Part II: Discrete-time signals

- Introduction (Course overview)
- Discrete-time signals
- Discrete-time systems
- Linear time-invariant systems

Part II: Discrete-time signals

- Sequences of numbers

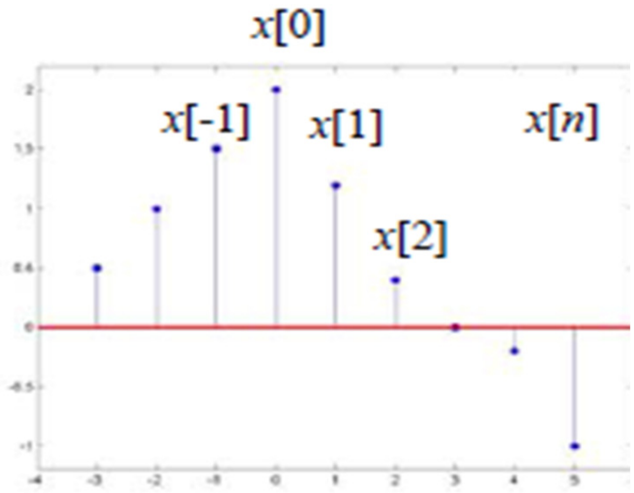
$$x = \{x[n]\}, \qquad -\infty < n < \infty$$

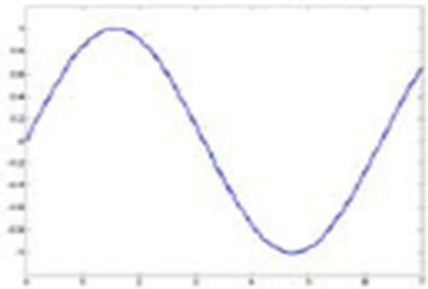
where n is an integer

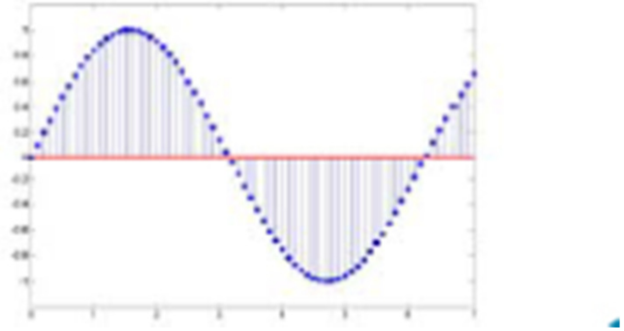
- Periodic sampling of an analog signal

$$x[n] = x_a(nT), \qquad -\infty < n < \infty$$

where T is called the sampling period.







Sequence operations

- The product and sum of two sequences x[n] and y[n]: sample-by-sample production and sum, respectively.
- Multiplication of a sequence x[n] by a number α : multiplication of each sample value by α.
- Delay or shift of a sequence x[n]

$$y[n] = x[n - n_0]$$

where n is an integer

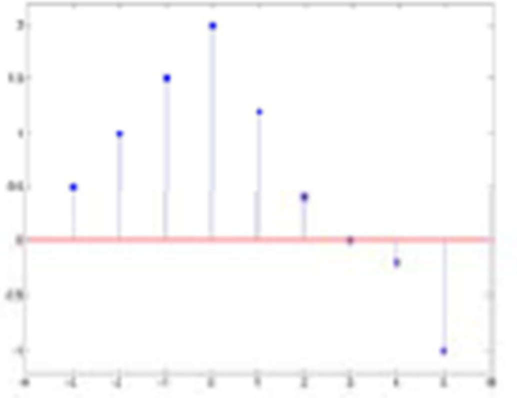
Basic sequences

- Unit sample sequence (discrete-time impulse, impulse)

