Chapter = 4 =

Sampling of Continuous-Time signals

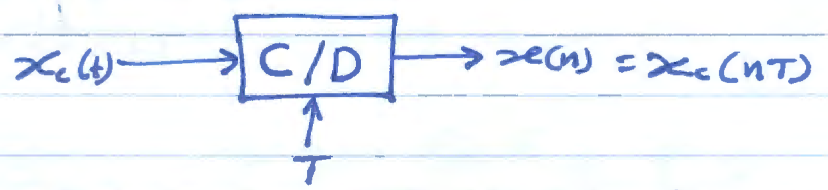
Periodic sampling:

$$x(n) = x_c(nT), \quad -\infty < n < \infty$$

$T \rightarrow$ sampling period

$f_s = \frac{1}{T} \rightarrow$ Sampling Frequency (samples/sec)

$\Omega_s = \frac{2\pi}{T}$ sampling frequency (radians/sec)

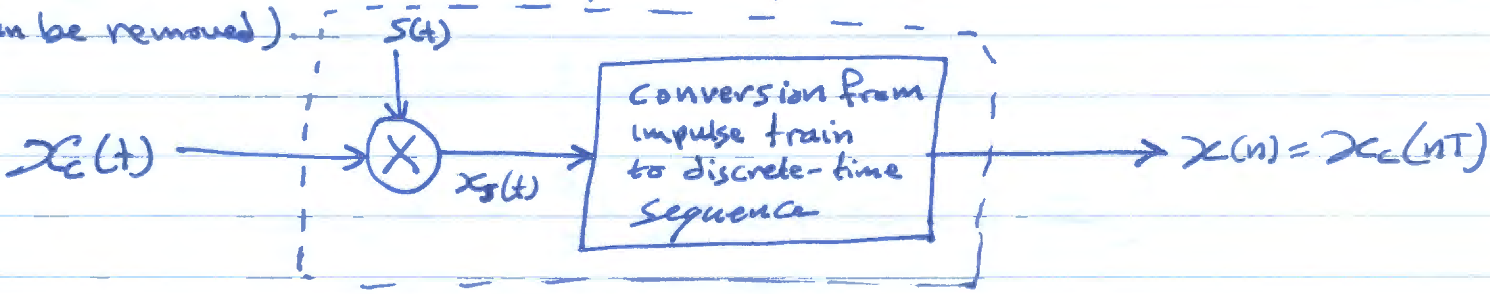


Ideal Continuous-to-Discrete-time Converter (C/D).

* In practical, sampling is implemented by Analog-to-Digital (A/D) Convertor.

* A/D
  - quantization of output samples?
  - linearity of quantization steps?
  - Need for sampling-and-hold circuits.
  - Limitations on the sampling rate.

* Sampling is not Invertible. i.e. we cannot re-generate $x_c(t)$ from $x(n)$. Since many continuous-time signals can have the same output samples $x(n)$. (This ambiguity can be removed).

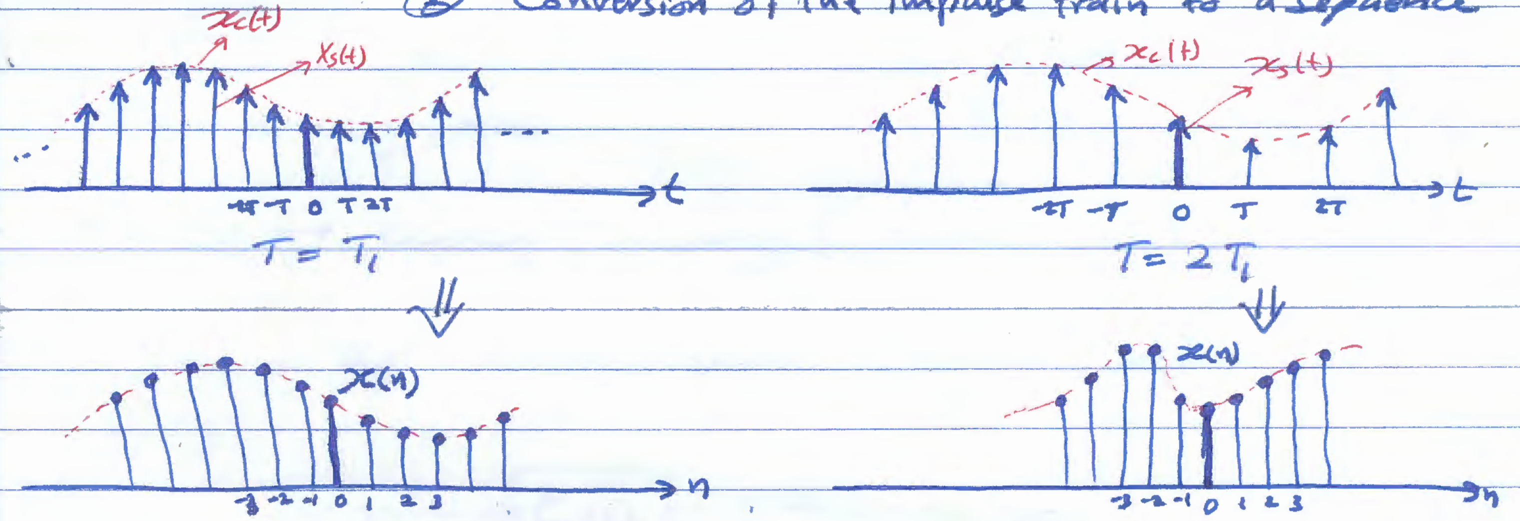


C/D Converter

Two ~~Signals~~ Stages:

Sampling → ① Impulse Train Modulator (followed by)

② Conversion of the impulse train to a sequence



Frequency-Domain Representation of Sampling:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$, $\delta(t) \rightarrow$ unit impulse function or Dirac Delta function

$$x_s(t) = x_c(t)\, s(t)$$
$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

* By shifting property of impulse function, $x_s(t)$ can be expressed as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT)$$

$F\{x_s(t)\}$?

Since $x_s(t) = x_c(t) \cdot s(t)$ (Multiplication in Time-domain).

$\Rightarrow$ $F\{x_s(t)\}$ is convolution of Fourier Transform $X_c(j\Omega)$ and $S(j\Omega)$.

* Fourier Transform of a periodic impulse train is a periodic impulse train:

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$\Omega_s = \frac{2\pi}{T} \rightarrow$ Sampling freq. in radians per second.

