Chapter -3-

Z - Transform

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn} \Rightarrow \text{Fourier Transform}$$

$$|X(e^{jw})| = \left|\sum x(n)\, e^{-jwn}\right|$$

$$\leq \sum |x(n)|\, \cancel{|e^{-jwn}|}$$

F.T. Converges if $\sum_{-\infty}^{\infty} |x(n)|$ converges (i.e. finite)

OR $\sum_{-\infty}^{\infty} |x(n)| < \infty$ (absolutly summable)

* If seq. $x(n)$ is absolutly summable, Then F.T. of $x(n)$ converges (i.e. exist).

* Stability of DT-system $\longrightarrow$ F.T. of unit sample response ($h(n)$) of the system $H(e^{jw})$ converges.
* Unit sample response of system is absolutly summable $\Rightarrow$ system is stable.

Example: ① $x(n) = (\frac{1}{2})^n u(n)$

$$\sum_{-\infty}^{\infty} |x(n)| = 2 \Rightarrow \text{F.T. Converges (exist)}$$

② $x(n) = 2^n u(n)$

$$\sum_{-\infty}^{\infty} |x(n)| = \infty \Rightarrow \text{F.T. is not exist.}$$

* exponential seq. which decays exponentially (To right or to left)
$\Rightarrow$ It's F.T. exist and The exponential seq. which grows exp.
$\Rightarrow$ It's F.T. doesn't exist.

* There is away around it by ~~using~~ Multiplying growing exp. by a decaying exponential (fast decaying) $\Rightarrow$

$\Rightarrow$ product has F.T.

$$X_r(e^{jw}) = \sum_{-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-jwn} \qquad r^{-n} \rightarrow \text{decaying complex exponential}$$

$\Rightarrow$ We pick r so that $(x(n) r^{-n})$ is absolutly summable.

$$X_r(e^{jw}) = \sum_{-\infty}^{\infty} x(n) \underbrace{(re^{jw})^{-n}}$$

$\rightarrow$ new complex variable called z

$$z = re^{jw}$$

$$|z| = r \,, \quad \angle z = w .$$

* going back to example 2 if we multiply 2^n by a decaying exponential (faster than 2^n) any choice of r which is greater than 2. $\Rightarrow$ so the product $2^n r^{-n}$ converges.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}} \quad z\text{-Transform}$$

* how z-transform related to Fourier Transform (if it is exist!)

$$X(e^{jw}) = X(z)\Big|_{z=e^{jw}} \quad \text{or when magnitude of } z = 1$$

$$|z| = 1$$

* When z-transform Converges?

$$\sum_{-\infty}^{\infty} |x(n)\, r^{-n}| < \infty \Rightarrow$$

$\Rightarrow$ for some values of r, z-transform Converges.
" " " " " , " diverges.

<u>Example:</u> $x(n) = (\frac{1}{2})^n u(n)$. [Called Right-sided exponential seq.]

$$X(z) = \sum_{n=0}^{\infty} (\tfrac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\tfrac{1}{2} z^{-1})^n$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}$$

Converges When $\sum |(\frac{1}{2} z^{-1})^n| < \infty \Rightarrow |\frac{1}{2} z^{-1}| < 1 \Rightarrow |z| > \frac{1}{2}$

So, ~~Zt~~ $X(z)$ converges when $|z|$ or r greater than $\frac{1}{2}$.

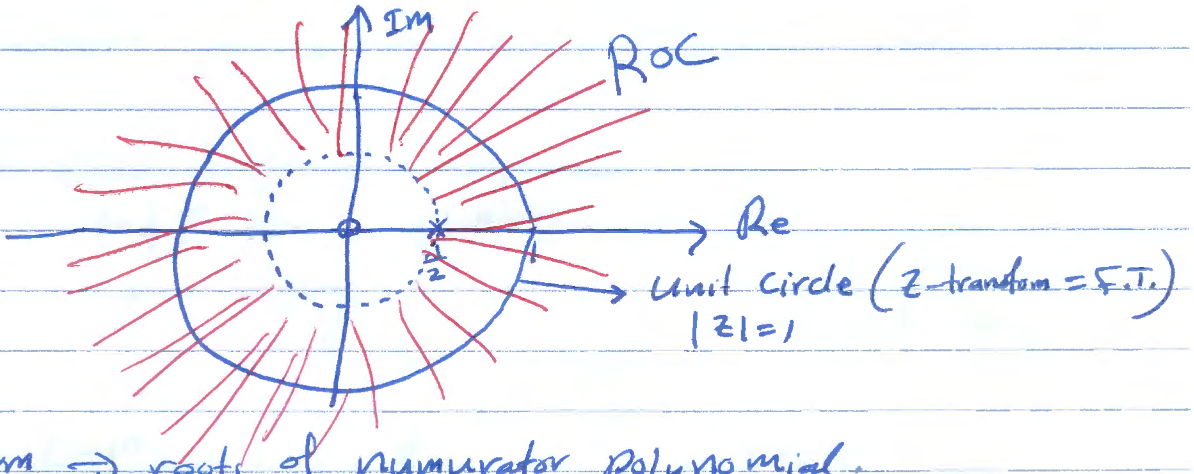
* all values of $|z|$ or r which makes $X(z)$ converges are called Region of Convergence (ROC).

