REVIEW OF COMPLEX NUMBERS

DEF: $j=\sqrt{-1}$. Use j not i in EECS since i=current in EECS!
DEF: A complex number z can be written as $z=x+jy=Me^{j\theta}=M\angle\theta$
where: $x+jy$=**rectangular** form of z; $Me^{j\theta}=M\angle\theta$=**polar** form of z.
AND: $x=Re[z]$; $y=Im[z]$; $M=|z|$; $\theta=arg[z]$.
Real part, Imag part, Magnitude, Argument or phase

LEMMA: **Euler's theorem:** $e^{j\theta}=\cos\theta+j\sin\theta$.
WHY? Insert $x=j\theta$ into the infinite series $e^x=1+x+x^2/2!+\ldots$.
ALSO: $\cos\theta=\frac{1}{2}(e^{j\theta}+e^{-j\theta})$; $\sin\theta=\frac{1}{2j}(e^{j\theta}-e^{-j\theta})$.

Converting rectangular form⇔polar form:

P→R: $Me^{j\theta}=x+jy$ where $x=M\cos\theta$ and $y=M\sin\theta$.
R→P: $x+jy=Me^{j\theta}$ where $M=\sqrt{x^2+y^2}$ and $\tan\theta=y/x$.
Summary: $\begin{cases} x=M\cos\theta \\ y=M\sin\theta \end{cases}$; $M=\sqrt{x^2+y^2}$; $\theta=\begin{cases}\arctan\frac{y}{x} & \text{if } x>0; \\ \arctan\frac{y}{x}\pm\pi & \text{if } x<0\end{cases}$.

EX#1: $+3+j4=5e^{+j0.927}$ since $5=\sqrt{3^2+4^2}$ and $+0.927=\arctan\frac{+4}{+3}$.
EX#2: $-3-j4=5e^{-j2.214}$ since $5=\sqrt{3^2+4^2}$ and $-2.214=\arctan\frac{-4}{-3}-\pi$.
HINT: Draw an *Argand diagram* ($Im[z]$ vs. $Re[z]$) to visualize.
Matlab: `abs(3+4j); angle(3+4j); real(3+4j); imag(3+4j)`

Adding, Subtracting, Multiplying, Dividing:

- **Add and subtract** complex numbers in **rectangular** form.

since: $(a+jb)+(c+jd)=(a+c)+j(b+d)$. **Note:** $Re[z+w]=Re[z]+Re[w]$.
EX: $5e^{j0.927}+\sqrt{2}e^{j0.785}=(3+j4)+(1+j)=4+j5=6.4e^{j0.896}$.

- **Multiply and divide** complex numbers in **polar** form.

since: $(M_1e^{j\theta_1})(M_2e^{j\theta_2})=(M_1M_2)e^{j(\theta_1+\theta_2)}$. **Note:** $|zw|=|z|\cdot|w|$.
EX: $(3+j4)(1+j)=(5e^{j0.927})(\sqrt{2}e^{j0.785})=5\sqrt{2}e^{j1.71}=-1+j7$.

- **Multiplying and dividing** in rectangular form:

$(a+jb)(c+jd)=(ac-bd)+j(ad+bc)$; $\frac{a+jb}{c+jd}=\frac{a+jb}{c+jd}\frac{c-jd}{c-jd}=\frac{ac+bd}{c^2+d^2}+j\frac{bc-ad}{c^2+d^2}$ (ugh)
EX: $(3+j4)(1+j)=(3\cdot1-4\cdot1)+j((4\cdot1)+(3\cdot1))=-1+j7=5\sqrt{2}e^{j1.71}$.

EX: Compute $2\sqrt{3}e^{j\pi/6}+2e^{-j\pi/3}$. Convert polar to rectangular:
$=(2\sqrt{3}\cos(\pi/6)+j2\sqrt{3}\sin(\pi/6))+(2\cos(\pi/3)+j2\sin(-\pi/3))$
$=(2\sqrt{3}(\sqrt{3}/2)+j2\sqrt{3}(1/2))+(2(1/2)-j2(\sqrt{3}/2))=4.$ (!)

EX: $(2\sqrt{3}e^{j\pi/6})/(2e^{-j\pi/3})=\frac{2\sqrt{3}}{2}e^{j(\pi/6-(-\pi/3))}=\sqrt{3}e^{j\pi/2}=j\sqrt{3}$.