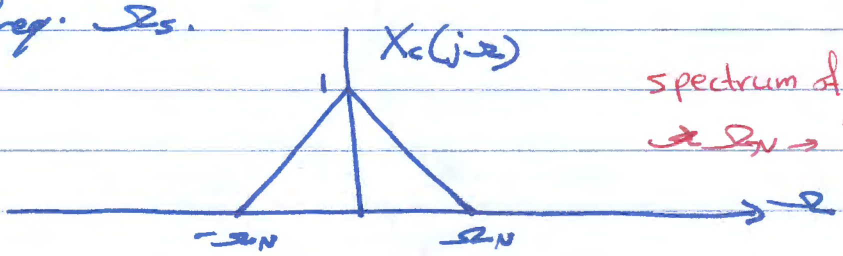
continuous convolution

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$X_s(j\Omega) = \underbrace{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))}_{\text{periodically repeated copies of } X_c(j\Omega)}$$

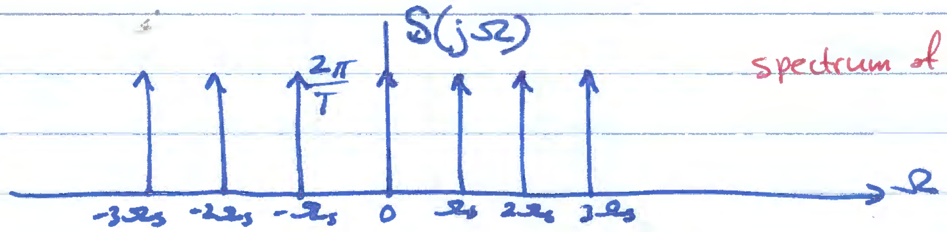
$F\{x_c(t)\}$

* Copies of $X_c(j\Omega)$ are shifted by Integer multiples of sampling freq. Ω_s.

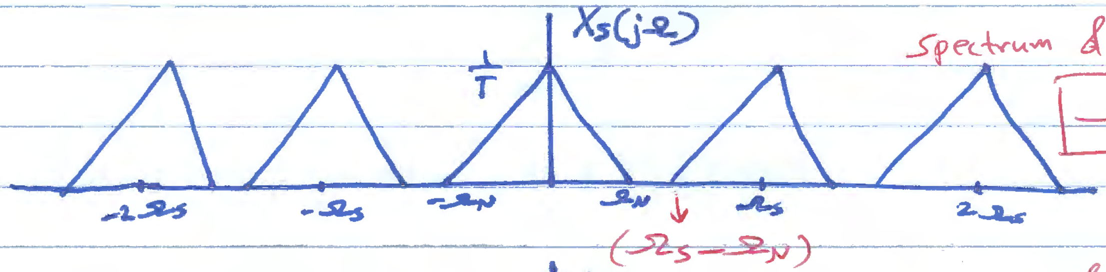


spectrum of original signal

$\Omega_N \rightarrow$ highest freq. in $x_c(t)$.

