.

Use common sense… looking at the signs of *a* and *b* will tell you what quadrant *z* is in… make sure your angle agrees with that!!! (See the examples)

Conjugate of *Z*

Denoted as
z^* or $\bar{z}$

$$z = a + jb \quad \Rightarrow \quad z^* = a - jb$$

$$z = re^{j\theta} \quad \Rightarrow \quad z^* = re^{-j\theta}$$

Properties of z^*

Imaginary parts cancel

1. $z + z^* = 2\,\mathrm{Re}\{z\}$

2. $z \times z^* = (a + jb)(a - jb) = a^2 + b^2 = |z|^2$

Summary of General Results

Polar to Rect

Given : $z = re^{j\theta}$

Convert : $z = r\cos\theta + jr\sin\theta$

Rect to Polar

Given : $z = a + jb$

Convert : $z = \sqrt{a^2 + b^2}\, e^{j\tan^{-1}(b/a)}$

Warning: If $a < 0$ calculator may give wrong angle… $\pm\pi$ to correct

For Rect Form

Add / Subtract : $(a + jb) \pm (c + jd) = (a \pm c) + j(b \pm d)$

Multiply : $(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$

Multiplying Using Polar Form

$$\left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

$$z^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta}$$

$$z^{1/n} = r^{1/n} e^{j\theta/n}$$

Dividing Using Polar Form

$$\frac{\left(r_1 e^{j\theta_1}\right)}{\left(r_2 e^{j\theta_2}\right)} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

$$\frac{1}{z_2} = \frac{1}{r_2 e^{j\theta_2}} = \frac{1}{r_2} e^{-j\theta_2}$$

Finding Magn/Angle of Rect

Given : $z = a + jb$

$|z| = \sqrt{a^2 + b^2}$ $\qquad \angle z = \tan^{-1}(b/a)$

Finding Magn/Angle of Products

$$|z_1 z_2| = |z_1||z_2|$$

$$\angle\{z_1 z_2\} = \angle\{z_1\} + \angle\{z_2\}$$

Finding Magn/Angle of Ratios

$$|z_1 / z_2| = |z_1| / |z_2|$$

$$\angle\{z_1 / z_2\} = \angle\{z_1\} - \angle\{z_2\}$$

A Few Tricks

$\boxed{\frac{1}{j} = -j}$ Proof: $\frac{1}{j} = \frac{1}{e^{j\pi/2}} = (1/1)e^{j(0-\pi/2)} = e^{-j\pi/2} = \cos(-\pi/2) + j\sin(\pi/2) = 0 - j$

$\boxed{-1 = e^{\pm j\pi}}$ Proof: $e^{\pm j\pi} = \cos(\pm\pi) + j\sin(\pm\pi) = -1 + j0$

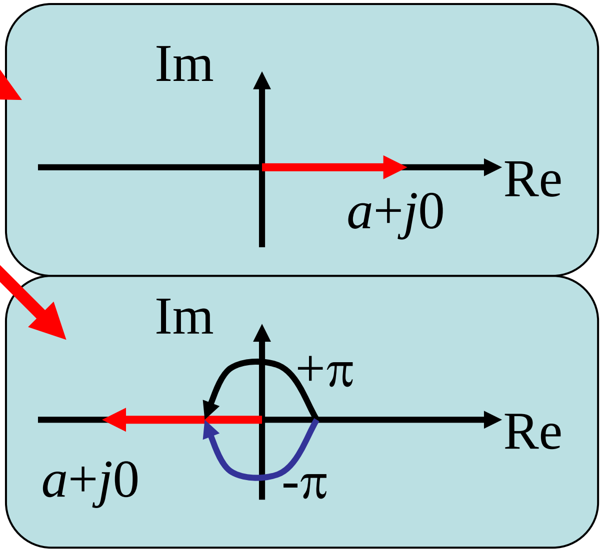
$\boxed{1 = e^{j0}}$ Proof: $e^{j0} = \cos(0) + j\sin(0) = 1 + j0$