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0, \end{cases}$$

- Any sequence can be represented as a sum of scaled, delayed impulses

$$x[n] = a_{-3}\delta[n+3] + a_{-2}\delta[n+3] + \ldots + a_5\delta[n-5]$$



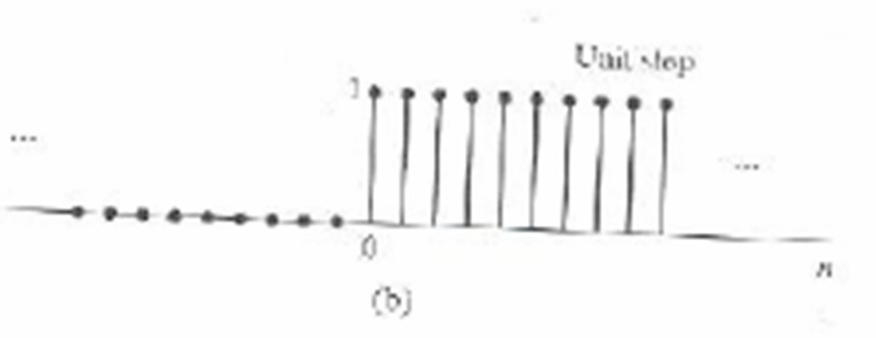
- More generally

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Unit step sequence

- Defined as

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases}$$



- Related to the impulse by

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \ldots$$

or

$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$$

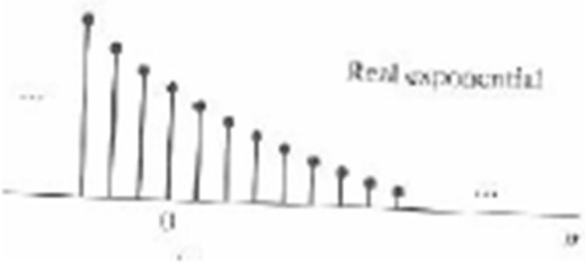
- Conversely,

$$\delta[n] = u[n] - u[n-1]$$

Exponential sequences

- Extremely important in representing and analyzing LTI systems.
- Defined as

$$x[n] = A\alpha^n$$



- If A and α are real numbers, the sequence is real.
- If $0 < \alpha < 1$ and A is positive, the sequence values are positive and decrease with increasing n.
- If $-1 < \alpha < 0$, the sequence values alternate in sign, but again decrease in magnitude with increasing n.
- If $|\alpha| > 1$, the sequence values increase with increasing n.

$x[n] = 2 \cdot (0.5)^n$

$x[n] = 2 \cdot (-0.5)^n$

$x[n] = 2 \cdot 2^n$

Combining basic sequences

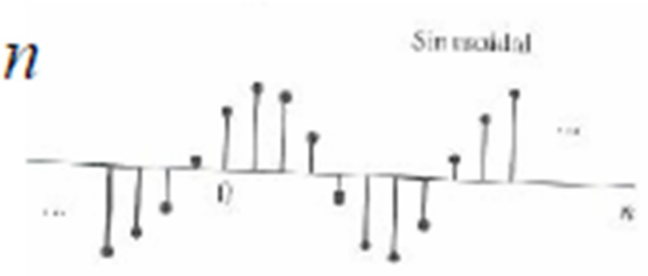
- An exponential sequence that is zero for n<0

$$x[n] = \begin{cases} A\alpha^n, & n \geq 0, \\ 0, & n < 0 \end{cases}$$

$$x[n] = A\alpha^n u[n]$$

Sinusoidal sequences

$$x[n] = A\cos(\omega_0 n + \phi), \qquad \text{for all } n$$



with A and ϕ real constants.

- The $A\alpha^n$ with complex α has real and imaginary parts that are exponentially weighted sinusoids.

If $\alpha = |\alpha| e^{j\omega_0}$ and $A = |A| e^{j\phi}$, then

$$x[n] = A\alpha^n = |A| e^{j\phi} |\alpha|^n e^{j\omega_0 n}$$

$$= |A||\alpha|^n e^{j(\omega_0 n+\phi)}$$

$$= |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi)$$

Complex exponential sequence

When $|\alpha| = 1$,

$$x[n] = |A| e^{j(\omega_0 n + \phi)} = |A| \cos(\omega_0 n + \phi) + j |A| \sin(\omega_0 n + \phi)$$

- By analogy with the continuous-time case, the quantity ω_0 is called the frequency of the complex sinusoid or complex exponential and ϕ is call the phase.
- *n* is always an integer → differences between discrete-time and continuous-time

An important difference – frequency range

- Consider a frequency $(\omega_0 + 2\pi)$

$$x[n] = Ae^{j(\omega_0 + 2\pi)n} = Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n}$$