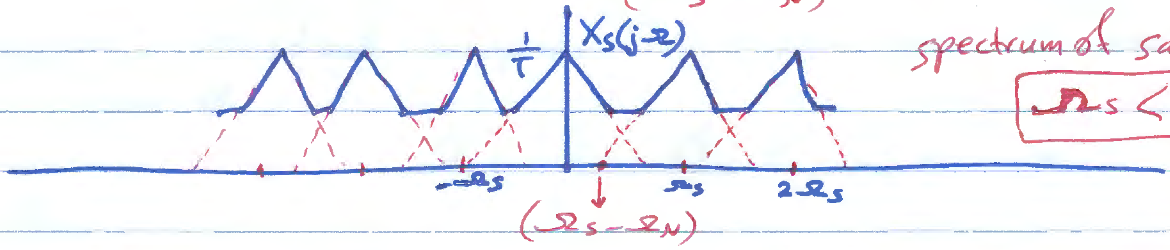


spectrum of Sampling function



Spectrum of Sampled Signal

$\boxed{\Omega_s > 2\Omega_N}$

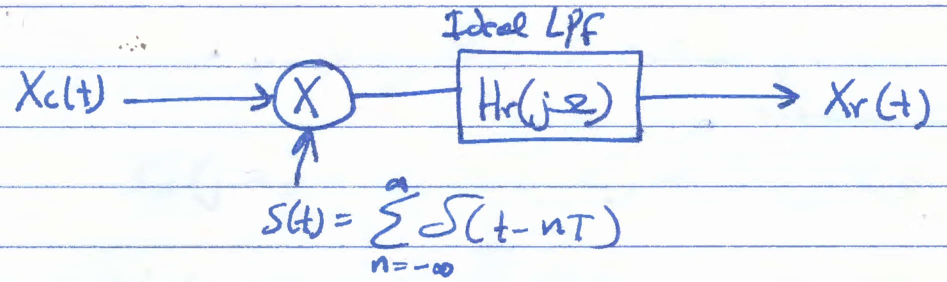


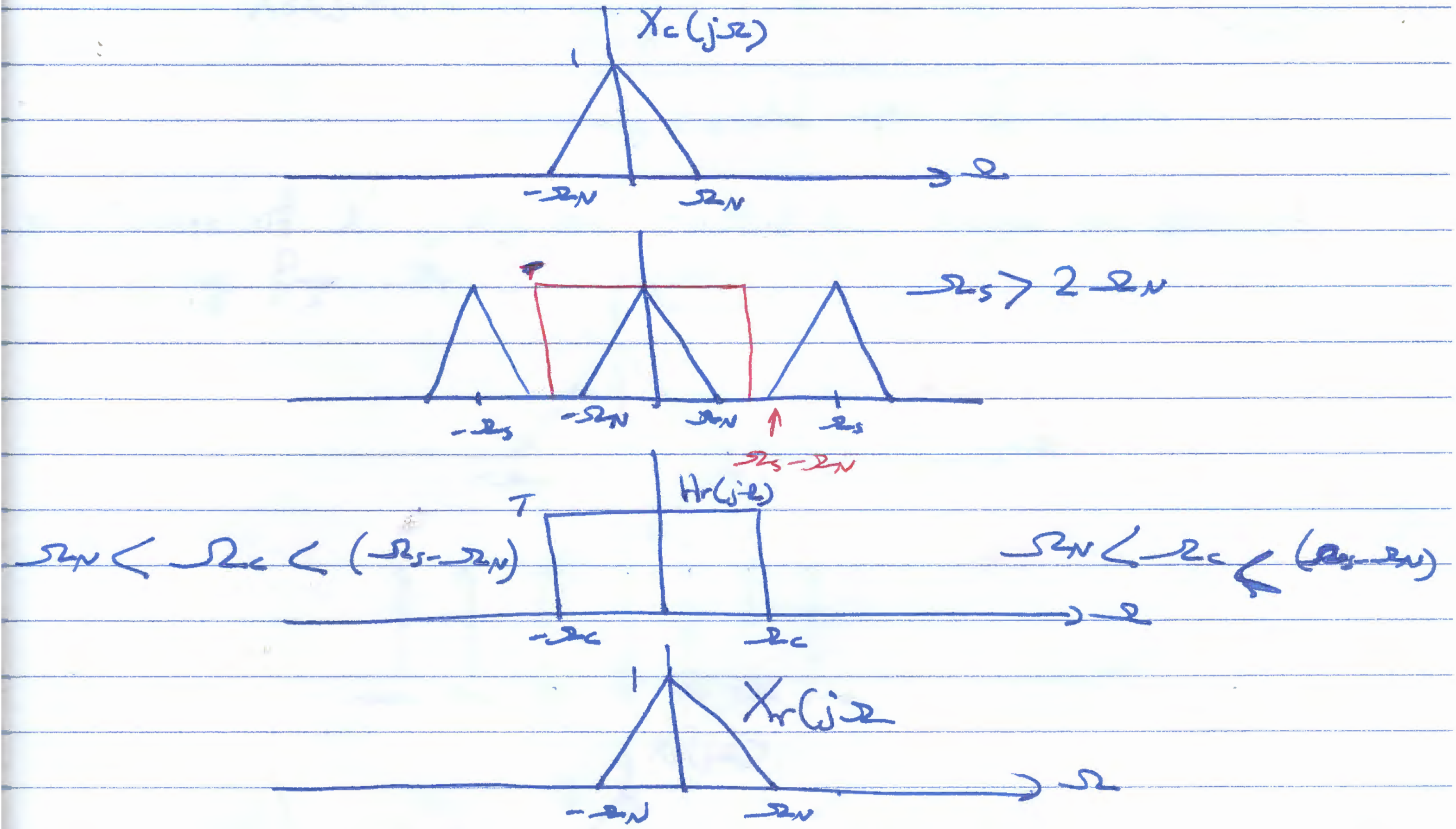
spectrum of sampled signal

$\boxed{\Omega_s < 2\Omega_N}$

When, $\Omega_s - \Omega_N > \Omega_N \Rightarrow$ or $\Omega_s > 2\Omega_N$

$\Rightarrow$ replicas of $X_c(j\Omega)$ don't ~~overlap~~ overlap $\Rightarrow x_c(t)$ can be recovered from $x_s(t)$ with an Ideal lowpass filter as shown:





$$X_r(j\Omega) = H_r(j\Omega)\, X_s(j\Omega)$$

$H_r(j\Omega) \rightarrow$ ideal Lowpass filter with gain T and cutoff freq. Ω_c such that:

$$\Omega_N < \Omega_c < (\Omega_s - \Omega_N)$$

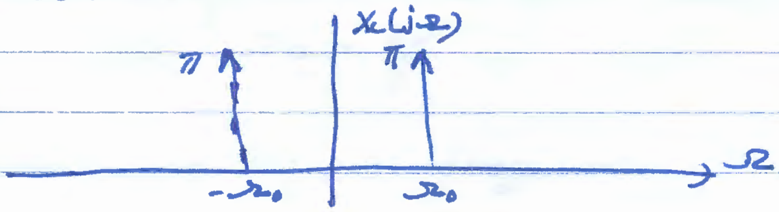
Then,

$$X_r(j\Omega) = X_c(j\Omega) \quad \text{and } x_r(t) = x_c(t).$$

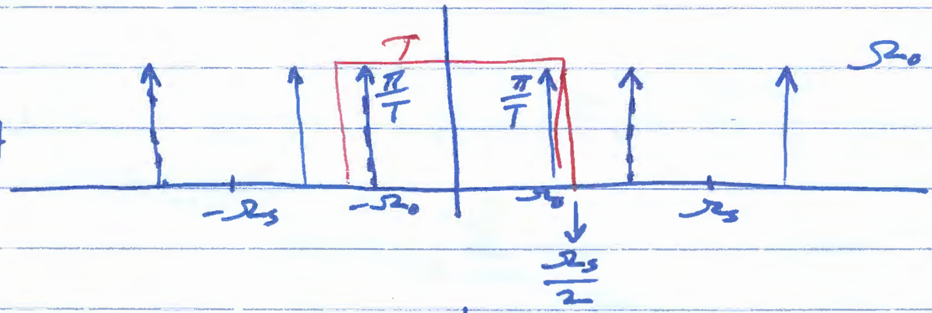
* If $\Omega_s \leq 2\Omega_N$, copies of $X_c(j\Omega)$ overlap, so that when they are added together, $X_c(j\Omega)$ is no longer recoverable by Lowpass filter.

* In this case, the reconstructed output $X_r(t)$ is related to the input $x_c(t)$ through a distortion refered to as <u>aliasing</u>.

* Assume $x_c(t) = \cos \Omega_0 t$

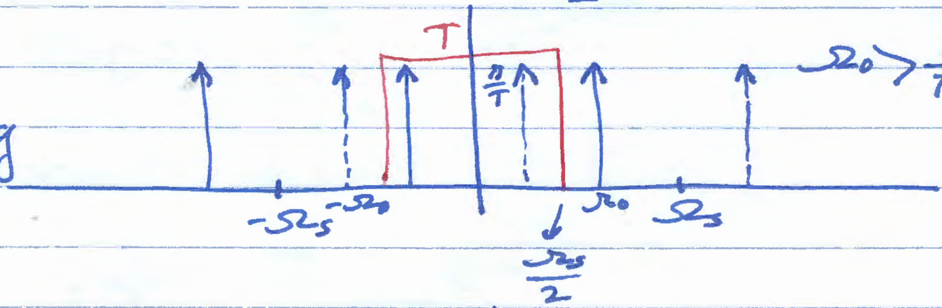


No aliasing



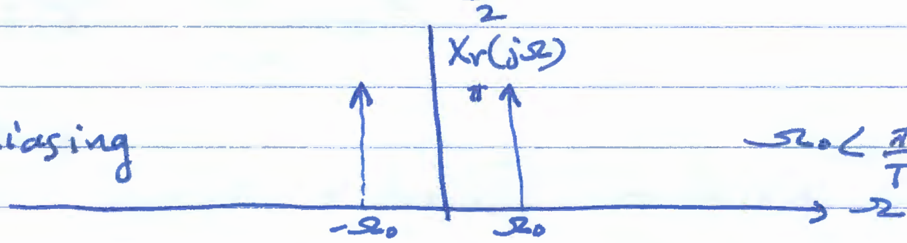
$\Omega_0 < \frac{\pi}{T} = \frac{\Omega_s}{2}$ or $\Omega_s > 2\Omega_0$