So, ROC of $X(z)$ is $|z| > \frac{1}{2}$.

Does F.T. of $X(z)$ Converges? $\Rightarrow$ F.T. converges (Exist) only if z-transform($X(z)$) converges at $|z| = 1$. $\Rightarrow$ i.e. ROC contains unit circle. ($|z| = 1$).

* Exponential sequences or sum of exponential sequence always have z-transform as a ratio of polynomials of z^{-1} or z. (i.e. rational function of z^{-1} or z).

* z-plane



- Zero's of transform $\Rightarrow$ roots of numerator polynomial.
- pole's of transform $\Rightarrow$ " " denominator " .

* Same as zeros and poles of Laplace-transform of the continuous signals.

Zeros $\longrightarrow$ denoted as circles $\circ$
poles $\longrightarrow$ " " cross $\times$

Example: $x(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$ [$\left(\frac{1}{2}\right)^n$ but for -ve values of n]
[this is called left-sided exponential sequence)

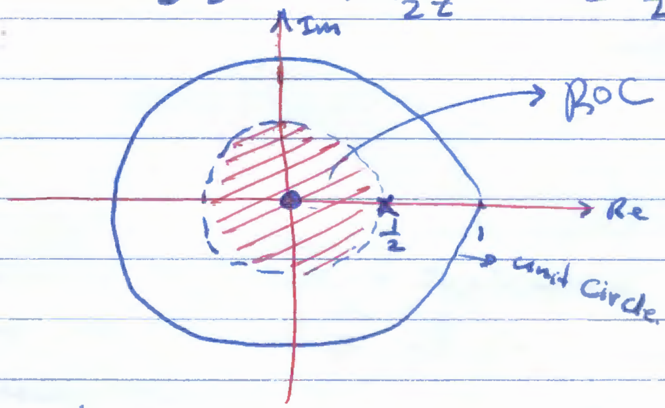
$$X(z) = -\sum_{-\infty}^{-1} \left(\tfrac{1}{2}\right)^n u(-n-1)\, z^{-n} = -\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^{-n} z^{n}$$

$$= 1 - \sum_{n=0}^{\infty} \left(\left(\tfrac{1}{2}\right)^{-1} z\right)^n$$

So, $X(z)$ converges if $\left|\left(\frac{1}{2}\right)^{-1} z\right| < 1 \Rightarrow |z| < \frac{1}{2}$

$$\Rightarrow X(z) = 1 - \frac{1}{1-\left(\frac{1}{2}\right)^{-1} z} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}, \quad |z| < \frac{1}{2}$$

zeros: at $z = 0$
poles: at $z = \frac{1}{2}$



* RoC of $X(z)$ doesn't contain unit circle $\Rightarrow$ F.T. doesn't exist.

Example: Two-sided Exponential sequence

$$x(n) = \left(-\tfrac{1}{3}\right)^n u(n) - \left(\tfrac{1}{2}\right)^n u(-n-1)$$

$x(n) \longrightarrow -\infty$ as $n \longrightarrow -\infty$

$$\left(-\tfrac{1}{3}\right)^n u(n) \overset{Z}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \tfrac{1}{3} \text{ RoC}$$

$$-\left(\tfrac{1}{2}\right)^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \tfrac{1}{2} \text{ RoC}$$

Thus, by linearity of Z-transform

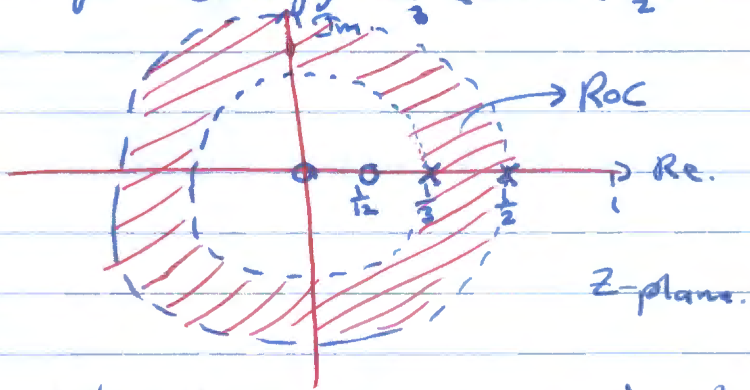
$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \tfrac{1}{3} < |z|, \ |z| < \tfrac{1}{2}$$

$$X(z) = \frac{2(1-\frac{1}{12}z^{-1})}{(1+\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})}$$

* RoC of $X(z)$ is annular region (ring) $\frac{1}{3} < |z| < \frac{1}{2}$

zeros: at $z=0$ and $z=\frac{1}{12}$

poles: at $z=-\frac{1}{3}$, $z=\frac{1}{2}$



* Note that RoC doesn't contain unit circle, so seq. $x(n)$ doesn't have F.T.