- More generally $(\omega_0 + 2\pi r)$, r being an integer,

$$x[n] = Ae^{j(\omega_0 + 2\pi r)n} = Ae^{j\omega_0 n}e^{j2\pi rn} = Ae^{j\omega_0 n}$$

- Same for sinusoidal sequences

$$x[n] = A\cos[(\omega_0 + 2\pi r)n + \phi] = A\cos(\omega_0 n + \phi)$$

- So, only consider frequencies in an interval of 2π such as

$$-\pi < \omega_0 \leq \pi \text{ or } 0 \leq \omega_0 < 2\pi$$

Another important difference – periodicity

- In the continuous-time case, a sinusoidal signal and a complex exponential signal are both periodic.
- In the discrete-time case, a periodic sequence is defined as

$$x[n] = x[n+N], \qquad \text{for all } n$$

where the period *N* is necessarily an integer.

- For sinusoid,

$$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi)$$

which requires that $\omega_0 N = 2\pi k$ or $N = 2\pi k / \omega_0$

where k is an integer.

Another important difference – periodicity

- Same for complex exponential sequence

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n},$$

which is true only for $\omega_0 N = 2\pi k$

- So, complex exponential and sinusoidal sequences
  - are not necessarily periodic in n with period $(2\pi / \omega_0)$
  - and, depending on the value of ω_0 , may not be periodic at all.
- Consider

$x_1[n] = \cos(\pi n / 4),$ with a period of $N = 8$

$x_2[n] = \cos(3\pi n / 8),$ with a period of $N = 16$

Increasing frequency → increasing period!

Another important difference – frequency

- For a continuous-time sinusoidal signal

$$x(t) = A\cos(\Omega_0 t + \phi),$$

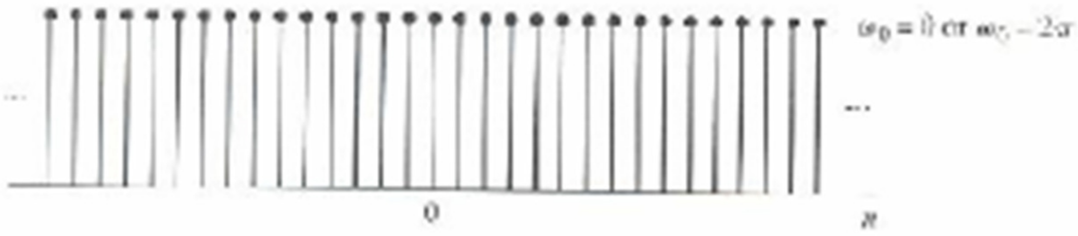
as Ω_0 increases, $x(t)$ oscillates more and more rapidly

- For the discrete-time sinusoidal signal

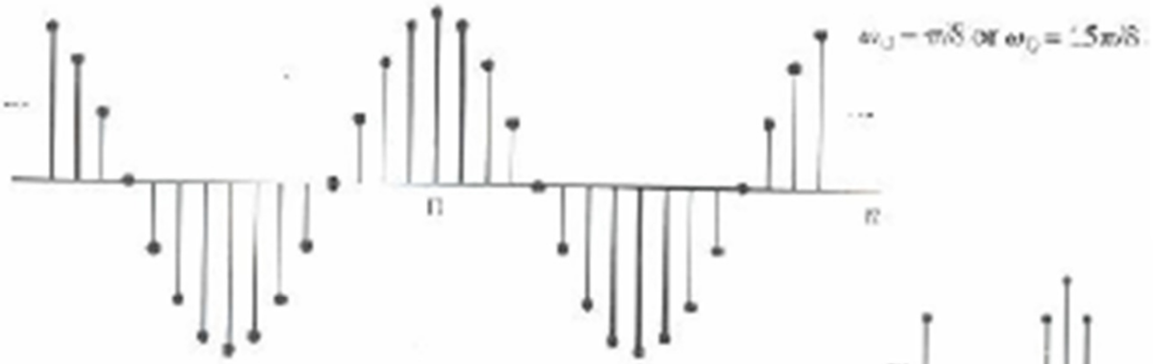
$$x[n] = A\cos(\omega_0 n + \phi),$$

as ω_0 increases from 0 towards π, $x[n]$ oscillates more and more rapidly

as ω_0 increases from π towards 2π, the oscillations become slower.