$\boxed{j = e^{j\pi/2}}$ Proof: $e^{j\pi/2} = \cos(\pi/2) + j\sin(\pi/2) = 0 + j1$

$\boxed{-j = e^{-j\pi/2}}$ Proof: $e^{-j\pi/2} = \cos(-\pi/2) + j\sin(-\pi/2) = 0 + j(-1)$

$$\boxed{\angle(a + j0) = \begin{cases} 0, & a > 0 \\ \pm\pi, & a < 0 \end{cases}}$$

a is real #



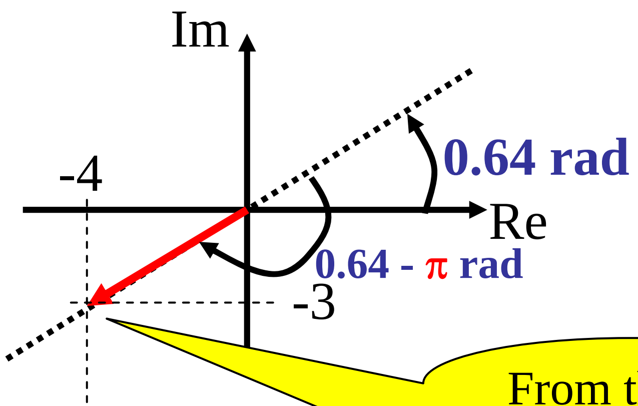
Example #1a: Given: $z = -4 - j3$ Convert to Polar Form

Use: $z = |z| e^{j\angle z}$

$$|z| = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\angle z = \tan^{-1}\left(\frac{-3}{-4}\right) = \tan^{-1}(0.75) - \pi = 0.64 - \pi \approx -2.5 \; rad$$

$\boxed{z = -4 - j3}$ ⇔ $\boxed{z = 5e^{-j2.5}}$



From this we see that z is in Quad III but our calculator gave us 0.64 which is in quadrant I. So if we subtract π we get an angle in Quad III and is between –π and + π

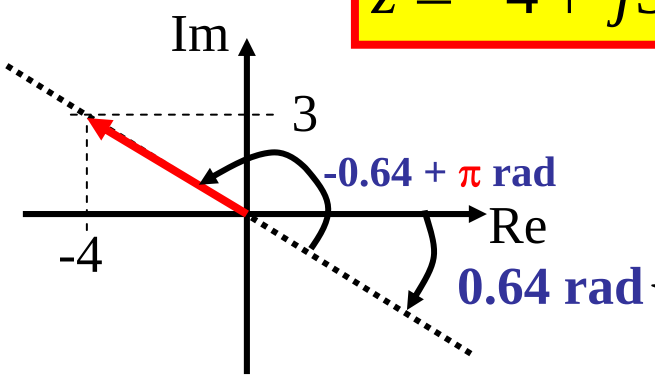
Example #1b: Given: $z = -4 + j3$ Convert to Polar Form

Use: $z = |z|e^{j\angle z}$

$$|z| = \sqrt{(-4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\angle z = \tan^{-1}\left(\frac{3}{-4}\right) = \tan^{-1}(-0.75) + \pi = -0.64 + \pi \approx 2.5\ rad$$

$\boxed{z = -4 + j3}$ ⇔ $\boxed{z = 5e^{j2.5}}$



From this we see that z is in Quad II but our calculator gave us -0.64 which is in quadrant IV. So if we add π we get an angle in Quad II and is between –π and + π

Comparing Ex. 1a and 1b we see that they are conjugates of each other… note how conjugation just changes the sign in front of *j* for both rect form and polar form!!!