* From the previous examples, ⇒ Infinitly long exponential seq's have z-transforms that can be expressed as a rational function of either z^{-1} or z. The case when the sequence has finite length also has a simple form. if $x(n)$ is nonzero only in the interval $N_1 \le n \le N_2$

$$X(z) = \sum_{n=N_1}^{N_2} x(n)\,z^{-n}$$ converges if each of the terms $|x(n)z^{-n}|$ is finite.

i.e. $|x(n)z^{-n}| < \infty$.

* For example, if $x(n) = \delta(n) + \delta(n-5)$, then $X(z) = 1 + z^{-5}$ which is finite for $|z| > 0$.

Example:

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha} \qquad N_2 \ge N_1$$

$$x(n) = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}}\,\frac{z^N - a^N}{z-a}$$

RoC?

$$\sum_{n=0}^{N-1} |a z^{-1}|^n < \infty$$

* Since there are only finite number of nonzero terms, the sum will be finite as long as $(a z^{-1})$ is finite $\Rightarrow$ $|a| < \infty$ and $z \neq 0$.

* Assume $|a|$ is finite $\Rightarrow$ RoC includes the entire Z-plane except the origin ($z=0$)

* For $N=16$ and a is real and $0 < |a| < 1$
$\Rightarrow$ $X(z)$ has N zeros at:

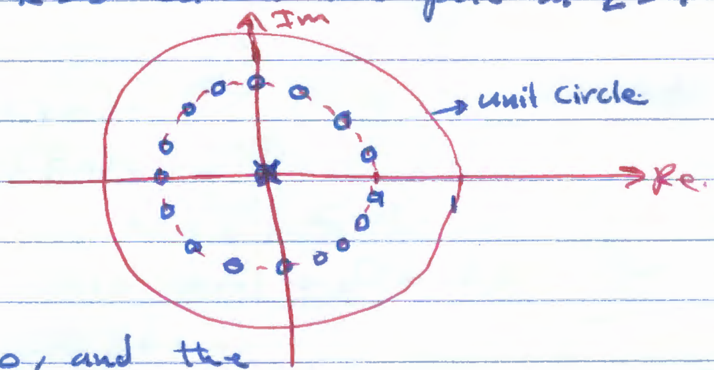
$$z_k = a e^{j\frac{2\pi k}{N}} \quad , k = 0, 1, 2, \ldots, N-1$$

Which satisfy $z^N = a^N$

When $a=1$, these complex values are the N^{th} zero's of unity (at unit circle). The zero at $k=0$ cancels the pole at $z=a$

poles: $z=0$, $\cancel{z=a}$

Canceled by zero at $k=0$



$\Rightarrow$ one pole at $z=0$, and the remaining zeros at $z_k = a e^{j\frac{2\pi k}{N}}$, $k = 1, 2, 3, \ldots, N-1$

Some Common Z-transform Pairs

	sequence	Z-transform	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4	$\delta(n-m)$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
8	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9	$\cos w_0 n \, u(n)$	$\frac{1-[\cos w_0]z^{-1}}{1-[2\cos w_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10	$\sin w_0 n \, u(n)$	$\frac{(\sin w_0)z^{-1}}{1-(2\cos w_0)z^{-1}+z^{-2}}$	$\lvert z\rvert < 1$
11	$[r^n \cos w_0 n]u(n)$	$\frac{1-[r\cos w_0]z^{-1}}{1-[2\cos w_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12	$(r^n \sin w_0 n)u(n)$	$\frac{(r\sin w_0)z^{-1}}{1-[2\cos w_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13	$\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Properities of ROC for Z-transform

$\boxed{1}$ ROC is a ring or disk in z-plane centred at the origin i.e. bounded by a pole or (∞ or 0).

$$0 \le r_R < |z| < r_L \le \infty$$

$\boxed{2}$ F.T. converges if and only if ROC of Z-transform Includes Unit circle.

$\boxed{3}$ ROC Cannot Contain any poles.

$\boxed{4}$ If $x(n)$ is a finite-duration seq. i.e. $x(n) = 0$ except in $-\infty < N_1 \le n \le N_2 < \infty$ interval, then ROC is the entire Z-plane except possibly $Z=0$ or $Z=\infty$.

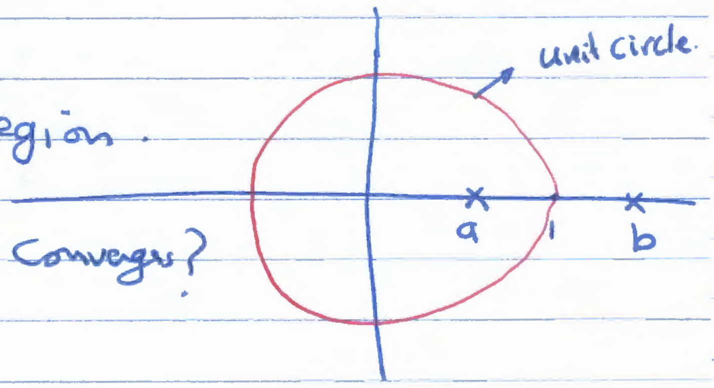
$\boxed{5}$ If $x(n)$ is right-sided seq., $x(n) = 0$, $n < 0$
$\Rightarrow$ ROC is outside of the outermost (i.e. Largest magnitude) finite pole to (and possibly Including) $Z=\infty$.

