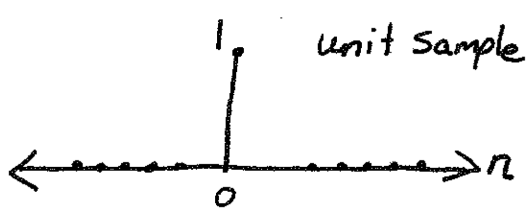
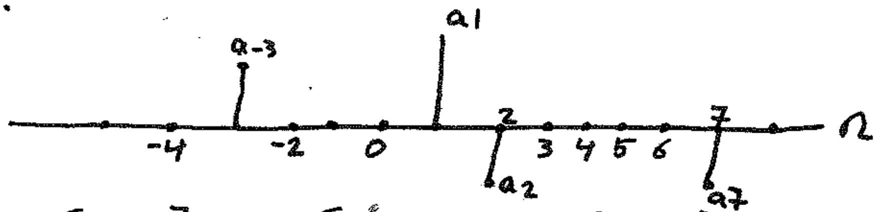
Basic sequence:

① The Unit sample sequance



$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

Called discrete time impulse on only impulse.
Any arbitray sequence can be represented a sum of scaled, delayed impulse. For example the sequence $p[n]$ in the following figure:



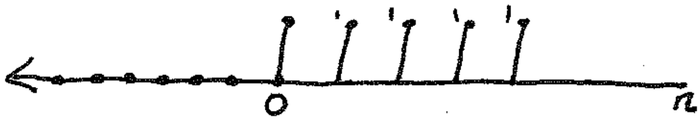
$$p[n] = a_{-3}\,\delta[n+3] + a_1\,\delta[n-1] + a_2\,\delta[n-2] + a_7\,\delta[n-7].$$

more generally, any sequence can be expressed as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].$$

② The unit step sequence is given by

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

→ The Value of unit step seq, at (time) Index n is equal to the accumulated sum of the Value at Index n and all previous Values of the Impulse Sequence.

* An alternative representation of unite step in terms of the impulse is sum of delayed impulses. (non-Zero Values are all unity).

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

conversely, The impulse sequence can

be expressed as first backward differance of the unit step sequence :. $\delta[n] = u[n] - u[n-1]$

③ exponential sequences (important for representing and analyzing linear time-invariant (LTI) discrete-time systems).

$$x[n] = A\,\alpha^n$$

If A and α are real numbers, then sequence is real.

* if $0 < \alpha < 1$ and A is positive, then sequence values are positive and decrease with increasing n.

* For $-1 < \alpha < 0$, the seq. values ~~alternate~~ in alternate in sign, But decrease in magnitude with increasing n. If $|\alpha| > 1$, then the seq. grows in magnitude as n increases.

Example: Combining Basic Sequences

we want an exponential sequence that is zero for $n < 0$,

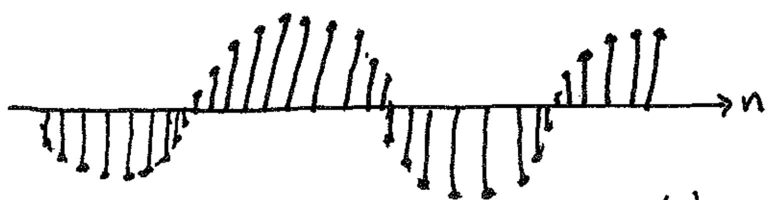
$$x[n] = \begin{cases} A\alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

A much simpler expression is $x[n] = A\,\alpha^n\,u[n]$.

④ Sinusoidal sequences :

$$x[n] = \underbrace{A}_{\text{Amplitude}} \cos(\underbrace{\omega_0}_{\text{freq.}} n + \underbrace{\phi}_{\text{phase shift}}) \text{, for all } n \text{ (periodic)}$$

Where, A and ϕ are real constants.



$\omega_0 = \pi/4$

$\phi = -\pi/8$

* The exponential sequence $A\alpha^n$ with complex α has real and imajinary parts that are exponentially weighted sinusiods.

$$\text{if } \alpha = |\alpha| e^{j\omega_0} \text{ and } A = |A| e^{j\phi},$$

the sequence $A\alpha^n$ can be expressed in any of the following forms:

$$
\begin{aligned}
x[n] = A\alpha^n &= |A| e^{j\phi} |\alpha|^n e^{j\omega_0 n} \\
&= |A| |\alpha|^n e^{j\omega_0 n} \\
&= |A| |\alpha|^n e^{j(\omega_0 n + \phi)} \\
&= |A| |\alpha|^n \cos(\omega_0 n + \phi) + j |A| |\alpha|^n \sin(\omega_0 n + \phi)
\end{aligned}
$$

Remider:
$$\boxed{e^{jx} = \cos(x) + j\sin(x)}$$

* If $|\alpha| > 1 \rightarrow$ sequence oscillates with an exponentially growing envelope.

If $|\alpha| < 1 \rightarrow$ seq. oscillates with an exponentially decaying envelope.

$\rightarrow$ As a simple example, consider $\omega_0 = \pi$.

When ~~|A|~~ $|\alpha| = 1$, the sequence is refered to as a "Complex exponential seq.".

$$x[n] = |A| e^{j(\omega_0 n + \phi)} = |A| \cos(\omega_0 n + \phi) + j |A| \sin(\omega_0 n + \phi) \quad \cdots (*)$$

* That is the real and imaginary parts of $e^{j\omega_0 n}$ vary sinusoidally with n.

* By analogy with continous-time case, $\omega_0 \rightarrow$ called frequency

$\phi \rightarrow$ phase

$n \rightarrow$ dimensionless Integer $\Longrightarrow$ ω_0 is in radians.

OR we can specify the unit of ω_0 to be radians per sample and unit of (n) to be samples.

* The fact that n is always Integer in eq. (*) leads to some important difference between the properties of continous-time and discrete-time complex exponential sequences and sinusoids sequences.

An important difference (sinusoids) is seen when we consider a frequency $(\omega_0 + 2\pi)$. In this case:

$$x[n] = A e^{j(\omega_0 + 2\pi)n}$$
$$= A e^{j\omega_0 n} \cancel{e^{j2\pi n}}^{1} = A e^{j\omega_0 n}$$

So, complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$, r is integer, are Indistinguishable from one another. (Same for sinusoids).

$$x[n] = A \cos((\omega_0 + 2\pi r)n + \phi)$$
$$= A \cos(\omega_0 n + \phi).$$

* So, we conclude that when discussing complex exponential signals $x[n] = A e^{j\omega_0 n}$ or real sinusoidal signals $x[n] = A \cos(\omega_0 n + \phi)$, we need only consider frequencies in an interval of length 2π, such as $-\pi \leq \omega_0 \leq \pi$ or $0 \leq \omega_0 \leq 2\pi$.

* Another important difference between Continuous-time (CT) and Discrete-time (DT) is in their periodicity.

— In CT case, complex exponential and sinusiods are both periodic with period equal to $2\pi/f$ $\left(T = \frac{2\pi}{f}\right)$.

— In DT case, a periodic sequence is a sequence for which

$$x[n] = x[n+N], \text{ for all } n.$$

Where, the period N is necessarily an Integer.