$\omega_0 = 0$ or $\omega_0 = 2\pi$
...
...
0
n

(a)


$\omega_0 = \pi/8$ or $\omega_0 = 15\pi/8$
...
...
0
n

(b)


$\omega_0 = \pi/4$ or $\omega_0 = 7\pi/4$
...
...
0
n

(c)


$\omega_0 = \pi$
...
...
...
...
0
n

(d)

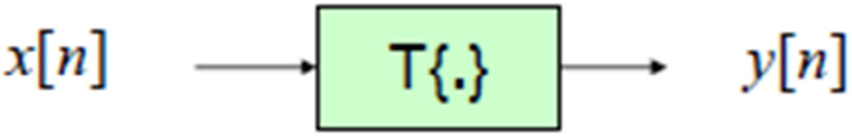
Part II: Discrete-time systems

- Introduction
- Discrete-time signals
- Discrete-time systems
- Linear time-invariant systems

Discrete-time systems

- A transformation or operator that maps input into output

$$y[n] = T\{x[n]\}$$



- Examples:
    - The ideal delay system

$$y[n] = x[n - n_d], \qquad -\infty < n < \infty$$

    - A memoryless system

$$y[n] = (x[n])^2, \qquad -\infty < n < \infty$$

Linear systems

- A system is linear if and only if

additivity property

$$T\{x_1[n]+x_2[n]\}=T\{x_1[n]\}+T\{x_2[n]\}=y_1[n]+y_2[n]$$

and

$$T\{ax[n]\}=aT\{x[n]\}=ay[n]$$ scaling property

where a is an arbitrary constant

- Combined into superposition

$$T\{ax_1[n]+bx_2[n]\}=aT\{x_1[n]\}+aT\{x_2[n]\}=ay_1[n]+ay_2[n]$$

- Example 2.6, 2.7 pp. 19

Time-invariant systems

- For which a time shift or delay of the input sequence causes a corresponding shift in the output sequence.

$$x_1[n] = x[n-n_0] \Rightarrow y_1[n] = y[n-n_0]$$

- Example 2.8 pp. 20

Causality

- The output sequence value at the index n=n_0 depends only on the input sequence values for n<=n_0.

- Example $y[n] = x[n - n_d], \qquad -\infty < n < \infty$
  - Causal for n_d>=0
  - Noncausal for n_d<0

Stability

- A system is stable in the BIBO sense if and only if every bounded input sequence produces a bounded output sequence.

- Example

$$y[n] = (x[n])^2, \qquad -\infty < n < \infty$$

stable

Part III: Linear time-invariant systems

- Course overview
- Discrete-time signals
- Discrete-time systems
- Linear time-invariant systems

Linear time-invariant systems

- Important due to convenient representations and significant applications
- A linear system is completely characterised by its impulse response