$\boxed{6}$ Left-sided seq. $x(n) = 0$ for $n > 0$. $\Rightarrow$
ROC extends inward from the innermost (i.e. smallest magnitude) nonzero pole to (and possibly Including) $Z=0$.

$\boxed{7}$ Two-sided seq. $\Rightarrow$ ROC is a ring in Z-plane, bounded on the Interior and exterior by a pole (and not containing any poles).

$\boxed{8}$ ROC must be connected region.

ROC? Which sided? F.T. converges?



* Thier are three choices of ROC

ROC	sided	F.T. Convergence
1. $\lvert z\rvert < a$	left	No
2. $a < \lvert z\rvert < b$	Two	Yes
3. $\lvert z\rvert > b$	Right	No

$$y(n) = x(n) * h(n)$$
$$\updownarrow \qquad \updownarrow \quad \downarrow \quad \updownarrow$$
$$Y(z) = X(z) \cdot H(z)$$

$H(z) \triangleq$ system function or Transfer function.

$h(n)$ is absolutly summable $\Rightarrow$ has F.T.

✓ Stable $\longleftrightarrow$ ROC Includes Unit Circle.

✓ Causal $\Longleftrightarrow$ $h(n) = 0$ for $n < 0$ $\Rightarrow$ Right-sided
$\Rightarrow$ ROC is outside outermost pole.

Example: $y(n) - \frac{1}{2} y(n-1) = x(n)$

To get system function, we use a useful property

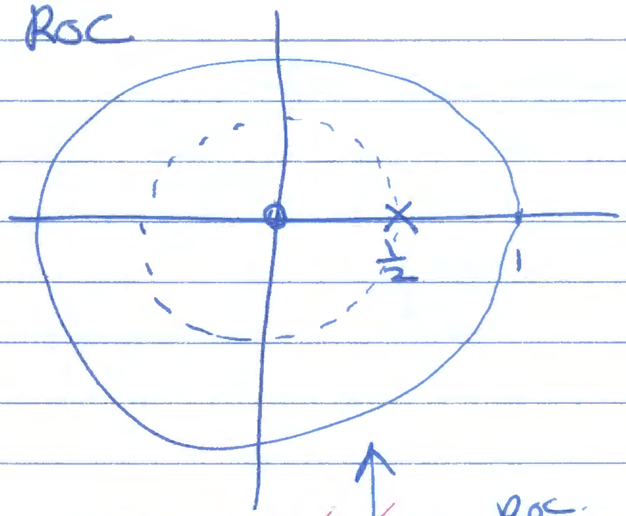
$$y(n) \overset{z}{\longleftrightarrow} Y(z)$$
$$y(n+n_0) \overset{z}{\longleftrightarrow} z^{n_0} Y(z)$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}} \qquad \text{ROC?}$$

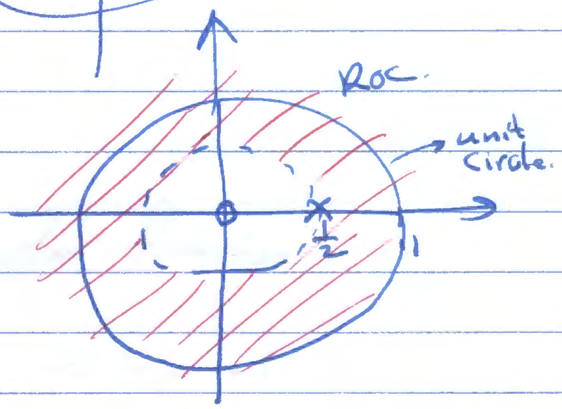
zeros: at $z = 0$
poles: at $z = \frac{1}{2}$

★ There are two choices for ROC



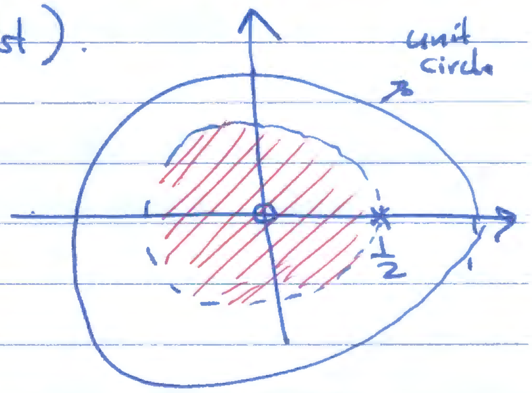
△1 $|z| > \frac{1}{2}$

- stable system (F.T. exist).
- Causal (Right-sided)
- $h(n) = (\frac{1}{2})^n u(n)$