* If we test this condition for DT sinusoid, then

$$A\cos(\omega_0 n+\phi) = A\cos(\omega_0 n+\omega_0 N+\phi)$$

Which requires that:

$$\omega_0 N = 2\pi k \quad , \quad k \text{ is an Integer}$$

* Samething for complex exponential seq. $Ce^{j\omega_0 n} \Rightarrow$ periodicity with period N. requires:

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \quad \text{Which is true}$$

only for $\omega_0 N = 2\pi k$.

* Consequently, Complex exponential and Sinusoidal sequences are not necessarily periodic in (n) with period $(\frac{2\pi}{\omega_0})$, and depending on the value of ω_0 maynot be periodic at all.

$\boxed{\text{Example}}$: Periodic and Aperiodic DT seq.

Let $x_1(n) =$ ~~cos(π(n+8)/n~~ $\cos(\pi n/4)$, this signal has a period of $N=8$. To show this:

$$x(n+8) = \cos(\pi(n+8)/4) = \cos(\pi n/4 + 2\pi) = \cos(\pi n/4)$$
$$= x(n).$$

$\therefore$ Thus satisfying the difinition of DT periodic signal.

$$\downarrow T = \frac{2\pi}{f\uparrow} \quad (\text{not necessarily true in DT}).$$

Let $x_2(n) = \cos(\frac{3\pi n}{8})$ has a higher frequency than $x_1(n)$.

However, $x_2(n)$ is not periodic with period 8, since

$$x_2(n+8) = \cos(3\pi(n+8)/8) = \cos(3\pi n/8 + 3\pi) = -x_2(n)$$

$x_2(n)$ has a period of $N=16$, thus Increasing freq. from $w_0 = \frac{2\pi}{8}$ to $w_0 = \frac{3\pi}{8}$ also Increase the period of the signal.

* This occurs because DT signals are defined only for Integers indices (n).

* The integer restriction on (n) causes some sinusoidal signals not to be periodic at all. For example, there is no Integer N such that the signal $x_3(n) = \cos(n)$ satisfies

$$X_3(n+N) = X_3(n) \quad \text{for all } n.$$

* High and low frequencies are different for CT and D.T. sinusoidal signals and complex exponential signals.

— For CT sinusoidal signal $x(t) = A\cos(\Omega_0 t + \phi)$, as Ω_0 increases, $x(t)$ oscillates more rapidly. For DT sinusoidal signal $x[n] = A\cos(w_0 n + \phi)$, As w_0 Increases from $w_0 = 0$ towards $w_0 = \pi$, $x[n]$ oscillates more and more rapidly. Howevere, as w_0 Increases from $w_0 = \pi$ towards $w_0 = 2\pi$, $x[n]$ oscillation becomes slower. Because of the periodicity in w_0 of sinusoidal and complex exp. sequences, $w_0 = 2\pi$ is indistinguishable from freq. $w_0 = 0$, and more generally, frequencies around $w_0 = 2\pi$ are indistinguishable from frequences around $w_0 = 0$.

* As A consequence, for sinusoidal and complex exp. signals values of w_0 in the vicinity of $w_0 = 2\pi k$ for any Integer k are typically reffered to as low frequencies (slow oscillation), while values of w_0 in the vicinity of $w_0 = (\pi + 2\pi k)$ for any Integer k are reffered to as high frequencies (rapid oscillation).

2.2 Discrete-Time Systems

$$x(n) \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y(n)$$

$y(n) = T\{x(n)\}$, $T \rightarrow$ System Transformation

$x(n) \longrightarrow y(n)$

*** Special Cases:**

* linear
* Shift-Invariant (or Time-Invariant)

CT → LTI , DT → LSI or LTI

linearity:

If $x_1(n) \longrightarrow y_1(n)$
and $x_2(n) \longrightarrow y_2(n)$
Then, $a x_1(n) + b x_2(n) \longrightarrow a y_1(n) + b y_2(n)$

* linear combination of inputs produces the same linear combination of the corresponding outputs.

$$\sum a_k x_k(n) \Longrightarrow \sum a_k y_k(n)$$

Shift-Invariant (or Time-Invariant)

$x(n) \longrightarrow y(n)$
$x(n-n_0) \longrightarrow y(n-n_0)$.

If we shift the input seq. in (n), we also shift output seq. in (n) (same amount of shift).

e.g.
(unit sample) $\delta(n) \longrightarrow h(n)$ (called unit sample response)
$\delta(n-k) \longrightarrow h(n-k)$

$$x(n) = \sum_{k=-\infty}^{\infty} \underbrace{x(k)\,\delta}_{\text{Linearity}}\underbrace{(n-k)}_{\text{Time-Invariant}}$$ linear combination of basic inputs.

(LTI)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$ by properity of LTI.

key Result For LTI systems / the response of an arbitrary system can be determined by knowing the response of unit sample, we can construct $y(n)$.

This refered to Convolution Sum

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$n-k = r \quad , \quad k = n-r$$

$$y(n) = \sum_{r} x(n-r)\,h(r)$$

$\Rightarrow$ This means that the system doesn't particularly care what we call input to the system and what we call the unit sample response of the system. $\Rightarrow$ Convolution is ccumulative

$$\text{i.e.} \quad y(n) = x(n) \overset{\text{convolve}}{*} h(n)$$
$$= h(n) * x(n)$$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$h(n) \longrightarrow \boxed{x(n)} \longrightarrow y(n)$$

This implies that in LTI Cascade system, the order of the cascaded systems is not important.

$$x(n) \longrightarrow \boxed{h_1(n)} \longrightarrow \boxed{h_2(n)} \longrightarrow y(n)$$

System response is $h_1(n) * h_2(n)$

$$x(n) \longrightarrow \boxed{h_2(n)} \longrightarrow \boxed{h_1(n)} \longrightarrow y(n)$$

sys. response is $h_2(n) * h_1(n) = h_1(n) * h_2(n)$

Example: Ideal delay system

$$y(n) = x(n - n_d) \quad , -\infty < n < \infty$$

$n_d \rightarrow$ Constant positive Integer, Called the delay of the sys. i.e. shifts the Input seq. to the right by n_d samples.

If n_d is fixed Negative Integer $\Rightarrow$ shifts Input Sequence to the left by $|n_d|$ samples, Corresponding to a time advance.

Example: Moving averger

$$y(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

$$= \frac{1}{M_1 + M_2 + 1} \{ x[n+M_1] + x[n+M_1-1] + \dots + x[n] + x[n-1] + \dots + x[n-M_2] \}$$

* For $M_1 = 0$, $M_2 = 5$

$$y(7) = \frac{1}{6}[x(7) + x(6) + x(5) + x(4) + x(3) + x(2))$$

$$y(8) = \frac{1}{6}\{x(8) + x(7) + x(5) + x(4) + x(3)]$$

* Memoryless Systems:

A System is referred to as memoryless if the output $y[n]$ at every value of n depends only on the Input $x[n]$ at the same value of n.

<u>Example</u>: $y[n] = (x[n])^2$, for each value of n.
This is memoryless system.

$\Rightarrow$ What is about Ideal Delay system ? Moving averge system?

$\Rightarrow$ When are they memoryless ?
($n_d = 0$ and $M_1 = M_2 = 0$).