$$y[n] = T\{x[n]\} = T\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\}$$

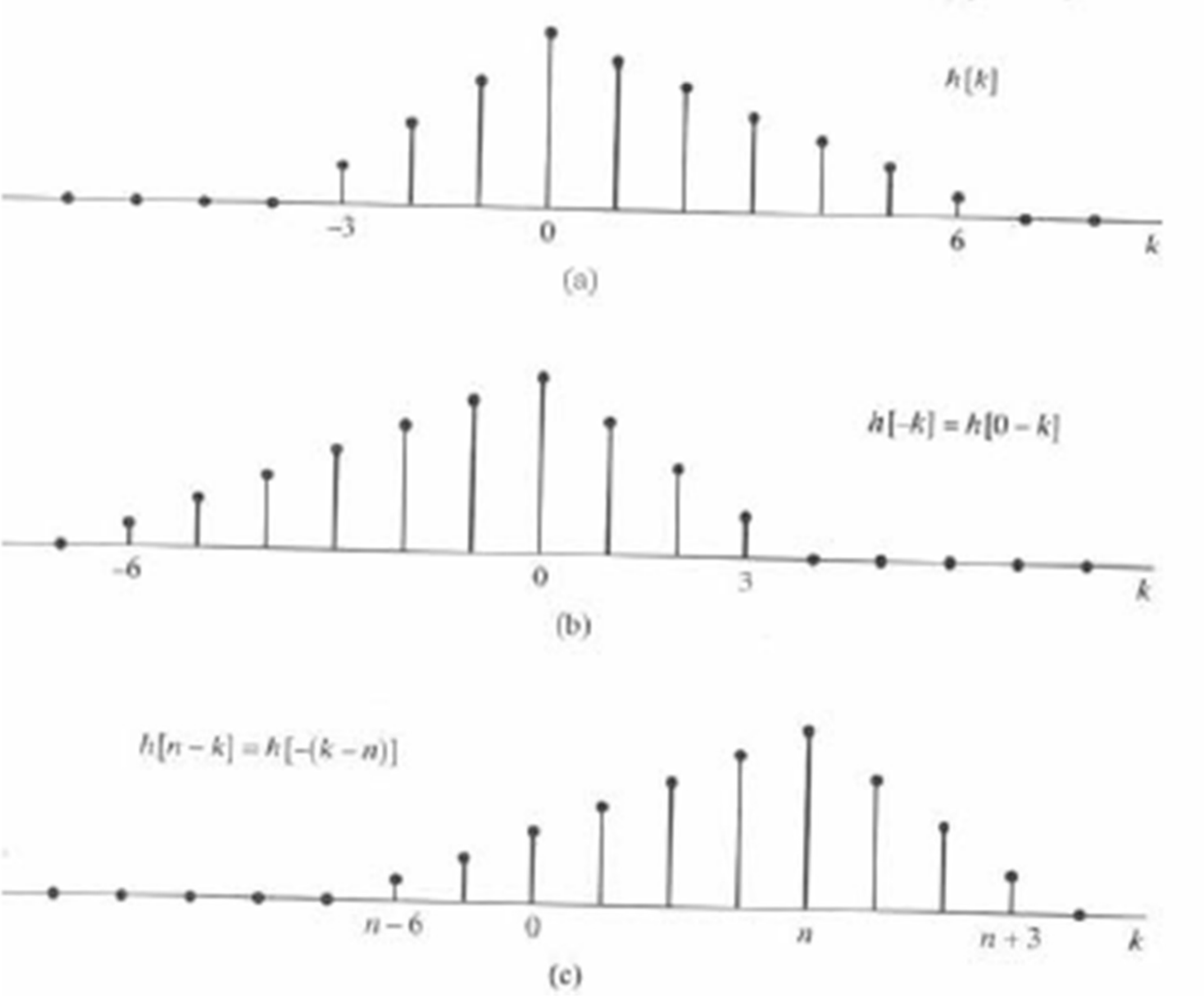
$$= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

- Time invariance $h_k[n] = h[n-k]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= x[n] * h[n]$$ Convolution sum

Forming the sequence h[n-k]



Computation of the convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Obtain the sequence h[n-k]
  - Reflecting h[k] about the origin to get h[-k]
  - Shifting the origin of the reflected sequence to k=n
- Multiply x[k] and h[n-k] for $-\infty < k < \infty$
- Sum the products to compute the output sample y[n]