Aliasing



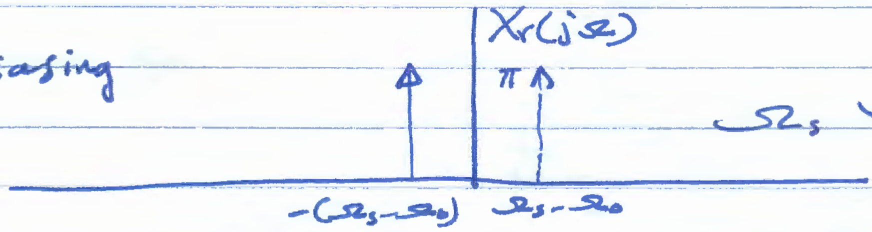
$\Omega_0 > \frac{\pi}{T} = \frac{\Omega_s}{2}$ or $\Omega_s < 2\Omega_0$

No aliasing



$\Omega_0 < \frac{\pi}{T}$

Aliasing



$\Omega_s > \frac{\pi}{T}$

* With No aliasing: $x_r(t) = \cos \Omega_0 t$
* with aliasing: $x_r(t) = \cos(\Omega_s - \Omega_0)t$

I.e. higher freq. signal $\cos \Omega_0 t$ has taken on the identity (alias) of lower freq. signal $\cos(\Omega_s - \Omega_0)t$.

Nyquist Sampling Theorem:

Let $x_c(t)$ be a bandlimited signal with $X_c(j\Omega) = 0$ for $|\Omega| \geq \Omega_N$. Then, $x_c(t)$ is uniquely determined by its samples $x(n) = x_c(nT)$, $n = 0, \pm 1, \pm 2, \ldots$ If $\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$

$\Omega_N \rightarrow$ refered to as Nyquist frequency.
$2\Omega_N \rightarrow$ " " " " rate.

* our objective is to express $X(e^{j\omega}) = \mathcal{F}\{x(n)\}$ in term of $X_s(j\Omega)$ and $X_c(j\Omega)$.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t - nT)$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\Omega T n}$$

Since, $x(n) = x_c(nT)$
and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

It follows that:

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega = \Omega T} = X(e^{j\Omega T})$$

From previous eq.

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

so,

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - k\Omega_s\right)\right)$$

or equivalently,

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

* As we see $X(e^{j\omega})$ is frequency-scaled version of $X_s(j\Omega)$ with freq. scaling specified by $\omega = \Omega T$.

* This frequency scaling can be thought as normalization of freq. axis, so $\Omega = \Omega_s$ in $X_s(j\Omega)$ is normalized to $\omega_s = 2\pi$ for $X(e^{j\omega})$

* This is due to time normalization in transformation from $x_s(t)$ to $x(n)$. $x_s(t)$ retains spacing between samples equal to sampling period T. In contrast, the "spacing" of sequence values $x(n)$ is always unity i.e. time axis is normalized by a factor of T. Correspondingly, in frequency domain, freq-axis is normalized by a factor of $f_s = \frac{1}{T}$.

<u>Example:</u> $x_c(t) = \cos(4000\pi t)$, $T = \frac{1}{6000}$ sec.

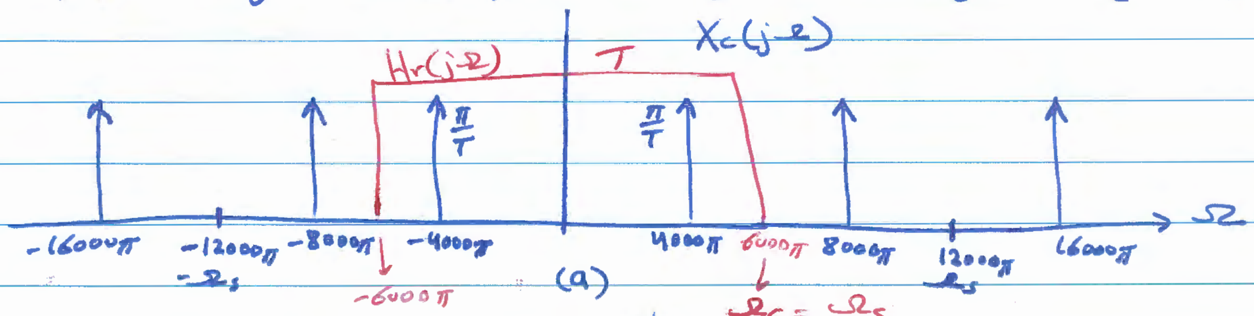
$x(n) = x_c(nT) = \cos(4000\pi T n) = \cos \omega_0 n$, where

$\omega_0 = 4000\pi T = \frac{2\pi}{3}$.

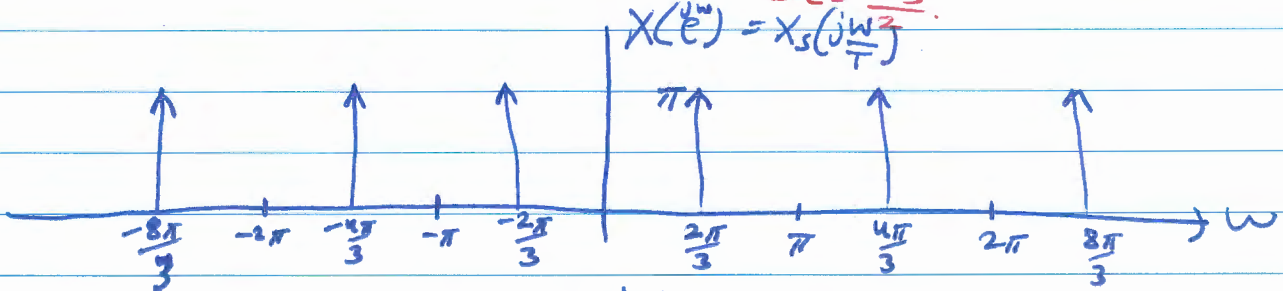
In this case,

$\Omega_s = \frac{2\pi}{T} = 12000\pi$, and highest freq. of $x_c(t)$ is $\Omega_0 = 4000\pi$, so conditions of Nyquist sampling theorem are satisfied. and there is no aliasing.

$$F\{x_c(t)\} = X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$



(a)



(b)

* Fig. (a) shows:

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

for $\Omega_s = 12000\pi$ rad/sec $\Rightarrow$ $X_c(j\Omega)$ is a pair of impulses at $\Omega = \mp 4000\pi$ and we see shifted copies of $X_c(j\Omega)$ centred at $\mp\Omega_s$, $\mp 2\Omega_s$, ... etc.