△2 $|z| < \frac{1}{2}$

- Not Stable. (F.T. doesn't exist).
- non-Causal
- $h(n) = -(\frac{1}{2})^n u(-n-1)$



Inverse z-transform

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-1} \Rightarrow x(n) = f^{?}(X(z))$$

1. Inspection Method (Informal)

$$a^n u(n) \stackrel{z}{\longleftrightarrow} \frac{1}{1 - a z^{-1}}, \quad |z| > |a|$$

$$-a^n u(-n-1) \stackrel{z}{\longleftrightarrow} \frac{1}{1 - a z^{-1}}, \quad |z| < |a|$$

E.g. $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

by Inspection (from table) $\Rightarrow$ $x(n) = \left(\frac{1}{2}\right)^n u(n)$

if ROC $|z| < \frac{1}{2} \Rightarrow x(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$

$\boxed{2.}$ Partial Fraction Expansion

$$X(z) = \frac{P(z^{-1})}{Q(z^{-1})} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

$\Rightarrow$ M zeros and N poles at nonzero locations in z-plane. In addition, there will be either $M-N$ poles at $z=0$ if $M>N$ or $N-M$ zeros at $z=0$ if $N>M$. i.e. $X(z)$ has the same number of zeros and poles and there are no poles or zeros at $z=\infty$.

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

c_k's are nonzero zero

d_k's " " poles.

* If $M<N$ and poles are all first-order, then $X(z)$ can be expressed as:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

Multiplying both sides by $(1-d_k z^{-1})$ and evaluating for $z=d_k$ shows that the coefficient A_k can be found by

(Residues) $$A_k = (1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Example: $$X(z)=\frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z|>\frac{1}{2}$$

$$X(z)=\frac{A_1}{1-\frac{1}{4}z^{-1}}+\frac{A_2}{1-\frac{1}{2}z^{-1}}$$

$$A_1=(1-\tfrac{1}{4}z^{-1})X(z)\Big|_{z=\frac{1}{4}} = -1$$

$$A_2=(1-\tfrac{1}{2}z^{-1})X(z)\Big|_{z=\frac{1}{2}} = 2$$

So, $$X(z)=\frac{2}{1-\frac{1}{2}z^{-1}}+\frac{-1}{1-\frac{1}{4}z^{-1}}$$

$\Rightarrow h(n) = 2(\frac{1}{2})^n u(n) - (\frac{1}{4})^n u(n)$. (Right-sided since ROC $|z|>\frac{1}{2}$).

* IF $M>=N$:

we divide $P(z^{-1})$ by $Q(z^{-1})$ by long division and we use partial fraction expansion for the remainder.

Example: $$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \quad |z|>1$$

$$X(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$, $B_0 = 2$ found by long division

$$A_1 = \left(1-\tfrac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = -9$$

$$A_2 = \left(1-z^{-1}\right)X(z)\Big|_{z=1} = 8$$

$$\begin{array}{r|l} & \quad 2 \\ \hline \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & -\underline{z^{-2}-3z^{-1}+2} \\ & \quad 5z^{-1}-1 \end{array}$$

So, $$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$$

by Inspection and using linearity of Z-transform, and ROC $|z|>1 \Rightarrow$ Right sided.

$$x(n) = 2\delta(n) - 9\left(\tfrac{1}{2}\right)^n u(n) + 8\,u(n).$$

3. Power series Expansion:

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \dots + x(-2)\,z^{2} + x(-1)\,z + x(0)\,z^{0} +$$
$$x(1)\,z^{-1} + x(2)\,z^{-2} + \dots$$

Example: $X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right)$

Although $X(z)$ is obviously a rational function, its only poles are at $z=0$, so Partional Fraction Expansion is not appropriate.

* by Multiplying factors:

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}$$

therefore, by Inspection, $x(n) = \begin{cases} 1, & n=-2 \\ -\frac{1}{2}, & n=-1 \\ -1, & n=0 \\ \frac{1}{2}, & n=1 \\ 0, & \text{otherwise} \end{cases}$

Equivelently, $x(n) = \delta(n+2) - \frac{1}{2}\delta(n+1) - \delta(n) + \frac{1}{2}\delta(n-1)$

Example: $X(z) = \log(1 + a z^{-1})$, $|z| > |a|$

by using power series expansion for $\log(x+1)$ with $|x|<1$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} \qquad \boxed{\mathrm{Log}(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}}$$

Therefore,

$$x(n) = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n} , & n \geq 1 \\ 0 , & \text{otherwise.} \end{cases}$$

Example:

$$X(z) = \frac{1}{1 - a z^{-1}} , \quad |z| > |a|$$

* since RoC is exterior of a circle $\Rightarrow$ Right-sided seq.

* since $X(z)$ approaches a finite constant as z approaches ∞
$\Rightarrow$ seq. $x(n)$ is causal.