Computing a discrete convolution

Example 2.13 pp.26

Impulse response

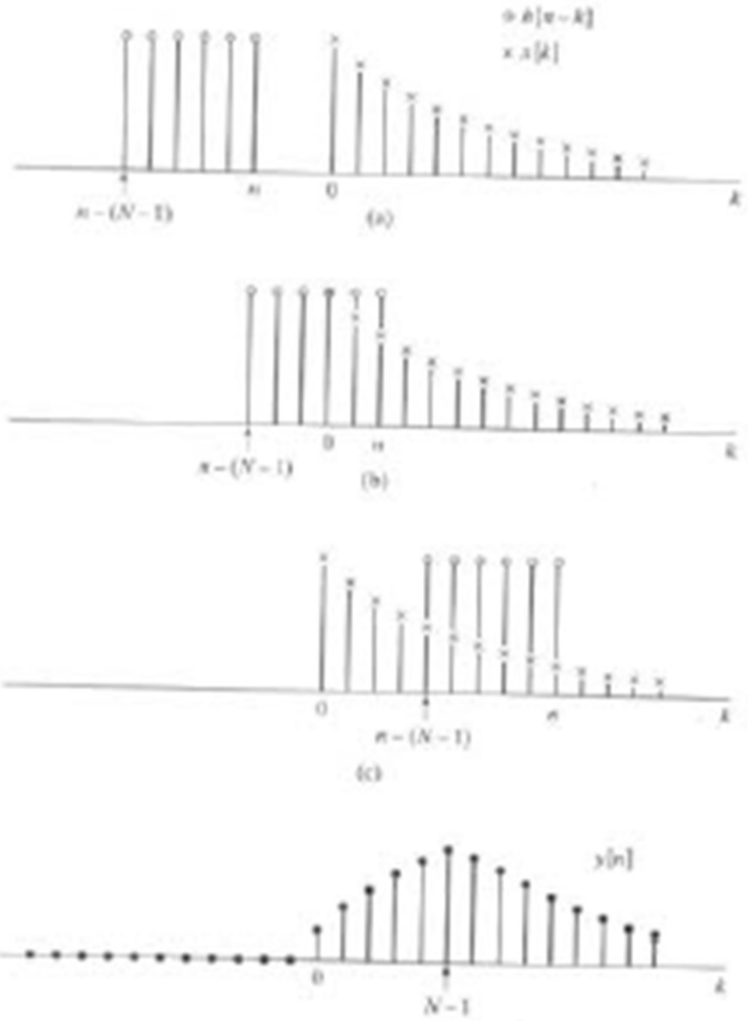
$$h[n] = u[n] - u[n-N]$$

$$= \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

input

$$x[n] = a^n u[n]$$

$$y[n] = \begin{cases} 0, & n < 0, \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \le n \le N-1, \\ a^{n-N+1}\left(\dfrac{1-a^N}{1-a}\right), & N-1 < n. \end{cases}$$



Computing discrete convolution - Tabular method

- Finite-discrete sequence: **y[n] = x[n] * g[n]**

x[n] = {x[0], x[1], x[2], x[3], x[4]}
g[n] = {g[0], g[1], g[2], g[3]}

n:	0	1	2	3	4			
x([n]	x[0]	x[1]	x[2]	x[3]	x[4]			
g([n]	g[0]	g[1]	g[2]	g[3]				
	x[0]g[0]	x[1]g[0]	x[2]g[0]	x[3]g[0]	x[4]g[0]			
	-	x[0]g[1]	x[1]g[1]	x[2]g[1]	x[3]g[1]	x[4]g[1]		
		-	x[0]g[2]	x[1]g[2]	x[2]g[2]	x[3]g[2]	x[4]g[2]	
			-	x[0]g[3]	x[1]g[3]	x[2]g[3]	x[3]g[3]	x[4]g[3]
	Σ	Σ						
	y[0]	y[1]	y[2]	y[3]	y[3]	y[4]	y[5]	y[6]

What is the maximum length of y[n]?

Computing discrete convolution - Tabular method

Example: consider the following two finite-length sequences,

$$x[n] = \{-2 \quad \underset{\substack{\uparrow \\ n=0}}{0} \quad 1 \quad -1 \quad 3 \quad \} \qquad\qquad h[n] = \{\underset{\substack{\uparrow \\ n=0}}{1} \quad 2 \quad 0 \quad -1 \quad \}$$

Find y[n] = x[n] * h[n]?

Properties of LTI systems

- Defined by discrete-time convolution
  - Commutative

$$x[n]*h[n]=h[n]*x[n]$$

  - Linear

$$x[n]*(h_1[n]+h_2[n])=x[n]*h_1[n]+x[n]*h_2[n]$$

  - Cascade connection (Fig. 2.11 pp.29)

$$h[n]=h_1[n]*h_2[n]$$

  - Parallel connection (Fig. 2.12 pp.30)

$$h[n]=h_1[n]+h_2[n]$$

Properties of LTI systems

- Defined by the impulse response
  - Stable

$$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

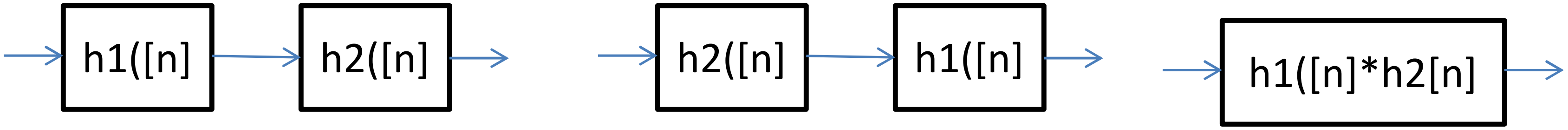
  - Causality

$$h[n] = 0, \qquad n < 0$$

- If x[n] and h[n] are causal sequences,
  then y[n] = x[n]*h[n] is also a causal sequence.