* Fig. (b) is a plot of $X(e^{jw}) = X_s(j\frac{w}{T})$ as a function of normalized freq. $w = \Omega T = \frac{\Omega}{f_s}$ (scaling the independent variable of an impulse also scales its area, i.e. $\delta(\frac{w}{T}) = T\delta(w)$.

* original freq. $\Omega_0 = 4000\pi \Rightarrow w_0 = 4000\pi T = \frac{2\pi}{3}$ which satisfies $w_0 < \pi \Leftrightarrow \Omega_0 = 4000\pi < \frac{\pi}{T} = 6000\pi$.

Fig. (a) also shows Ideal reconstruction filter $H_r(j\Omega)$ for given sampling frequency of $\Omega_s = 12000\pi$.

Example: Suppose $x_c(t) = \cos(16000\pi t)$, Sampling period $T = \frac{1}{6000}$

This sampling period T fails to satisfy the Nyquist Criterion, since

$\Omega_s = \frac{2\pi}{T} = 12000\pi < 2\Omega_0 = 32000\pi$.

$\Rightarrow$ we expect to see aliasing !!

$X_s(j\Omega)$ for this case is identical to that of the previous example., but now the impulse located at $\Omega = -4000\pi$ is from $X_c(j(\Omega - \Omega_s))$ rather than from $X_c(j\Omega)$, and impulse at $\Omega = 4000\pi$ is from $X_c(j(\Omega + \Omega_s))$.

* plotting $X(e^{jw}) = X_s(j\frac{w}{T})$ as a function of w yields the same graph in the previous example Fig. (b). Since we are normalizing by the same sampling period.

* Reason for this: sequence of samples, $x(n)$, in both Cases is the same. $\therefore \cos(16000\pi n/6000) = \cos(2\pi n + 4000\pi n/6000) = \cos(\frac{2\pi}{3} n)$.

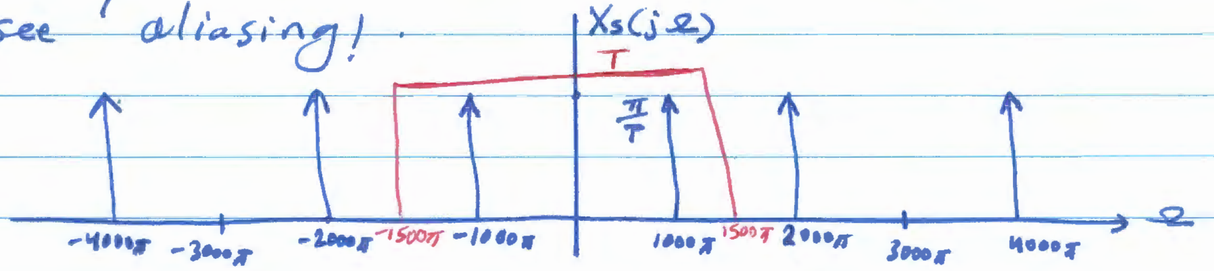
* (Note we can add any Integer multiple of 2π to argument of Cosine with Changing its value).

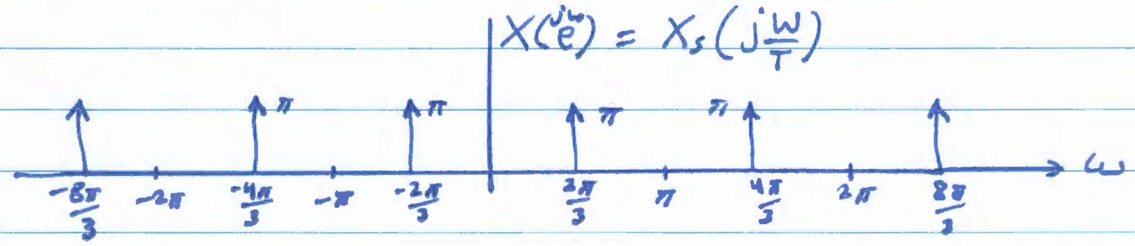
* So, we got the same seq. $x(n) = \cos(\frac{2\pi}{3} n)$ by sampling two different continuous-time signals with the same sampling freq.

* In the second case, output of filter $H_r(j\Omega)$ is $\cos(4000\pi t)$ which is not the original signal $x_c(t) = \cos(16000\pi t)$.

<u>Example:</u> suppose $x_c(t) = \cos(4000\pi t) \Rightarrow \Omega_0 = 4000\pi$

$T = \frac{1}{1500} \Rightarrow$ fails to satisfy Nyquist rate, since $\Omega_s = \frac{2\pi}{T} = 3000\pi < 2\Omega_0 = 8000\pi \Rightarrow$ So we expect to see aliasing!





$\Omega = 1000\pi$ comes from $X_c(j(\Omega + \Omega_s))$ and $-1000\pi \rightarrow$ comes from $X_c(j(\Omega - \Omega_s))$

$X(e^{jw})$ plot $\Rightarrow$ Corresponds to seq. $x(n) = \cos(\frac{2\pi}{3} n)$

* Same discrete-time signal ($x(n)$) may result from sampling the same continuous-time signal at different sampling rate if of those sampling rates fails to satisfy sampling theorem.

* Re-constructed signal from $H_r(j\Omega)$ is $\cos(1000\pi t)$ and $\cos(4000\pi t)$.

$\boxed{\text{Sec. 4.3}}$ Reconstruction of Bandlimited signal from its samples.