<u>Example</u>: Linear system "Accumulator system"

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

* output at time (n) is the sum of the present and all past input samples. $\Rightarrow$ Accumulator sys. is a linear system !

<u>Proof</u>:
~~Let~~ $x_1[n]$ and $x_2[n]$ are two arbitrary Inputs to the system.
So, their corresponding outputs:

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] \quad , \quad y_2[n] = \sum_{k=-\infty}^{n} x_2[k]$$

$\Rightarrow$ When the input $x_3[n] = a x_1[n] + b x_2[n]$, the superposition principle requires that $y_3[n] = a y_1[n] + b y_2[n]$ for all possible choices of a and b

$$\Rightarrow y_3(n) = \sum_{k=-\infty}^{n} x_3[k]$$

$$= \sum_{k=-\infty}^{n} \left(a x_1(k) + b x_2(k) \right)$$

$$= a \sum_{k=-\infty}^{n} x_1(k) + b \sum_{k=-\infty}^{n} x_2(k)$$

$$= a y_1(n) + b y_2(n) \quad \because$$

Example: A nonlinear system

$$w[n] = \log_{10}(|x(n)|)$$

This system is nonlinear?? prove?

* To prove this we only need to find one counterexample.

$$x_1(n) = 1, \quad x_2(n) = 10$$

$$\Rightarrow w_1[n] = 0, \quad w_2[n] = 1$$

now, since $x_2[n] = 10\, x_1[n]$ if system is linear, it must be that $w_2[n] = 10\, w_1[n]$. Since this is not true for this set of inputs and outputs so the system is not linear.

* Is Accumulator system, $y(n) = \sum_{k=-\infty}^{n} x(k)$, a Time-Invariant sys.?

Proof: let $x_1[n] = x[n-n_0]$, To show shift-Invariant, we solve for both $y(n-n_0)$ and $y_1(n)$ and compare them.

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x(k)$$

$$y_1(n) = \sum_{k=-\infty}^{n} x_1(k)$$

$$= \sum_{k=-\infty}^{n} x[k-n_0]$$

Substituting change of variables $k_1 = k - n_0$ into the summation gives

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y(n-n_0)$$

$\therefore$ Thus Accumulator is a Time-Invariant system.

Example: "Compressor System" Not Time-Invariant system.

$$y(n) = x[Mn], \quad -\infty < n < \infty$$

with M a positive Integer. It discards $(M-1)$ samples out of M

Let $y_1(n) \longrightarrow x_1(n) = x[n-n_0]$

In order for system to be Time-Invariant, the output of the system when the input is $x_1(n)$ must be equal to $y(n-n_0)$

$$y_1(n) = x_1(Mn) = x(Mn-n_0)$$

$\Rightarrow$ Delaying $y(n)$ by n_0 samples yeilds:

$$y(n-n_0) = x(M(n-n_0))$$

comparing these two outputs, we see $y(n-n_0) \neq y_1(n)$ for all M and n_0, therfore this system is not Time-Invariant.

* Alternativily, we can prove this by finding one Counterexample.
When $M=2$, $x(n) = \delta(n)$, and $x_1(n) = \delta(n-1)$
For this choices of inputs and M, $y(n) = \delta(n)$, but $y_1(n) = 0$
thus, $y_1(n) \neq y(n-1)$ for this system.

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* Two Additional Constraints: Causality and Stability.

$$x(n) \longrightarrow \boxed{T\{\ \}} \longrightarrow y(n)$$

General: $y(n) = T[x[n]]$

LTI: $$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$
$$= \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \longrightarrow \text{Convolution Sum}$$

* Stability: [Bounded-Input Bounded-Output (BIBO)].
general: If $x(n)$ bounded, i.e. $|x(n)| < B_x < \infty$, all (n)
then, $y(n)$ is bounded i.e. $|y(n)| < B_y < \infty$ all (n)

LTI: $$\sum_{k=-\infty}^{+\infty} |h(k)| < \infty \text{ (absolutly Summable)}.$$

e.g. $h(n) = 2^n u(n) \longrightarrow$ Unstable
$h(n) = (\frac{1}{2})^n u(n) \longrightarrow$ Stable.

* Causility: $y(n)$ for $n = n_1$ depends on $x(n)$ only for $n \leq n_1$ (non-anticipante system).

for LTI: $h(n) = 0 \quad , n < 0$

e.g: $h(n) = 2^n u(-n) \xrightarrow{\text{non-zero for negative } n} \text{ non-Causal system, stable.}$

e.g2: Forward difference system

$$y(n) = x(n+1) - x(n)$$

This system is not Causal, Since the current value of the output depends on a future value of input.

Set $x_1(n) = \delta(n-1)$ and $x_2(n) = 0 \Rightarrow$

$y_1(n) = \delta(n) - \delta(n-1)$ and $y_2(n) = 0$.

Note that $x_1(n) = x_2(n)$ for $n \leq 0$, so $y_1(n) = y_2(n)$ for $n < 0$ which is not for $n = 0$.

$\Rightarrow$ Thus, by this Counterexample, this system is not Causal.

Simple Interconnection Schemes:

* Two common schemes are used to develop complex LTI system from simple LTI systems.

1. Cascade Connection

$$\rightarrow \boxed{h_1(n)} \rightarrow \boxed{h_2(n)} \rightarrow \quad \equiv \quad \rightarrow \boxed{h_2(n)} \rightarrow \boxed{h_1(n)} \rightarrow \quad \equiv \quad \rightarrow \boxed{h_1(n) * h_2(n)} \rightarrow$$

(Commulative property of Convolution)

$$h(n) = h_1(n) * h_2(n) = h_2(n) * h_1(n).$$

* Cascade of stable systems is stable, likewise the cascade of Passive (Lossless) systems is also passive (lossless).

* An application of cascade connection scheme is developing of an Inverse system. If two LTI cascade systems are such that:

$$h_1(n) \circledast h_2(n) = \delta(n) \quad , \text{ then}$$

$h_2(n)$ is the Inverse of $h_1(n)$, and vice versa. So, If the input to the cascaded system is $x(n)$, its output is also $x(n)$.

<u>Example:</u> The impulse response of the DT accumulator is the unit step sequence $u(n)$. Therefore, the Inverse system must satisfy the condition

$$u(n) * h_2(n) = \delta(n)$$

$h_2(n) = 0$ for $n < 0$ and

$h_2(n) = 1$, $\quad \sum_{L=0}^{n} h_2(L) = 0, \quad n \geq 1$

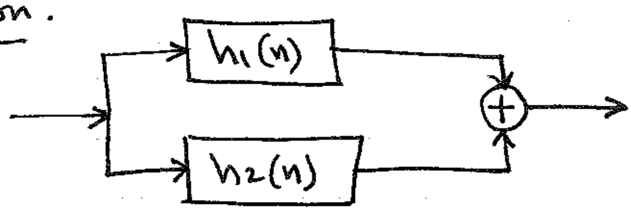
As a result: $\quad h_2(n) = -1$, and $h_2(n) = 0$, $n \geq 2$

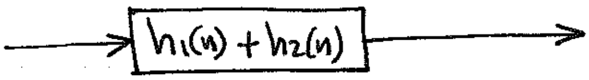
Thus, the impulse response of the Inverse system is given by

$$h_2(n) = \delta(n) - \delta(n-1)$$

which is called a backward difference system.