* by using long division

$$\begin{array}{r|l} & 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} \\ \hline 1 - a z^{-1} & 1 \\ & -\;(1 - a z^{-1}) \\ \hline & a z^{-1} \\ & -\;(a z^{-1} - a^2 z^{-2}) \\ \hline & a^2 z^{-2} \\ & a^2 z^{-2} - a^3 z^{-3} \\ \hline & a^3 z^{-3} \\ & a^3 z^{-3} - a^4 z^{-4} \\ \hline & a^4 z^{-4} \\ & \vdots \end{array}$$

$$X(z) = 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + \ldots$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Rightarrow x(n) = a^n u(n).$$

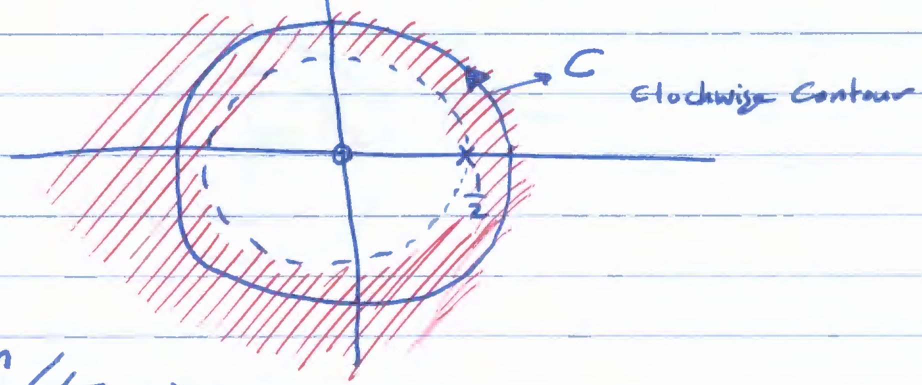
4. Contour Integration (Formal method)

$$x(n) = \frac{1}{2\pi j} \oint_C \underbrace{X(z)\, z^{n-1}\, dz}_{\text{Complex integration}}$$

$= \sum$ (residues of $X(z) z^{n-1}$ at poles of $(X(z) z^{n-1})$ inside C.

Example:

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2}$$



$X(z) z^{n-1} = z^n / (z - \frac{1}{2})$

$n \geq 0 \Rightarrow$ 1 pole at $z = \frac{1}{2}$

Residue of $\frac{z^n}{z - \frac{1}{2}}$ at $z = \frac{1}{2}$

$$\left. \frac{z^n}{(z - \frac{1}{2})} (z - \tfrac{1}{2}) \right|_{z = \frac{1}{2}} = \left(\frac{1}{2}\right)^n$$

$n \geq 0 \Rightarrow x(n) = \left(\frac{1}{2}\right)^n$

for $n < 0$ one pole at $z = \frac{1}{2}$
n poles at $z = 0$

Easy way (we know $x(n) = 0$ for $n < 0$)

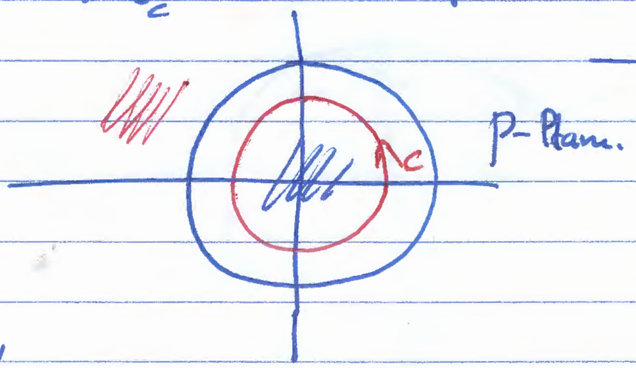
$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

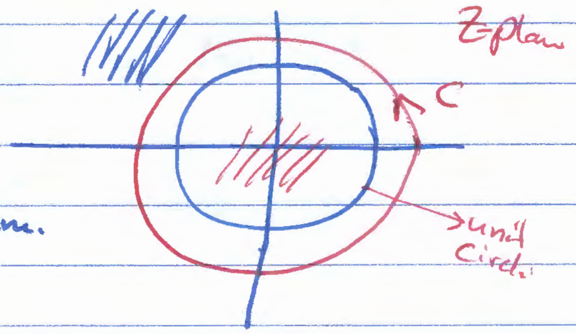
let $z = p^{-1} \Rightarrow z^{n-1} = p^{-n+1}$