Example #2: Given: $z = 3e^{j\pi/4}$ Convert to Rect Form

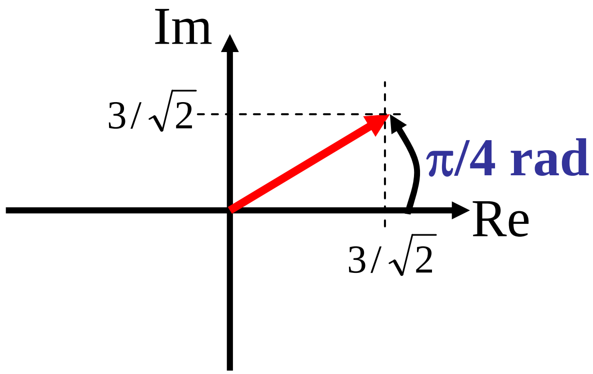
Use: $z = |z|\cos(\angle z) + j|z|\sin(\angle z)$

By Inspection: $|z| = 3 \quad \angle z = \pi/4$

$$\cos(\pi/4) = 1/\sqrt{2} \qquad \sin(\pi/4) = 1/\sqrt{2}$$

Your calculator will give 0.707
but more precisely it is 1/sqrt(2)

$$\boxed{z = 3e^{j\pi/4}} \Leftrightarrow \boxed{z = \frac{3}{\sqrt{2}} + j\frac{3}{\sqrt{2}}}$$



Example #3: Given: $z = je^{-j\pi/2}$ <u>Write it in Polar Form</u>

Isn't it ALREADY in polar form!!!??? **NO!!!!!!!**

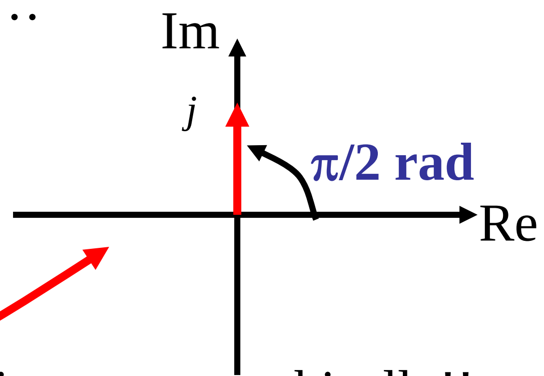
View it as a product of two complex numbers…
and note that the first is in rect form: 0 + j

$$z = \underbrace{[j]}_{z_1}\underbrace{\left[e^{-j\pi/2}\right]}_{z_2}$$

Since multiplication is easier with polar form…
convert the rect form # into polar form

$$|j| = \sqrt{0^2 + 1^2} = 1$$

$$\angle j = \tan^{-1}(1/0) = \pi/2$$



Easier to see graphically!!

$$z = je^{-j\pi/2} = \left[e^{j\pi/2}\right]\left[e^{-j\pi/2}\right] = e^{j(\pi/2-\pi/2)} = e^{j0} = 1$$

$$\boxed{z = je^{-j\pi/2}} \Leftrightarrow \boxed{z = 1}$$

Example #4: Given: $z = \dfrac{2+j3}{-3+j2}$ Find magnitude and angle

Use: $|z_1/z_2| = |z_1|/|z_2|$

$\angle\{z_1/z_2\} = \angle\{z_1\} - \angle\{z_2\}$

$$\left|\frac{2+j3}{-3+j2}\right| = \frac{|2+j3|}{|-3+j2|} = \frac{\sqrt{2^2+3^2}}{\sqrt{(-3)^2+2^2}} = \frac{\sqrt{13}}{\sqrt{13}} = 1$$

$$\angle\left\{\frac{2+j3}{-3+j2}\right\} = \angle\{2+j3\} - \angle\{-3+j2\} \approx 0.983 - (-0.588+\pi) = 1.57\ rad$$