If $x(n)$ is given $\Rightarrow$ we can form an impulse train $x_s(t)$ in which successive impulses are assigned an area equal to sequence value i.e.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n)\,\delta(t-nT)$$

The n^{th} sample is associated with impulse at $t = nT$, when T is sampling period associated with $x(n)$.

$$x_s(t) \longrightarrow \boxed{h_r(t)} \longrightarrow x_r(t)$$

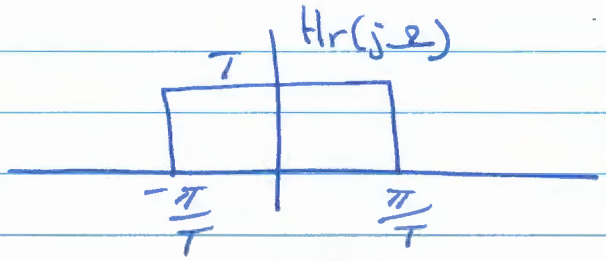
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n)\, h_r(t-nT)$$

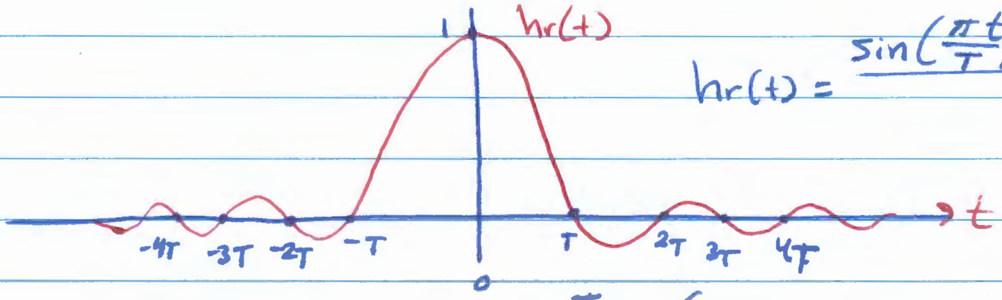
* Recall ideal reconstruction filter has a gain of T (to compensate for a factor $\frac{1}{T}$) and cutoff freq. Ω_c between Ω_N and $\Omega_s - \Omega_N$.

Ideal reconstruction system

$$\longrightarrow \boxed{\begin{array}{c}\text{Convert from seq.}\\ \text{to impulse train}\end{array}} \xrightarrow{x_s(t)} \boxed{\begin{array}{c}\text{ideal reconstruction}\\ \text{filter } H_r(j\Omega)\end{array}} \longrightarrow$$

T





$hr(t) = \dfrac{\sin(\frac{\pi t}{T})}{}$

Common choices for $\Omega_c = \frac{\Omega_s}{2} = \frac{\pi}{T}$ (No aliasing as long as $\Omega_s > 2\Omega_N$).

So, $$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}}$$

* As we know if $x(n) = x_c(nT)$, when $X_c(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ then $x_r(t)$ is equal to $x_c(t)$.

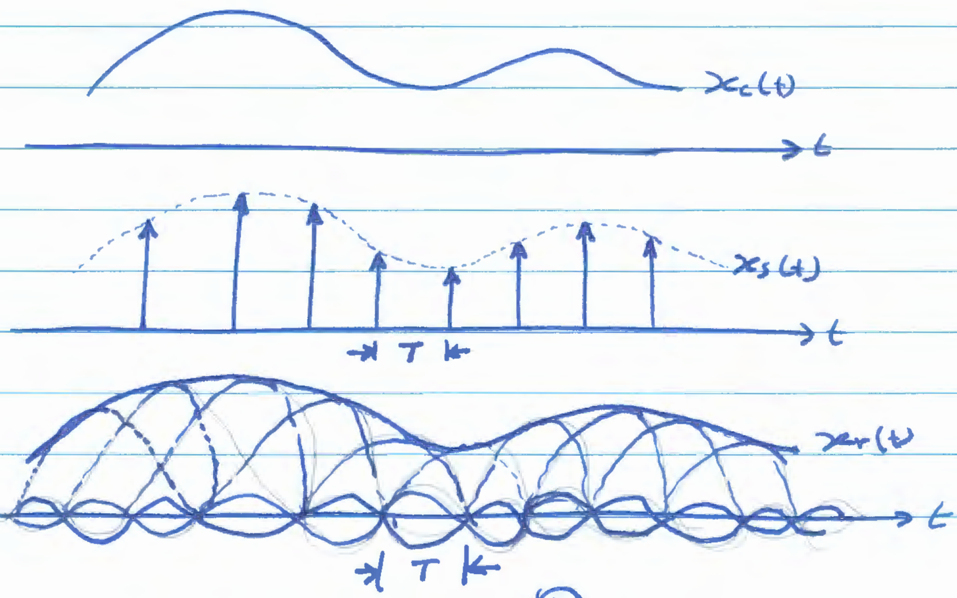
More illustration:

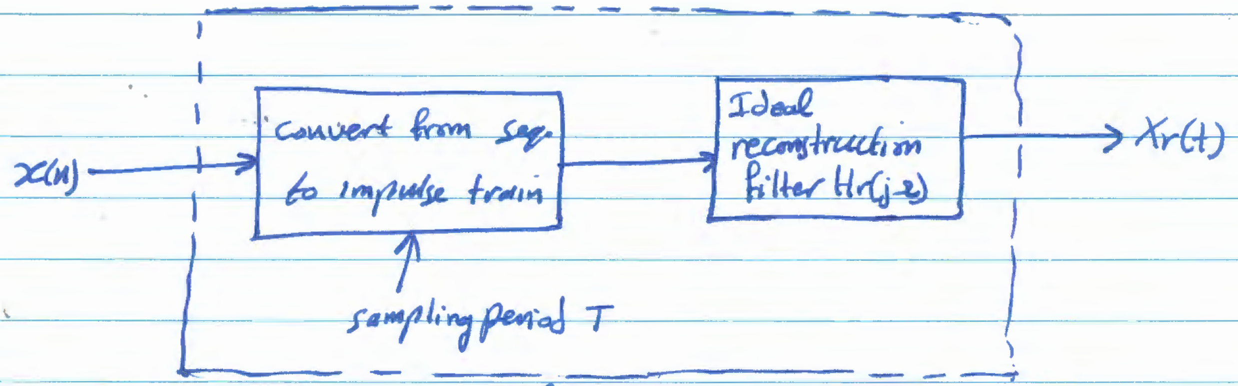
by looking at $hr(t)$ above, we note $hr(0) = 1$

$hr(nT) = 0$ for $n = \mp 1, \mp 2, \ldots$

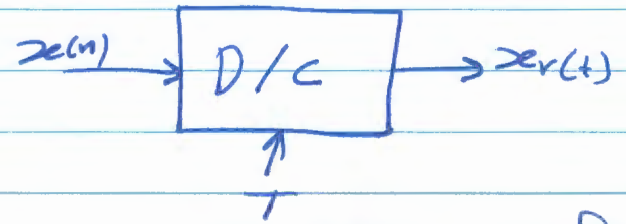
So, that $\Rightarrow$ if $x(n) = x_c(nT)$, then $x_r(mT) = x_c(mT)$ for all integer m.

$\Rightarrow$ That is, the reconstructed signal has the same values at sampling times as the original continuous-time signal, independently of T.





Ideal discrete-to-continuous-time convertor.