If $z = r e^{j\theta}$ thus $p = \left(\frac{1}{r}\right) e^{-j\theta}$

$$x(n) = \frac{-1}{2\pi j} \oint_c X\left(\frac{1}{p}\right) p^{-n-1}\, dp$$

$$= \frac{1}{2\pi j} \oint_c X\left(\frac{1}{p}\right) p^{-n-1}\, dp$$

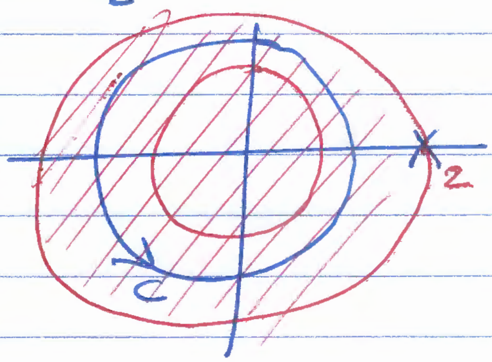




$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} , \quad |z| > \frac{1}{2}$$

$$X\left(\frac{1}{p}\right) = \frac{1}{1 - \frac{1}{2} p} , \quad |p| < \frac{1}{2}$$

$$= \frac{-2}{p-2}$$



$$X\left(\frac{1}{p}\right) p^{-n-1}$$

$n < 0 \Rightarrow$ 1 Pole at $p = 2$

$\therefore x(n) = 0$ for $n < 0$

$n \geq 0$: 1 Pole at $p = 2 \rightarrow$ this case is already handled.

$(n+1)$ poles at $p = 0$

Finally $x(n) = \left(\frac{1}{2}\right)^n u(n)$

Z-Transform Properities

$$x(n) \xleftrightarrow{Z} X(Z), \quad ROC = R_x$$
$$x_1(n) \xleftrightarrow{Z} X_1(Z), \quad ROC = R_{x_1}$$
$$x_2(n) \xleftrightarrow{Z} X_2(Z), \quad ROC = R_{x_2}$$

1. linearity

$$a x_1(n) + b x_2(n) \xleftrightarrow{Z} a X_1(Z) + b X_2(Z)$$
$$ROC = R_{x_1} \cap R_{x_2}$$

2. Time shifting

$$x(n-n_0) \xleftrightarrow{Z} Z^{-n_0} X(Z), \quad ROC = R_x \text{ (except for possible addition or deletion of } Z=0 \text{ or } Z=\infty)$$

Proof:

$$Y(Z) = \sum_{n=-\infty}^{\infty} x(n-n_0) Z^{-n}$$

substitute $m = n - n_0$

$$Y(Z) = \sum_{m=-\infty}^{\infty} x(m) Z^{-(m+n_0)}$$

$$= Z^{-n_0} \sum_{m=-\infty}^{\infty} x(m) Z^{-m} = Z^{-n_0} X(Z)$$

Example:

$$X(Z) = \frac{1}{Z - \frac{1}{4}}, \quad |Z| > \frac{1}{4}$$

from ROC $[|Z| > \frac{1}{4}] \Rightarrow x(n)$ is right-sided sequence.

$$X(Z) = \frac{Z^{-1}}{1 - \frac{1}{4}Z^{-1}}, \quad |Z| > \frac{1}{4}$$

This Z-transform is of the form

$$X(Z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k Z^{-1})}{\prod_{k=1}^{N}(1 - d_k Z^{-1})}$$

with $M = N \Rightarrow$ by long division

$$\begin{array}{r|l} & -4 \\ \hline -\frac{1}{4}Z^{-1}+1 & Z^{-1} \\ & -Z^{-1} + 4 \\ \hline & 4 \end{array}$$

So, $\dfrac{z^{-1}}{1-\frac{1}{4}z^{-1}} = -4 + \dfrac{4}{1-\frac{1}{4}z^{-1}}$

$\Rightarrow x(n) = -4\delta(n) + 4\left(\frac{1}{4}\right)^n u(n).$

* Alternatively, $x(n)$ can be obtained directly by applying time-shifting property.

$$X(z) = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right), \quad |z| > \frac{1}{4}$$

$$\Rightarrow x(n) = \left(\frac{1}{4}\right)^{n-1} u(n-1).$$

3. Multiplying by exponential sequence:

$$z_0^n x(n) \overset{z}{\longleftrightarrow} X(z/z_0), \quad ROC = R_x |z_0|$$

Similar to Fourier Transform $\quad e^{jw_0 n} x(n) \overset{F.T}{\longleftrightarrow} X(e^{j(w-w_0)}).$

Example:

$$u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$x(n) = r^n \cos(w_0 n)\, u(n)$$

First, $x(n)$ can be expressed as:

$$x(n) = \frac{1}{2}(re^{jw_0})^n u(n) + \frac{1}{2}(re^{-jw_0})^n u(n)$$

$$\Rightarrow \frac{1}{2}(re^{jw_0})^n u(n) \overset{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1-re^{jw_0}z^{-1}}, \quad |z| > r$$

$$\frac{1}{2}(re^{-jw_0})^n u(n) \overset{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1-re^{-jw_0}z^{-1}}, \quad |z| > r$$

$$X(z) = \frac{\frac{1}{2}}{1-re^{jw_0}z^{-1}} + \frac{\frac{1}{2}}{1-re^{-jw_0}z^{-1}}, \quad |z| > r$$

$$= \frac{(1-r\cos w_0 z^{-1})}{1-2r\cos w_0 z^{-1}+r^2z^{-2}}, \quad |z| > r$$

4. Differentiation of $X(z)$

$$n\,x(n) \xleftrightarrow{Z} -z\frac{dX(z)}{dz}, \quad ROC = R_x$$

proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \Rightarrow \frac{dX(z)}{dz} = \sum_{-\infty}^{\infty} -n\,x(n)\, z^{-n-1}$$

$$\frac{dX(z)}{dz} = -z^{-1}\sum_{-\infty}^{\infty} n\,x(n)\, z^{-n}$$

$$-z\frac{dX(z)}{dz} = \sum_{-\infty}^{\infty} (n\,x(n))\, z^{-n} = Z\{n\,x(n)\} \quad \therefore$$