Correcting for case when real part is negative (i.e., quads II & III)

$$\left|\frac{2+j3}{-3+j2}\right| = 1$$

$$\angle\left\{\frac{2+j3}{-3+j2}\right\} \approx 1.57\ rad$$

Exact value is π/2

<u>Example #5a:</u>

Given: $z = \dfrac{R_1 + 1/jC}{R_2 + jL}$ <u>Find magnitude and angle</u>

First some common manipulations:

$$z = \frac{R_1 + 1/jC}{R_2 + jL} = \frac{jC}{jC}\frac{(R_1 + 1/jC)}{(R_2 + jL)} = \frac{jR_1C + jC/jC}{jR_2C + jCjL} = \frac{1 + jR_1C}{-LC + jR_2C}$$

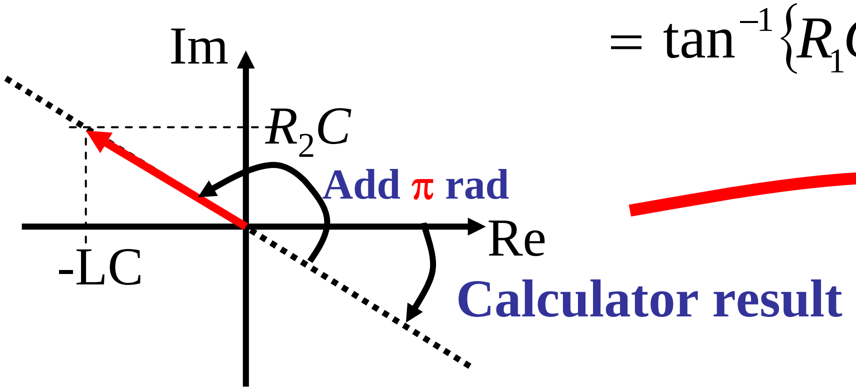
Now to find magnitude:

$$|z| = \frac{|1 + jR_1C|}{|-LC + jR_2C|} = \frac{\sqrt{1^2 + (R_1C)^2}}{\sqrt{(-LC)^2 + (R_2C)^2}} = \frac{\sqrt{1 + (R_1C)^2}}{\sqrt{(LC)^2 + (R_2C)^2}} = \frac{\sqrt{1 + (R_1C)^2}}{C\sqrt{L^2 + R_2^2}}$$

Now to find angle:

$$\angle z = \angle\{1 + jR_1C\} - \angle\{-LC + jR_2C\}$$

$$= \tan^{-1}\{R_1C\} - \left[\tan^{-1}\{-R_2/L\} + \pi\right]$$



Example #5b:

Given: $z = \dfrac{R_1 + 1/jC}{R_2 + jL}$ Find magnitude and angle

A slightly different way to do it:

$$z = \frac{R_1 + 1/jC}{R_2 + jL} = \frac{R_1 - j/C}{R_2 + jL}$$

Now to find magnitude:

$$|z| = \frac{|R_1 - j/C|}{|R_2 + jL|} = \frac{\sqrt{R_1^2 + (-1/C)^2}}{\sqrt{R_2^2 + L^2}} = \frac{\sqrt{R_1^2 + 1/C^2}}{\sqrt{R_2^2 + L^2}}$$

Now to find angle:

$$\angle z = \angle\{R_1 - j/C\} - \angle\{R_2 + jL\}$$

$$= \tan^{-1}\left\{\frac{-1/C}{R_1}\right\} - \tan^{-1}\{L/R_2\}$$

$$= \tan^{-1}\left\{\frac{-1}{R_1 C}\right\} - \tan^{-1}\{L/R_2\}$$

Even though these have a different form than the Ex 5a results they give the exact same <u>numerical</u> values!!!

Example #6:

Given: $$z = \frac{-1}{R_2 + jL}$$ Find magnitude and angle

Find magnitude:

$$|z| = \frac{|-1|}{|R_2 + jL|} = \frac{1}{\sqrt{R_2^2 + L^2}}$$

Now to find angle:

$$\angle z = \angle\{-1\} - \angle\{R_2 + jL\}$$

$$= \pm\pi - \tan^{-1}\{L / R_2\}$$