2 Parallel Connection:





$$h(n) = h_1(n) + h_2(n).$$

* Parallel of stable systems is stable. Howevere, the parallel of passive (lossless) systems may or may not be passive (lossless).

the pair $y(n)$ is shifted by L samples with respect to the reference seq. $x(n)$ to the right for positive values of L and shifted by L samples to the left for negative values of L.

* The ordering of subscripts xy specifies that $x(n)$ is the reference sequence that remains fixed in time. Whereas, $y(n)$ is being shifted with respect to $x(n)$

$$r_{yx}(L) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-L)$$

$$= \sum_{m=-\infty}^{\infty} y(m+L)\, x(m) = r_{xy}(-L)$$

Thus, $r_{yx}(L)$ is obtained by time-reversing the sequence $r_{xy}(L)$

The autocorrelation sequence of $x(n)$ is

$$r_{xx}(L) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-L)$$

Note that $r_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n) = \mathcal{E}_x$, the energy of signal $x(n)$

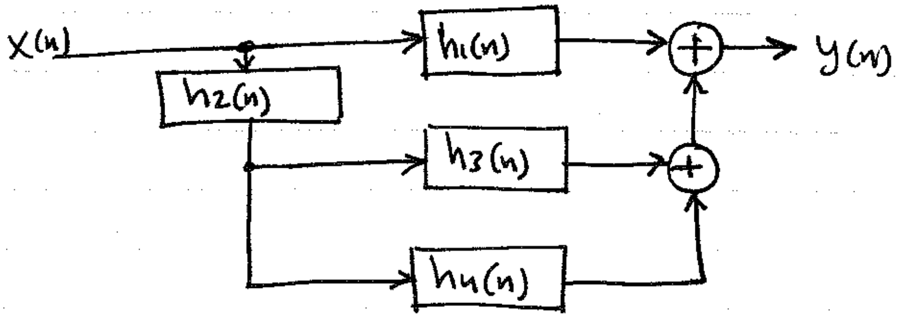
also, note that $r_{xx}(L) = r_{xx}(-L)$, implying that $r_{xx}(L)$ is an even function for real $x(n)$.

* Expression for cross-correlation looks very similar to that of the convolution. This similarity is much clearer if we re-write $r_{xy}(L)$ as

$$r_{xy}(L) = \sum_{n=-\infty}^{\infty} x(n)\, y(-L-n) = x(L) * y(-L)$$

Thus, cross-correlation of sequence $x(n)$ with reference sequence $y(n)$ can be computed by processing $x(n)$ with an LTI DT system of impulse response $y(-n)$. Likewise, the autocorrelation of $x(n)$ can be determined by passing it through an LTI DT system of impulse response $x(-n)$.

Example: consider a DT system



Where, $h_1(n) = \delta(n) + \frac{1}{2}\delta(n-1)$

$h_2(n) = \frac{1}{2}\delta(n) - \frac{1}{4}\delta(n-1)$

$h_3(n) = 2\delta(n)$

$h_4(n) = -2\left(\frac{1}{2}\right)^n u(n)$.

The overall impulse response $h(n)$ is given by:

$$h(n) = h_1(n) + h_2(n) * \big(h_3(n) + h_4(n)\big)$$
$$= h_1(n) + h_2(n) * h_3(n) + h_2(n) * h_4(n).$$

Now,

$$h_2(n) * h_3(n) = \left(\tfrac{1}{2}\delta(n) - \tfrac{1}{4}\delta(n-1)\right) * \left(2\delta(n)\right)$$
$$= \delta(n) - \tfrac{1}{2}\delta(n-1)$$

$$h_2(n) * h_4(n) = \left(\tfrac{1}{2}\delta(n) - \tfrac{1}{4}\delta(n-1)\right) * \left(-2\left(\tfrac{1}{2}\right)^n u(n)\right)$$

$$= \left(\tfrac{1}{2}\right)^n u(n) + \tfrac{1}{2}\left(\tfrac{1}{2}\right)^{n-1} u(n-1)$$
$$= -\left(\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{1}{2}\right)^n u(n-1)$$

$$= -\left(\tfrac{1}{2}\right)^n \delta(n) = -\delta(n)$$

Therefore,

$$h(n) = \delta(n) + \tfrac{1}{2}\delta(n-1) + \delta(n) - \tfrac{1}{2}\delta(n-1) - \delta(n)$$
$$= \delta(n)$$

* Correlation of signals:

A measure of similarity between a pair of signals, $x(n)$ and $y(n)$ is given by cross-correlation sequence $r_{xy}(L)$ defined by:

$$r_{xy}(L) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-L) \quad , \quad L = 0, \mp 1, \mp 2, \ldots$$

The parameter L called lag, indicates the time-shift between

* <u>properities of Autocorrelation and Cross-Correlation sequences:</u>

Suppose we have two finite-energy sequences $X(n)$ and $y(n)$. Now the energy of the combined sequence $aX(n)+y(n-L)$ is also finite and non-negative that is:

$$\sum_{n=-\infty}^{\infty}\left(ax(n)+y(n-L)\right)^2 = a^2\sum_{n=-\infty}^{\infty} x^2(n) + 2a\sum_{n=-\infty}^{\infty} x(n)\,y(n-L) + $$

$$+ \sum_{n=-\infty}^{\infty} y^2(n-L)$$

$$= a^2 r_{xx}(0) + 2a\, r_{xy}(L) + r_{yy}(0) \geq 0$$

Where, $r_{xx}(0) = \mathcal{E}_x > 0$ and $r_{yy}(0) = \mathcal{E}_y > 0$ are energies of $X(n)$ and $y(n)$. The previous eq. can be written as:

$$[a \quad 1]\begin{bmatrix} r_{xx}(0) & r_{xy}(L) \\ r_{xy}(L) & r_{yy}(0) \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix} \geq 0$$

for any finite value of a or in other words, the matrix is positive semi-define. This implies

$$r_{xx}(0)\, r_{yy}(0) - r_{xy}^2(L) \geq 0$$

or equivelently,

$$|r_{xy}(L)| \leq \sqrt{r_{xx}(0)\, r_{yy}(0)} = \sqrt{\mathcal{E}_x \mathcal{E}_y}$$

This inequality provides an upper bound for the cross-correlation sequence samples. IF we set $y(n) = X(n)$ =>

$$|r_{xx}(L)| \leq r_{xx}(0) = \mathcal{E}_x$$

* *This is a significant result that at zero lag (i.e. $L=0$), the sample* Value of the autocorrelation sequence has its maximum value.