Simple Interconnection Schemes

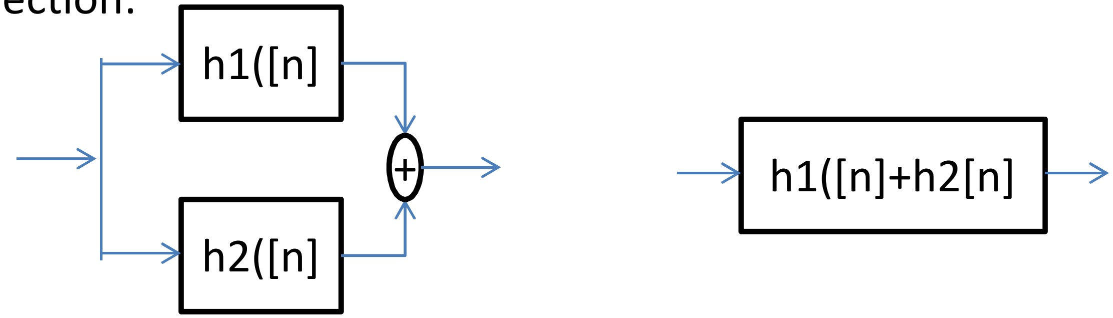
- Cascade connection:



h1([n] h2([n] h2([n] h1([n] h1([n]*h2[n]

Cascade of stable systems is stable

- Parallel connection:



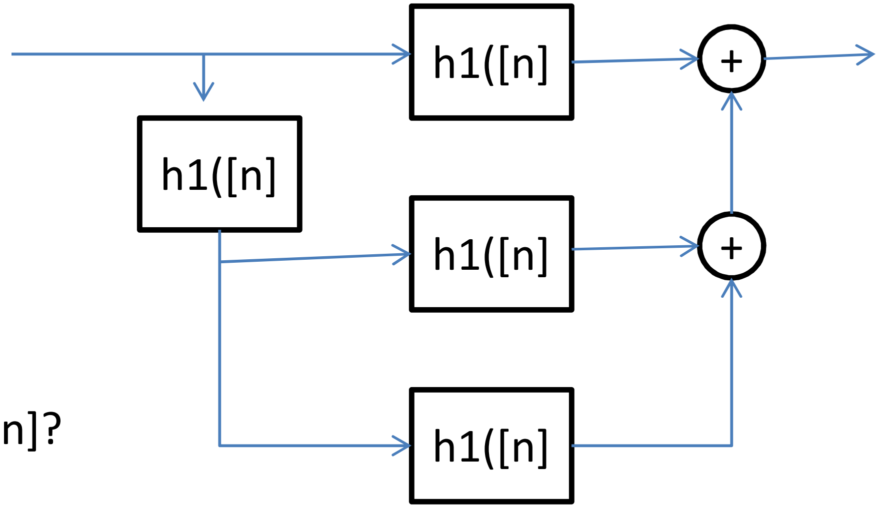
h1([n] h2([n] + h1([n]+h2[n]

Parallel of stable systems is also stable

Simple Interconnection Schemes -example

h1[n] = δ[n] + 0.5δ[n-1]
h2[n] = 0.5δ[n] – 0.25δ[n-1]
h3[n] = 2δ[n]
h4[n] = -2(0.5)^n u[n]

Find the overall impulse response h[n]?



Linear Constant-Coefficient Difference Equations (LCCDE)

Illustration On the board:

LCCDE

Frequency Response of LTI systems:

Why it is important:
A- easily obtained directly from unit sample response.
B – Freq. Resp. allows us to obtain the response of system to sinusoidal excitation. And an arbitrary sequence can be represented as a linear combination of complex exponential or sinusoidal sequences.

Properties of Freq. Response:
A- function of continuous variable ω (changes continuously)
B – periodic function of ω. period = 2π

Matlab Functions

Plot signals:

-stem(xn)
-plot(n,xn)

Computing Convolution:

- conv(xn,hn)

Computing Cross-correlation (Rxy) and auto-correlation (Rxx):

- Rxy = conv(xn,fliplr(yn))
- Rxy = xcorr(xn,yn)
- -Rxx = xcorr(xn)