Example:

$$X(z) = \log(1 + a z^{-1}), \quad |z| > |a|$$

$$\frac{dX(z)}{dz} = \frac{-a z^{-2}}{1 + a z^{-1}}$$

$$n\,x(n) \xleftrightarrow{Z} -z\frac{dX(z)}{dz} = \frac{a z^{-1}}{1 + a z^{-1}}, \quad |z| > |a|$$

$$n\,x(n) = a(-a)^{n-1} u(n-1)$$

$$x(n) = (-1)^{n+1}\frac{a^n}{n} u(n-1) \xleftrightarrow{Z} \log(1 + a z^{-1}), \quad |z| > |a|$$

Example: Find $Z\{n a^n u(n)\}$, $x(n) = n\,a^n u(n) = n(a^n u(n))$

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1 - a z^{-1}}\right), \quad |z| > |a|$$

$$= \frac{a z^{-1}}{(1 - a z^{-1})^2}, \quad |z| > |a|$$

Therefore,

$$n\,a^n u(n) \xleftrightarrow{Z} \frac{a z^{-1}}{(1 - a z^{-1})^2}, \quad |z| > |a|$$

5 Conjugate of a complex sequence.

$$x^*(n) \overset{Z}{\longleftrightarrow} X^*(z^*) \quad , \quad ROC = R_x$$

6 Time Reversal

$$x^*(-n) \overset{Z}{\longleftrightarrow} X^*(1/z^*) \quad , \quad ROC = \frac{1}{R_x}$$

i.e. ROC is inverted $\Rightarrow$ if $R_x \rightarrow r_R < |z| < r_L \Rightarrow$ ROC becomes

$\frac{1}{r_L} < |z| < \frac{1}{r_R}$, thus if z_0 is in ROC of $x(n)$, then

$\frac{1}{z_0^*}$ is in ROC of $Z\{x^*(-n)\}$.

$*$ If $x(n)$ is real sequence $\Rightarrow$

$$x(-n) \overset{Z}{\longleftrightarrow} X(\tfrac{1}{z}) \quad , \quad ROC = \frac{1}{R_x}$$

Example:

$$x(n) = a^{-n} u(-n) \quad , \quad X(z)\,?$$

$x(n)$ is time-reversal of $a^n u(n)$.

$$X(z) = \frac{1}{1 - az} = \frac{-a^{-1} z^{-1}}{1 - a^{-1} z^{-1}} \quad , \quad |z| < |a^{-1}|$$

7 Convolution of sequences:

$$x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(z) X_2(z) \quad , \quad ROC \text{ contains } R_{x_1} \cap R_{x_2}$$

Proof:

$$y(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] z^{-n}$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n}$$

Let $m = n-k$

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{m=-\infty}^{\infty} x_2(m)\, z^{-m} \right] z^{-k}$$

$$\Rightarrow Y(z) = X_1(z)\, X_2(z). \quad \#$$

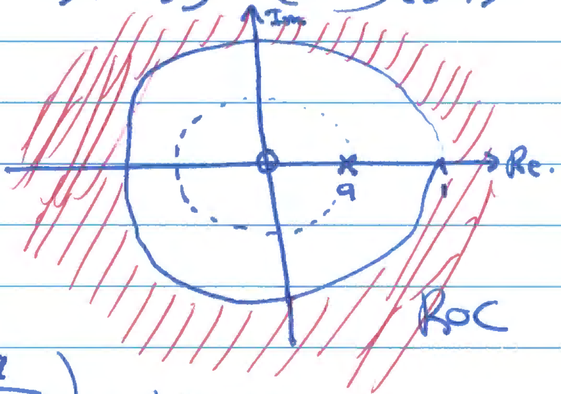
Example: $x_1(n) = a^n u(n)$ and $x_2(n) = u(n)$

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

If $|a| < 1$

$$\Rightarrow Y(z) = \frac{1}{(1-az^{-1})(1-z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \quad |z| > 1$$



Z-plane.

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}} \right), \quad |z| > 1$$

$$\Rightarrow y(n) = \frac{1}{1-a}\left(u(n) - a^{n+1} u(n) \right)$$

See table 3.2 page 126 which summarizes Z-transform properties.

8 Intial Value Theorem

If $x(n) = 0$ for $n < 0$ (i.e. $x(n)$ is causal), then

$$x(0) = \lim_{z \to \infty} X(z)$$

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{-\infty}^{\infty} x(n) \lim_{z\to\infty} z^{-n} = x(0)$$