Computing Auto-Correlation and Cross-Correlation using MATLAB

$r_{xy}(L) =$ conv(x, fliplr(y)); or $r_{xy}(L) =$ xcorr(x,y);

$r_{xx}(L) =$ xcorr(x);

Normalized forms of Correlation:

$$\rho_{xx}(L) = \frac{r_{xx}(L)}{r_{xx}(0)} , \qquad |\rho_{xx}(L)| \leq 1$$

$$\rho_{xy}(L) = \frac{r_{xy}(L)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}} , \qquad |\rho_{xy}(L)| \leq 1$$

2.3 Linear Time-Invariant Systems:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

$x(n) = \delta(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = h(n)$

$$y(n) = x(n) * h(n).$$

* Computation of the Convolution Sum:

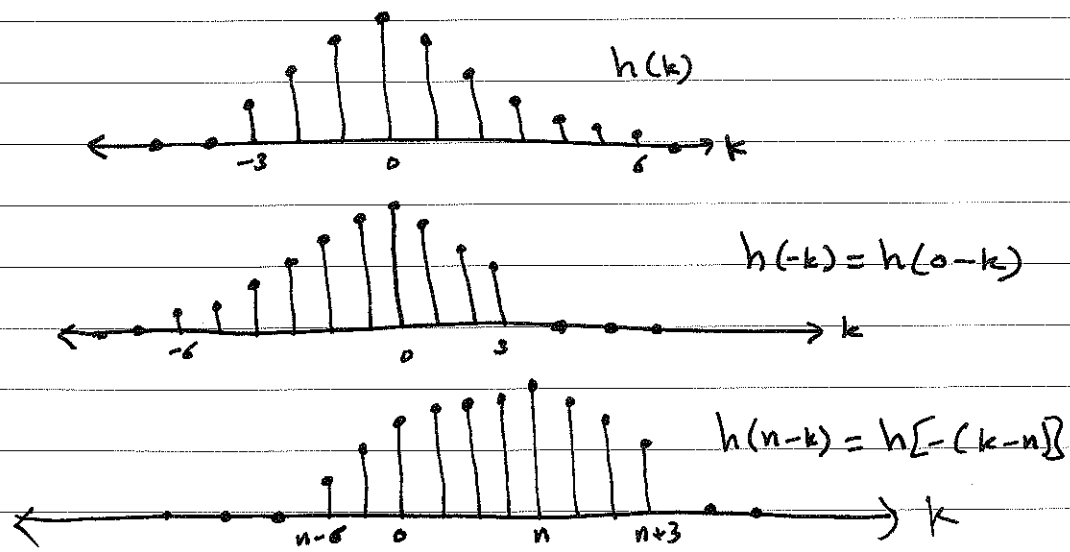
output at n:

$y(n)$ is obtained by multiplying input sequence $x(k)$ by a sequence $h(n-k)$, $-\infty < k < \infty$, and then for any fixed value of n, summing all the values of the products $x(k)\,h(n-k)$, k counting index in the summation. Therefore, this computation should be done for all values of n. i.e. $y(n)$, $-\infty < n < \infty$.

* key is to understand how to form $h(n-k)$, $-\infty < k < \infty$.

$$h(n-k) = h(-(k-n))$$

Example:



Two Steps:

1. Reflect $h(k)$ about the origin to obtain $h[-k]$.
2. Shifting the origin of the reflected sequence to $k=n$.

show slides

Example: $h(n) = u(n) - u(n-N) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

$x(n) = a^n u(n)$

find $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$ (slides)

$y(n) = 0$ for $n<0$

for $0 \le n \le N-1$

$$x(k)\,h[n-k] = a^k$$

$$y(n) = \sum_{k=0}^{n} a^n \,, \quad \text{for } 0 \le n \le N-1$$

$$\boxed{\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \quad N_2 \geq N_1} \text{ geometric series.}$$

So,

$$y(n) = \frac{1-a^{n+1}}{1-a}, \quad 0 \le n \le N-1$$

$$x(k)h(n-k) = a^k, \quad n-N+1 \le k \le n$$

$$\Rightarrow y(n) = \sum_{k=n-N+1}^{n} a^k, \quad \text{for } N-1 < n$$

$$y(n) = \frac{a^{n-N+1} - a^{n+1}}{1-a}$$

$$y(n) = a^{n-N+1}\left(\frac{1-a^N}{1-a}\right).$$

Thus,

$$y(n) = \begin{cases} 0, & n<0 \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \le n \le N-1 \\ a^{n-N+1}\left(\dfrac{1-a^N}{1-a}\right), & N-1 < n \end{cases}$$

* Tabular Method for Computing discrete convolution:

■ Finite - discrete sequence : $y[n] = x[n] * g[n]$

$$x[n] = \{x[0], x[1], x[2], x[3], x[4]\}, \text{ Length} = 5$$
$$g[n] = \{g[0], g[1], g[2], g[3]\}, \text{ length} = 4$$

n:	0	1	2	3	4			
$x[n]$	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$			
$g[n]$	$g[0]$	$g[1]$	$g[2]$	$g[3]$				
$\sum$	$x(0)g(0)$	$x(1)g(0)$	$x(2)g(0)$	$x(3)g(0)$	$x(4)g(0)$			
	—	$x(0)g(1)$	$x(1)g(1)$	$x(2)g(1)$	$x(3)g(1)$	$x(4)g(1)$		
	—	—	$x(0)g(2)$	$x(1)g(2)$	$x(2)g(2)$	$x(3)g(2)$	$x(4)g(2)$	
	—	—	—	$x(0)g(3)$	$x(1)g(3)$	$x(2)g(3)$	$x(3)g(3)$	$x(4)g(3)$
$\sum$ $y(n)$	$y(0)$	$y(1)$	$y(2)$	$y(3)$	$y(4)$	$y(5)$	$y(6)$	$y(7)$

* length of $y(n)$ $(x(n) * g(n))$ = length of $x(n)$ + length of $g(n)$ − 1
$= 5 + 4 - 1 = 8$

* Example: consider the following two finite-length sequences,

$$x(n) = \{\underset{\substack{\uparrow \\ n=0}}{-2}, 0, 1, -1, 3\} \text{ and } h(n) = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 0, -1\}$$

Find $y(n) = x(n) * h(n)$?

* Using Tabular Method

n:	0	1	2	3	4			
$x(n)$:	−2	0	1	−1	3			
$h(n)$:	1	2	0	−1				
$\sum$	−2	0	1	−1	3			
	—	−4	0	2	−2	6		
	—	—	0	0	0	0	0	
	—	—	—	2	0	−1	1	−3

$y(n)$: $\{-2, -4, 1, 3, 1, 5, 1, -3\}$, $0 \le n \le 7$

* Tabular method can also be used to evalute convolution sum of two finite-length two-sided sequences:

Example:

$$g(n) = \{ 3, \underset{\uparrow n=0}{-2}, 4 \}$$
$$h(n) = \{ \underset{\uparrow n=0}{4}, 2, -1 \}$$

Solution:

$g(n)$:			3	$-2_\bullet$	4
$h(n)$:			$4_\bullet$	2	-1
			-3	2	-4
		6	-4	8	—
	12	-8	18	—	—
	12	$-2_\bullet$	9	10	-4

So, $y(n) = \{ 12, \underset{\uparrow n=0}{-2}, 9, 10, -4 \}$, $-1 \leq n \leq 3$

* For Infinite sequences: $x(n) = \alpha^n u(n)$, $h(n) = \beta^n u(n)$, $|\alpha| < 1$, $|\beta| < 1$

$$y(n) = x(n) * h(n)$$
$$= \left(\beta^n u(n)\right) * \left(\alpha^n u(n)\right)$$

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k) = \sum_{k=0}^{\infty} \alpha^k \beta^{n-k}$$

$$y(n) = \beta^n \sum_{k=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^k = \frac{\beta^n}{1 - \frac{\alpha}{\beta}} = \left(\frac{1}{\beta - \alpha}\right) \beta^{n+1}, \quad n \geq 0$$

section 2.5

linear Constant-Coefficient Difference Equations (LCCDE):

N^{th} order system

$$\underbrace{\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{r=0}^{M} b_r \, x(n-r)}_{\text{constants}}$$

linear

order $\equiv$ no. of delays in output sequence.