PHASORS: COMPLEX NUMBERS REPRESENT SINUSOIDS

Phasors: Represent sinusoid $x(t) = M\cos(\omega t+\theta)$ with complex no. $X = Me^{j\theta}$.
Note: $x(t) = Re[Xe^{j\omega t}] = Re[Me^{j\theta}e^{j\omega t}] = Re[Me^{j(\omega t+\theta)}] = M\cos(\omega t+\theta)$.
Why? $A\cos(\omega t+\theta) + B\cos(\omega t+\phi) = Re[Ae^{j(\omega t+\theta)} + Be^{j(\omega t+\phi)}]$
$= Re[e^{j\omega t}(Ae^{j\theta} + Be^{j\phi})] = Re[e^{j\omega t}(Ce^{j\psi})] = C\cos(\omega t+\psi)$.
where: $Ae^{j\theta} + Be^{j\phi} = Ce^{j\psi}$. Add sinusoids $\Leftrightarrow$ add complex numbers!

EX#1: Simplify $x(t) = 3\cos(\omega t) + 3\cos(\omega t+120^o) + 3\cos(\omega t+240^o)$.
Hard: Use cosine addition formula→mess. If do it right, get $x(t) = 0$ (?!)
Easy: Phasors: $X = 3e^{j0} + 3e^{j120^o} + 3e^{j240^o} = 0 \rightarrow x(t) = Re[Xe^{j\omega t}] = 0$!
Why? Draw picture in complex plane: easy to see resultant of these=0!

EX#2: Show that $5\cos(\omega t+53^o) + \sqrt{2}\cos(\omega t+45^o) = 6.4\cos(\omega t+51^o)$.
Hard: Use cosine addition formula twice. Left as exercise for student.
Easy: $5e^{j53^o} + \sqrt{2}e^{j45^o} = (3+j4) + (1+j) = (4+j5) = 6.4e^{j51^o}$. QED.

---

1. Sinusoids must **all be at the same frequency** to add their phasors. If different frequencies: partition into sums at same frequency.
2. Use $\sin(\omega t+\theta) = \cos(\omega t+\theta-\pi/2)$ as necessary.
3. Multiplying sinusoids is NOT equivalent to multiplying phasors!

OTHER COMPLEX NUMBER FACTS:

DEF: Complex conjugate z^* of z is $z^* = x - jy = Me^{-j\theta} = M\angle -\theta$.

1. $|z^*| = |z|$; $|z|^2 = zz^*$; $arg[z^*] = -arg[z]$; $(zw)^* = z^*w^*$.
2. $\frac{1}{z} = \frac{1}{z}\frac{z^*}{z^*} = \frac{z^*}{|z|^2}$. Compare to rectangular division formula.
3. $z = Me^{j\theta} \rightarrow z^* = Me^{-j\theta}$; $\frac{1}{z} = \frac{1}{M}e^{-j\theta}$; $-z = Me^{j(\theta\pm\pi)}$.
4. $A\cos(\omega t+\theta) = \frac{1}{2}(X + X^*)$ where $X = Ae^{j(\omega t+\theta)} = Ae^{j\theta}e^{j\omega t}$.
5. $Re[z] = \frac{1}{2}(z+z^*)$ and $|e^{j|z|}| = 1$ for **any** complex z.

OTHER COMPLEX NUMBER TRICKS:

1. $Im[3+j4] = 4$ **NOT** $j4$! **VERY common mistake!**.
2. $e^{\pm j\pi} = -1$. $j = e^{j\pi/2}$ and $-j = e^{j3\pi/2} \Leftrightarrow \cos(\omega t \pm 90^o) = \mp\sin(\omega t)$.
3. $j(3+j4) = -4+j3 = (1e^{j90^o})(5e^{j53^o}) = 5e^{j143^o}$: 90^o rotation.
4. Compute $\left|\frac{5(8+j)(8+j6)(5+j12)(5+j10)}{26(7+j4)(7+j24)(2+j11)}\right| = \sqrt{\frac{(25)(65)(100)(169)(125)}{(26)^2(65)(625)(125)}} = 1$.
5. The following identity is true to an overall sign. Why?

$$\frac{(a+jb)(c+jd)}{(e+jf)(g+jh)} = \sqrt{\frac{(a^2+b^2)(c^2+d^2)}{(e^2+f^2)(g^2+h^2)}}\exp j[\tan^{-1}\frac{b}{a} + \tan^{-1}\frac{d}{c} - \tan^{-1}\frac{f}{e} - \tan^{-1}\frac{h}{g}]$$