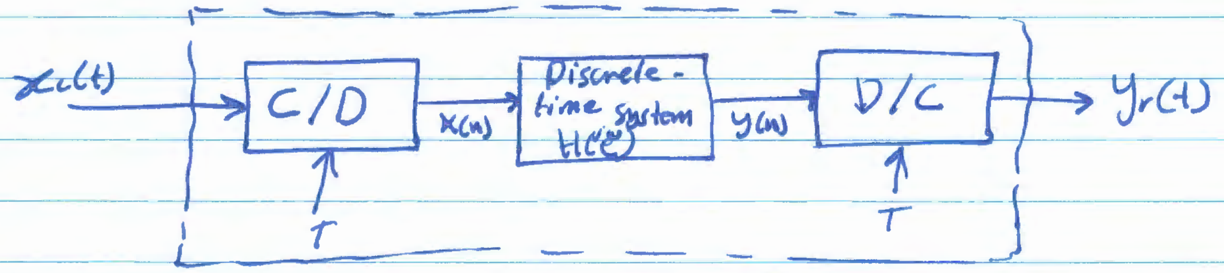
* relationship between input / output in frequency domain

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, H_r(j\Omega)\, e^{-j\Omega T n}$$

$$X_r(j\Omega) = H_r(j\Omega)\, X(e^{j\Omega T})$$

normalized freq. ω is replaced by ΩT.

$\boxed{\text{sec 4.4}}$ Discrete-time processing of continuous-time signals



* we assume sampling period of C/D and D/C are the same.
[but it is not necessary always]

C/D → produces $x(n) = x_c(nT)$

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \quad ---- \quad (*)$$

D/C → creates a continuous-time output

$$y_r(t) = \sum_{n=-\infty}^{\infty} y(n) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

$$Y_r(j\Omega) = H_r(j\Omega)\, Y(e^{j\Omega T})$$

$$= \begin{cases} T\cancel{X}(e^{j\Omega T}), & |\Omega| < \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

LTI systems:

$$Y(e^{j\omega}) = \underbrace{H(e^{j\omega})}_{\text{freq. response of LTI system.}} X(e^{j\omega})$$

So,

$$Y_r(j\Omega) = H_r(j\Omega)\, H(e^{j\omega})\, X(e^{j\Omega T})$$

substitute $\omega = \Omega T$ in eq. (*) above.

$$Y_r(j\Omega) = H_r(j\Omega)\, H(e^{j\Omega T}) \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

* If $X_c(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$, then Ideal lowpass reconstruction filter $H_r(j\Omega)$ cancels the factor $1/T$ and selects only the term for $k=0$.

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})\, X_c(j\Omega), & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| \geq \frac{\pi}{T} \end{cases}$$

* If $X_c(j\Omega)$ is bandlimited and sampling rate is above Nyquist rate the output-input relation:

$$Y_r(j\Omega) = H_{eff}(j\Omega)\, X_c(j\Omega)$$

When,

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| \geq \frac{\pi}{T} \end{cases}$$

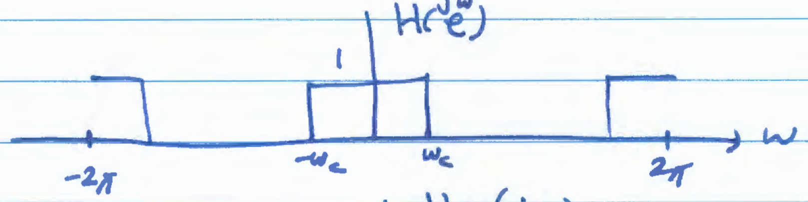
Two conditions for this relationship:

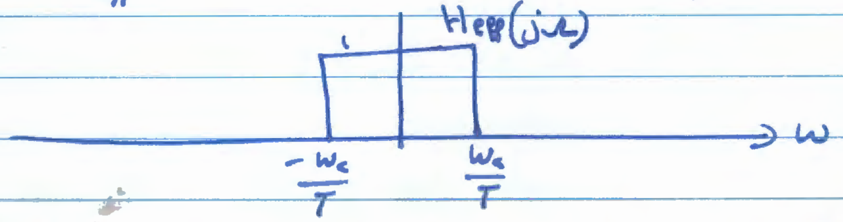
① Discrete-time system must be LTI

② Input signal must be bandlimited, and sampling rate must be high enough, so that any aliased components are removed by discrete-time system.

Example:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$ → periodic with period 2π