* If $N=0$, $a_0=1$

$$\Rightarrow y(n) = \sum_{r=0}^{M} b_r \, x(n-r) \rightarrow \text{This eq. is identical to convolution sum.}$$

$$h(n) = \begin{cases} b_n, & n=0,1,\ldots,M \\ 0, & \text{otherwise} \end{cases}$$

* If $N \neq 0$, and still $a_0=1$

$$y(n) = \sum_{r=0}^{M} b_r \, x(n-r) - \sum_{k=1}^{N} a_k \, y(n-k)$$

past (k) outputs to calculate $y(n)$.

In this case, we need an initial conditions.

* First Order

$$y(n) - a\,y(n-1) = x(n)$$

let $x(n) = \delta(n)$

Initial Cond.: Assume $y(n)=0$, $n<0 \rightarrow$ Same Condition for Causality.

$$\Rightarrow y(n) = \delta(n) + a\,y(n-1)$$

$y(-1) = 0 \rightarrow$ Initial Condition

$y(0) = 1$

$y(1) = a$

$y(2) = a^2$

$\vdots$

$a^n u(n)$

if $|a|<1 \Rightarrow$ stable system.

* Assume $x(n) = \delta(n)$ and $y(n)=0$ for $n>0$ (corresponds to non-Causal sys.)

$$y(n-1) = a^{-1}[y(n) - \delta(n)]$$

$n=2 \Rightarrow y(1) = 0$

$n=1 \Rightarrow y(0) = 0$

$n=0 \Rightarrow y(-1) = -a^{-1}$

$n=-1 \Rightarrow y(-2) = -a^{-2}$

$-a^n u(-n-1)$

$|a|<1$ Unstable.

Frequency Response of LTI systems :

* if input to LTI sys. is a complex exponential, then the output is also complex exponential ($e^{j\omega n}$) → called "eigen Function"

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

let $x(n) = e^{j\omega n}$

$$\Rightarrow y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)} \rightarrow e^{j\omega n} \cdot e^{-j\omega k}$$

$$= e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}}_{H(e^{j\omega})}$$

→ doesn't depend on (n) so, it is just number !

So, $\boxed{y(n) = H(e^{j\omega})\, e^{j\omega n}}$

$e^{j\omega n}$ → is eigen Function for LTI system.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

$\triangleq$ Frequency Response of LTI system.

* Why Freq. Response is important ?
  - ⇒ Easly obtained directly from Unit sample Response.
  - ⇒ Allows us to obtain the response of system to sinusoidal Excitation. Arbitrary sequence can be represented as linear combination of complex exponential or sinusoidal sequences.

* Sinusoidal Response :

$$x(n) = A\cos(\omega_o n + \phi)$$

$$= \frac{A}{2} e^{j\phi} e^{j\omega_o n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_o n}$$

$$H(e^{j\omega_o}) = |H(e^{j\omega_o})|\, e^{j\theta(\omega_o)}$$

$$y(n) = A\,|H(e^{j\omega_o})| \cdot \cos(\omega_o n + \phi + \theta)$$

Example: $y(n) - a\,y(n-1) = x(n)$

Causal: $h(n) = a^n u(n)$, $0 < a < 1$ (stable)

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n = \frac{1}{1 - a e^{-j\omega}}$$

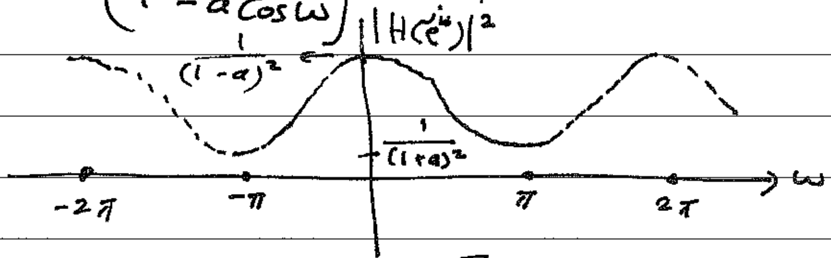
$$\boxed{\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \quad , 0 < \alpha < 1}$$

sum of geometric series.

$$|H(e^{j\omega})|^2 = \frac{1}{1 - a e^{j\omega}} \cdot \frac{1}{1 - a e^{-j\omega}}$$ (square of magnitude)

$$= \frac{1}{1 + a^2 - 2a\cos\omega}$$

$$\measuredangle H(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)$$ (phase-shift).



* Both $|\ |$ and $\measuredangle$ are periodic $[-\pi$ to $\pi]$.

Properities of freq. Response:

① Function of Continuous variable ω → change continuously

② Periodic function of ω; period $= 2\pi$

why periodic? ⟹ $e^{j(\omega + 2\pi k)n} = e^{j\omega n}\, \cancel{e^{j2\pi kn}}^{\,1}$, k, n integers

* Generalization of Frequency Response is Fourier Transform

Fourier Transform for DT

Freq. Response

$$e^{j\omega n} \longrightarrow \underbrace{H(e^{j\omega})}_{\text{Freq. Resp.}} e^{j\omega n}$$

$$\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{h(n)}_{\text{Unit Sample Response of the System.}} e^{-j\omega n}} \Longrightarrow \text{Fourier Series.}$$

$\omega \rightarrow$ Continuous Variable.

$n \rightarrow$ discrete Variable.

$$\boxed{h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n}\, d\omega}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{ \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \right\} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{k} h(k) \int_{-\pi}^{\pi} e^{j\omega(n-k)}\, d\omega$$

$n \neq k, \int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega = 0$

$n = k, \int_{-\pi}^{\pi} = 2\pi$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-k)}\, d\omega = \frac{\sin \pi(n-k)}{\pi(n-k)} = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} = \delta(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\,\delta(n-k) = h(n).$$

Fourier Transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

Example Ideal Low-Pass Filter

$$H_{LP}(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & w_c \leq |w| \leq 2\pi \end{cases}$$

$$h(n) = \frac{1}{2\pi jn}\left(e^{jw_c n} - e^{-jw_c n}\right)$$

$$= \frac{\sin w_c n}{\pi n} \quad (\text{non-Causal}).$$

* Convoution Properity:

$$x(n) * y(n) \xleftrightarrow{F.T.} X(e^{jw})\,Y(e^{jw})$$

$$e^{jw_0 n} \longrightarrow \boxed{LTI} \longrightarrow H(e^{jw_0})\,e^{jw_0 n}$$

* linear system

$$\sum_k A_k e^{jw_k n} \longrightarrow \sum A_k H(e^{jw_k})\,e^{jw_k n} \quad \text{---} \; (*)$$

* arbitrary input

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\,e^{jwn}\,\partial w$$

(*) This is a linear Combination of complex exponentials of the input. So What is the output?

output:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\,H(e^{jw})\,e^{jwn}\,\partial w = y(n).$$

key proparity of LTI system: $\boxed{Y(e^{jw}) = X(e^{jw})\,H(e^{jw})}$

This justifies Convolution properity $y(n) = x(n) * h(n)$.

Symmetry Properities:

$x(n)$ is real

$X(e^{jw}) = X^*(e^{-jw})$ Conjugate symmetric

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-jwn}$$

$$X(e^{-jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{+jwn}$$

$$X^*(e^{-jw}) = \sum x^*(n)\, e^{-jwn} \quad , \; x(n) \text{ is real} \Rightarrow x^*(n) = x(n).$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

F.T. is conjugate symmetric function

$$X(e^{jw}) = X_R(e^{jw}) + j\, X_I(e^{jw})$$

$$X^*(e^{-jw}) = X_R(e^{-jw}) - j\, X_I(e^{-jw})$$

So, $X_R(e^{jw}) = X_R(e^{-jwn}) \Rightarrow X_R(e^{jw})$ is even function

$\Rightarrow$ real part of $X(e^{jw})$ is the same if w is replaced by $-w$ $\Rightarrow$ even

$X_I(e^{jw}) = -X_I(e^{-jw}) \Rightarrow X_I(e^{jw})$ is odd function

$$\left.\begin{array}{lll} |X(e^{jw})| & \text{is} & \text{even function of } w \\ \measuredangle X(e^{jw}) & \text{is} & \text{odd} \quad \text{"} \quad \text{"} \quad \text{"} \end{array}\right\} \text{Periodic Functions}$$

<u>Example</u>: We have seen that the Fourier Transform (FT) of the real sequence $x(n) = a^n u(n)$ is

$$X(e^{jw}) = \frac{1}{1 - a e^{-jw}} \; , \text{ if } |a| < 1$$

Then, from the properties of complex numbers,

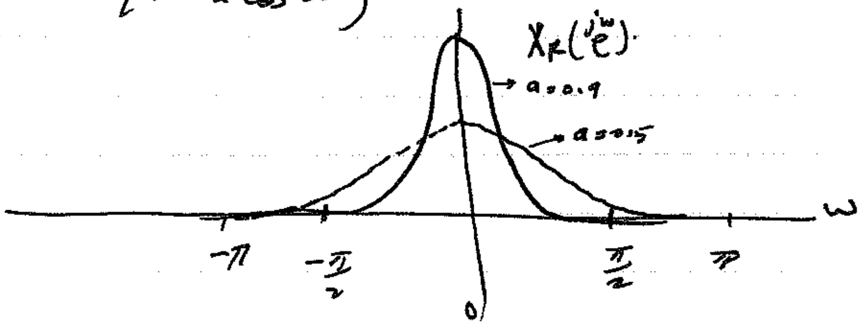
$$X(e^{jw}) = \frac{1}{1 - a e^{-jw}} = X^*(e^{-jw})$$

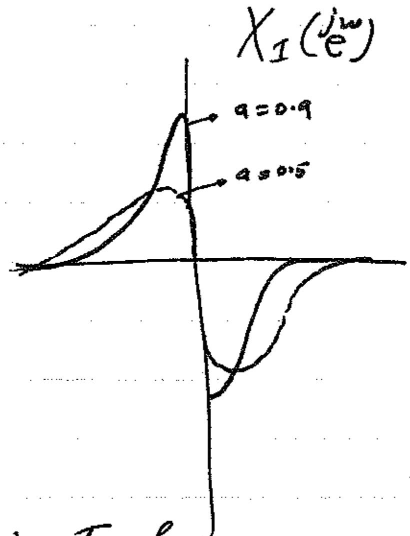
$$X_R(e^{jw}) = \frac{1 - a \cos w}{1 + a^2 - 2a \cos w} = X_R(e^{-jw}) \qquad (\text{even}).$$

$$X_I(e^{jw}) = \frac{-a \sin w}{1 + a^2 - 2a \cos w} = -X_I(e^{-jw}) \qquad (\text{odd}).$$

$$|X(e^{j\omega})| = \frac{1}{(1+a^2-2a\cos\omega)^{1/2}} = |X(e^{-j\omega})|$$

$$\measuredangle X(e^{j\omega}) = \tan^{-1}\left[\frac{-a\sin\omega}{1-a\cos\omega}\right] = -\measuredangle X(e^{-j\omega})$$





Fourier Transform Theorems:

$X(e^{j\omega}) = F\{x(n)\}$, $F\{\cdot\}$ is Fourier Transform.

$x(n) = F^{-1}\{X(e^{j\omega})\}$

$$x(n) \xleftrightarrow{F.T.} X(e^{j\omega})$$

1. linearity: if $x_1(n) \xleftrightarrow{F} X_1(e^{j\omega})$

and $x_2(n) \xleftrightarrow{F} X_2(e^{j\omega})$

Then,

$$a x_1(n) + b x_2(n) \xleftrightarrow{F} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

2. Time shifting and Frequency shifting:

If $x(n) \xleftrightarrow{F} X(e^{j\omega})$

then, $x(n-n_d) \xleftrightarrow{F} e^{-j\omega n_d} X(e^{j\omega})$

$$e^{j\omega_0 n} x(n) \xleftrightarrow{F} X(e^{j(\omega-\omega_0)})$$

3 Time Reversal:

if $x(n) \xleftrightarrow{F} X(e^{j\omega})$

then, if sequence $x(n)$ is time reversed

$$x(-n) \xleftrightarrow{F} X(e^{-j\omega})$$

if $x(n)$ is real, then

$$x(-n) \xleftrightarrow{F} X^*(e^{jw})$$

4. Differential in Freq.

$$n\,x(n) \xleftrightarrow{F} j\,\frac{\partial X(e^{jw})}{\partial w}$$

5. Parseval's theorem:

$$x(n) \xleftrightarrow{F} X(e^{jw})$$

then,

(energy) $$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{jw})|^2 \, dw$$

Function $|X(e^{jw})|^2$ is called the Energy Density Spectrum (EDS)

EDS determines how energy is distributed in the frequency domain.

6. Convolution Theorem:

if $x(n) \xleftrightarrow{F} X(e^{jw})$

and $\cancel{x}\,h(n) \xleftrightarrow{F} H(e^{jw})$

and if

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) = x(n) * h(n).$$

Then,

$$Y(e^{jw}) = X(e^{jw})\,H(e^{jw}).$$

7. Modulation OR Windowing Theorem:

if $x(n) \xleftrightarrow{F} X(e^{jw})$

and $w(n) \xleftrightarrow{F} W(e^{jw})$

and if $y(n) = x(n)\,w(n)$

then,

$$Y(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\,W(e^{j(w-\theta)})\,d\theta$$

* $\delta(n-n_d) \xleftrightarrow{F} e^{-jwn_d}$

if $h(n) = \delta(n-n_d)$; then $y(n) = x(n) * \delta(n-n_d) = x(n-n_d)$

Therefore, $H(e^{jw}) = e^{-jwn_d}$ and $Y(e^{jw}) = e^{-jwn_d}\,X(e^{jw})$

Recall also that if $x(n) = e^{jwn} \Rightarrow$ then $y(n) = H(e^{jw})\,e^{jwn}$.