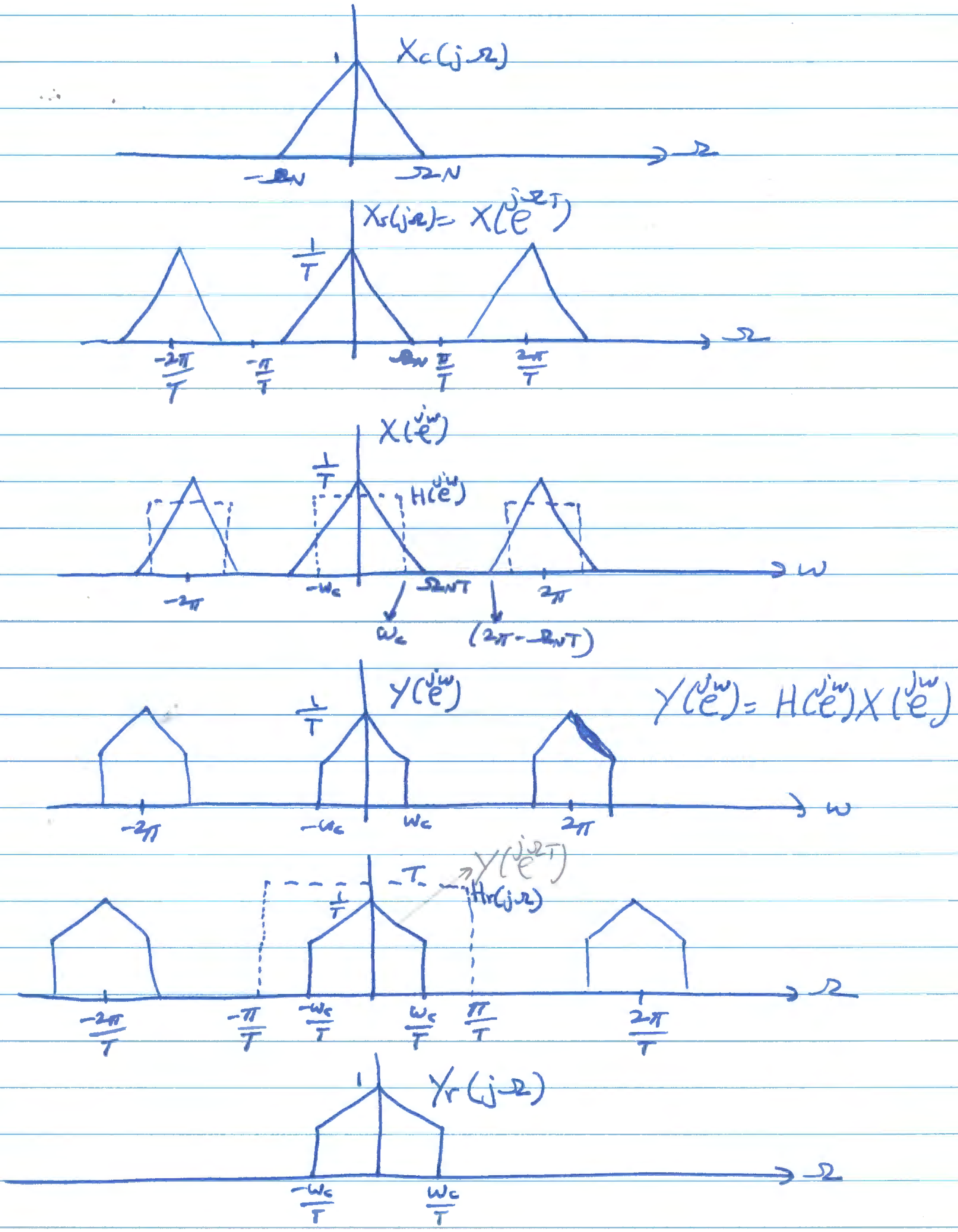




* For bandlimited input sampled above Nyquist rate, overall sys. is LTI continuous-time system with freq. response

$$H_{eff}(j\Omega) = \begin{cases} 1, & |\Omega T| < \omega_c \text{ or } |\Omega| < \frac{\omega_c}{T} \\ 0, & |\Omega T| > \omega_c \text{ or } |\Omega| > \frac{\omega_c}{T} \end{cases}$$

The effective freq. response → ideal lowpass filter with cutoff freq. $\Omega_c = \frac{\omega_c}{T}$.



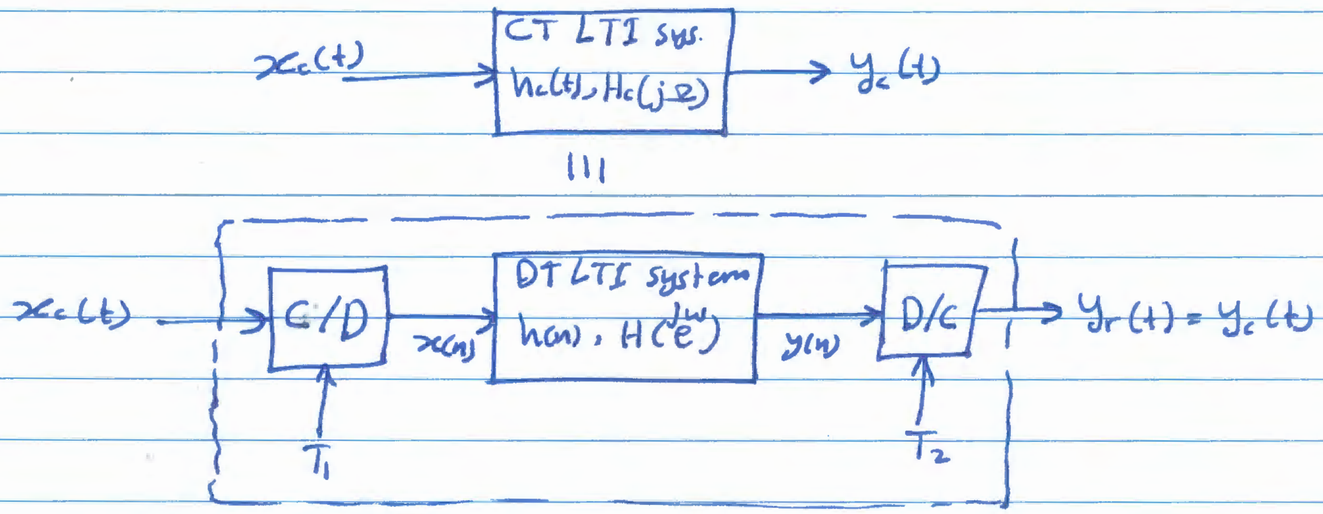
* Important points:

* Lowpass discrete-time filter with cutoff freq. w_c has effect of an ideal lowpass filter with cutoff $\Omega_c = \frac{w_c}{T}$ when used in the overall system (This cutoff depends on w_c and T).

* To get the same lowpass filter, we can fix the discrete-time Lowpass filter cutoff (w_c) and vary sampling freq. F_s or T.

E.g. If T were chosen so that $\Omega_N T < w_c \Rightarrow y_r(t) = x_c(t)$.
For no aliasing $\Rightarrow (2\pi - \Omega_N T) > w_c$
compared with Nyquist requirements that
$$(2\pi - \Omega_N T) > \Omega_N T$$

Impulse Invariance



$$H_{eff}(j\Omega) = H_c(j\Omega)$$

recall,
$$H_{eff} = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| \geq \frac{\pi}{T} \end{cases}$$

* This specifies how to choose $H(e^{jw})$ so that $H_{eff}(j\Omega) = H_c(j\Omega)$
i.e.
$$H(e^{jw}) = H_c\left(j\frac{w}{T}\right), \quad |w| < \pi \quad \text{—— } (1)$$
and T should be chosen such that,
$$H_c(j\Omega) = 0, \quad |\Omega| \geq \frac{\pi}{T} \quad \text{—— } (2)$$
Under these two constrains ((1) and (2)), also $h(n) = T h_c(nT)$
i.e. impulse response of DT system is a scaled, sampled version of $h_c(t)$. proof in page 161 (Textbook).

<u>Example</u>: We want Ideal LPF (Discrete-time) with $w_c < \pi$.
We can do this by sampling a continuous-time ideal lowpass filter with cutoff frequency $\Omega_c = \frac{w_c}{T} < \frac{\pi}{T}$ defined by:

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & |\Omega| \geq \Omega_c \end{cases}$$

$$\Rightarrow h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

So, we define impulse response of DT system to be:

$$h(n) = T h_c(nT) = T \frac{\sin(\Omega_c nT)}{\pi n T} = \frac{\sin(w_c n)}{\pi n}$$

Where, $w_c = \Omega_c T$

* We have already shown that Fourier Transform of $h(n)$ is

$$H(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & w_c < |w| \leq \pi \end{cases}$$

Which is identical to $H_c(j\frac{w}{T})$.