Example: suppose that

$$X(e^{jw}) = \frac{1}{(1 - a e^{-jw})(1 - b e^{-jw})}$$

$$\Rightarrow \quad X(e^{jw}) = \frac{a/(a-b)}{1 - a e^{-jw}} - \frac{b/(a-b)}{1 - b e^{-jw}}$$

$$\Rightarrow \quad x(n) = \left(\frac{a}{a-b}\right) a^n u(n) - \left(\frac{b}{a-b}\right) b^n u(n)$$

Example: The Frequency Response of high-pass filter with delay is

$$H(e^{jw}) = \begin{cases} e^{-jwn_d}, & w_c < |w| < \pi \\ 0, & |w| < w_c \end{cases}$$

$$\Rightarrow \quad H(e^{jw}) = e^{-jwn_d}\left(1 - H_{lp}(e^{jw})\right) = e^{-jwn_d} - e^{-jwn_d} H_{lp}(e^{jw})$$

Where,

$H_{lp}(e^{jw})$ is periodic with period 2π, and

$$H_{lp}(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & w_c < |w| < \pi \end{cases}$$

$$\Rightarrow \quad h(n) = \delta(n - n_d) - r(n - n_d)$$

$$= \delta(n - n_d) - \frac{\sin w_c(n - n_d)}{\pi(n - n_d)}$$

Example: Impulse Response for difference equation.

$$y(n) - \tfrac{1}{2} y(n-1) = x(n) - \tfrac{1}{4} x(n-1)$$

Let's set $x(n) = \delta(n)$, with $h(n)$ denoting the impulse response

$$\Rightarrow \quad h(n) - \tfrac{1}{2} h(n-1) = \delta(n) - \tfrac{1}{4}\delta(n-1)$$

Applying F.T. on both sides:

$$H(e^{jw}) - \tfrac{1}{2} e^{-jw} H(e^{jw}) = 1 - \tfrac{1}{4} e^{-jw}$$

$$H(e^{jw}) = \frac{1 - \frac{1}{4} e^{-jw}}{1 - \frac{1}{2} e^{-jw}}$$

To get $h(n)$, we take Inverse F.T. of $H(e^{jw})$

* Fourier Transform Pairs

Sequence	Fourier Transform
1. $\delta(n)$	1
2. $\delta(n-n_0)$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega+2\pi k)$
4. $a^n u(n) \quad (\lvert a\rvert < 1)$	$\frac{1}{1-a e^{-j\omega}}$
5. $u(n)$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega+2\pi k)$
6. $(n+1)\,a^n u(n) \quad (\lvert a\rvert<1)$	$\frac{1}{(1-a e^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p}\,u(n)\,,\ \lvert r\rvert<1$	$\frac{1}{1-2r\cos\omega_p\, e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1\,, & \lvert\omega\rvert<\omega_c \\ 0\,; & \omega_c<\lvert\omega\rvert<\pi \end{cases}$
9. $x(n) = \begin{cases} 1\,, & 0\le n\le M \\ 0\,, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}\, e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty}[\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

Example: Find F.T. of $x(n) = a^n u(n-5)$?

~~Let~~ $x_1(n) = a^n u(n) \Rightarrow X_1(e^{j\omega}) = \frac{1}{1-a e^{-j\omega}}$

To obtain $x(n)$ from $x_1(n)$, we first delay $x_1(n)$ by 5 samples, i.e.

$$x_2(n) = x_1(n-5) \Rightarrow X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega}),\ \text{so}$$

$$X_2(e^{j\omega}) = \frac{e^{-j5\omega}}{1-a e^{-j\omega}}$$

In order to get $x(n)$ from $x_2(n)$, we need to multiply by a constant a^5.
i.e. $x(n) = a^5 x_2(n)$.

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1-a e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$\left(\tfrac{1}{2}\right)^n u(n) \overset{F}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$\left(-\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u(n-1) \overset{F}{\longleftrightarrow} \frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

So, $h(n) = \left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u(n-1)$

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* Condition for Fourier Transform:

* condition for the convergence of infinite sum

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right|$$

$$\leq \sum_{-\infty}^{\infty} |x(n)| \, |e^{-j\omega n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

If $x(n)$ is absolutly summable, then its Fourier transform exists.

Summary of DT systems Properities:

$$x(n) \overset{H}{\longmapsto} y(n)$$

$$x_k(n) \overset{H}{\longmapsto} y_k(n)$$

Linearity:

$$\sum_k C_k x_k(n) \overset{H}{\longmapsto} \sum_k C_k y_k(n)$$

Time-Invariance: $x(n-n_0) \overset{H}{\longmapsto} y(n-n_0)$

Stability: $|x(n)| \leq M_x < \infty \overset{H}{\longmapsto} |y(n)| \leq M_y < \infty$

Causality: $x(n) = 0$ for $n < n_0 \overset{H}{\longmapsto} y(n) = 0$ for $n \leq n_0$

Summary of Convolution properities:

Identity : $x(n) * \delta(n) = x(n)$

Delay : $x(n) * \delta(n-n_0) = x(n-n_0)$

Commutative : $x(n) * h(n) = h(n) * x(n)$

Associative : $(x(n) * h_1(n)) * h_2(n) = x(n) * (h_1(n) * h_2(n))$

Distributive : $x(n) * (h_1(n) + h_2(n)) = x(n) * h_1(n) + x(n) * h_2(n)$

Response of LTI systems to some test sequences:

- Impulse : $x(n) = \delta(n) \xmapsto{H} y(n) = h(n)$ (Impulse Response)
- Step : $x(n) = u(n) \xmapsto{H} y(n) = s(n) = \sum_{k=-\infty}^{\infty} h(k)$ (step Response)
- Exponential : $x(n) = a^n$, all n $\xmapsto{H} y(n) = H(a)\, a^n$, all n
- Complex Sinusoidal : $x(n) = e^{jwn}$, all n $\xmapsto{H} y(n) = H(e^{jw})\, e^{jwn}$, all n

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Numerical Computation of Convolution:

$y(n) = x(n) * h(n)$

e.g. Length of $x(n) = 6$, Length of $h(n) = 3$

Matrix Convolution :

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} x(0) & 0 & 0 \\ x(1) & x(0) & 0 \\ x(2) & x(1) & x(0) \\ x(3) & x(2) & x(1) \\ x(4) & x(3) & x(2) \\ x(5) & x(4) & x(3) \\ 0 & x(5) & x(4) \\ 0 & 0 & x(5) \end{bmatrix} \cdot \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$$

$y = \mathrm{Convmtx}(x, N+M-1) * h$

Computation Difference Equation:

$y = \mathrm{filter}(b, a, x);$

$b = [b_0 \ b_1 \ b_2 \ \dots \ b_M]$

$a = [1 \ a_1 \ a_2 \ \dots \ a_N]$

$x = [x(1) \ x(2) \ \dots \ x(L)] = [x(0) \ x(1) \ x(2) \ \dots \ x(L-1)]$

$$y = [y(1)\ y(2)\ \dots\ y(L)] = [y(0)\ y(1)\ \dots\ y(L-1)]$$

$$y(n) + \sum_{k=1}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

Matlab Functions:

- Stem(x);
- plot(x);
- Conv(x,y); Convolution
- Rxy = Conv(x, fliplr(y)); Cross-Correlation using Convolution Function
- Rxy = xcorr(x,y); Cross-Correlation
- Rxx = xcorr(x); Auto-Correlation