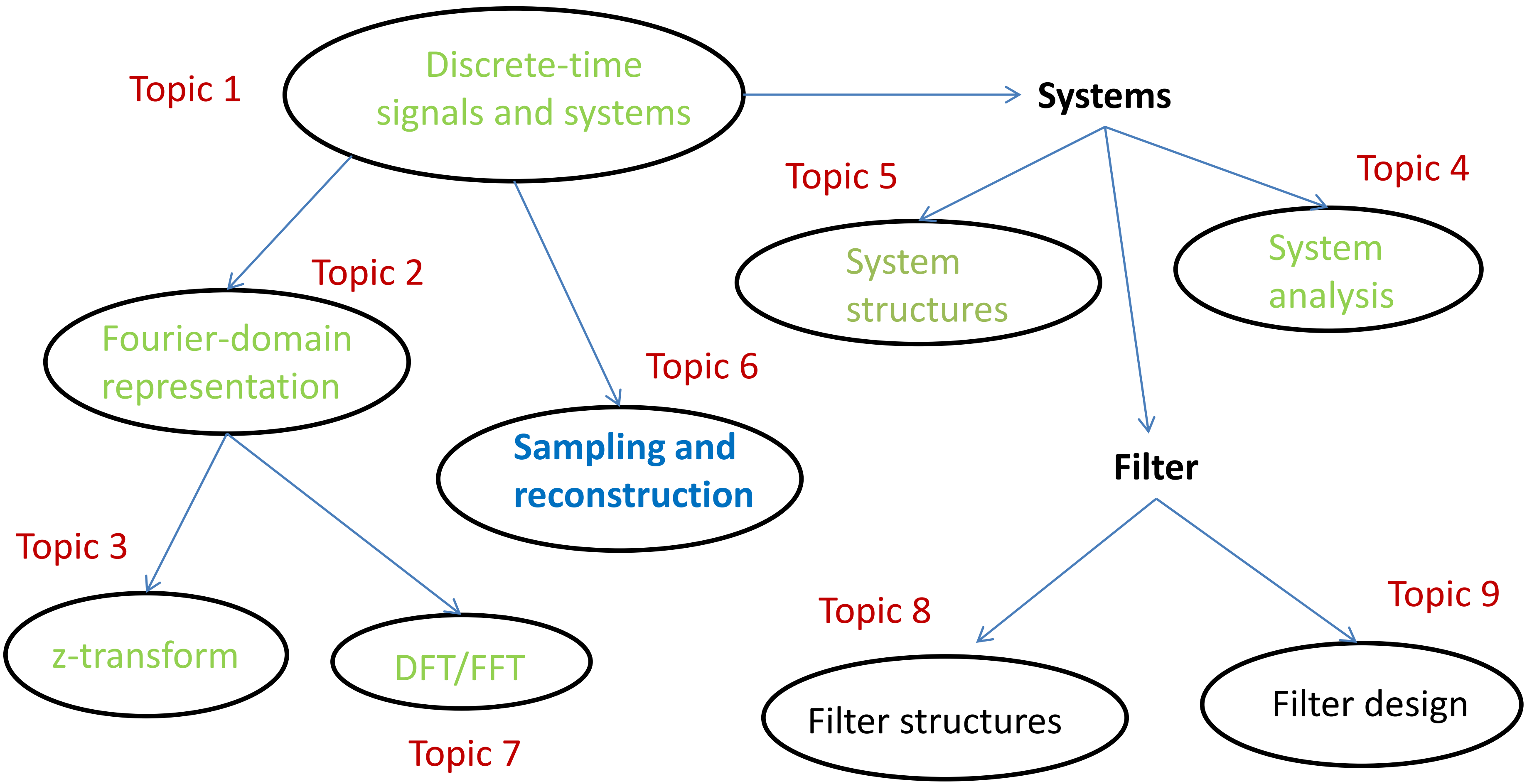
Sampling and Reconstruction

Chapter 4 in the textbook

Sec 4.0 – sec 4.6

Course at a glance



Periodic sampling

- From continuous-time $x_c(t)$ to discrete-time $x[n]$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

  - Sampling period T
  - Sampling frequency $f_s = 1/T$

    $\Omega_s = 2\pi / T$

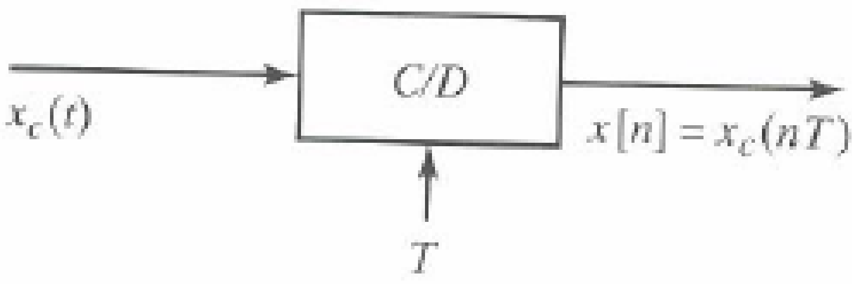


Figure 4.1 Block diagram representation of an ideal continuous-to-discrete-time (C/D) converter.

Two stages

In practice?

- Mathematically
  - Impulse train modulator
  - Conversion of the impulse train to a sequence

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) = x_c(t)s(t)$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

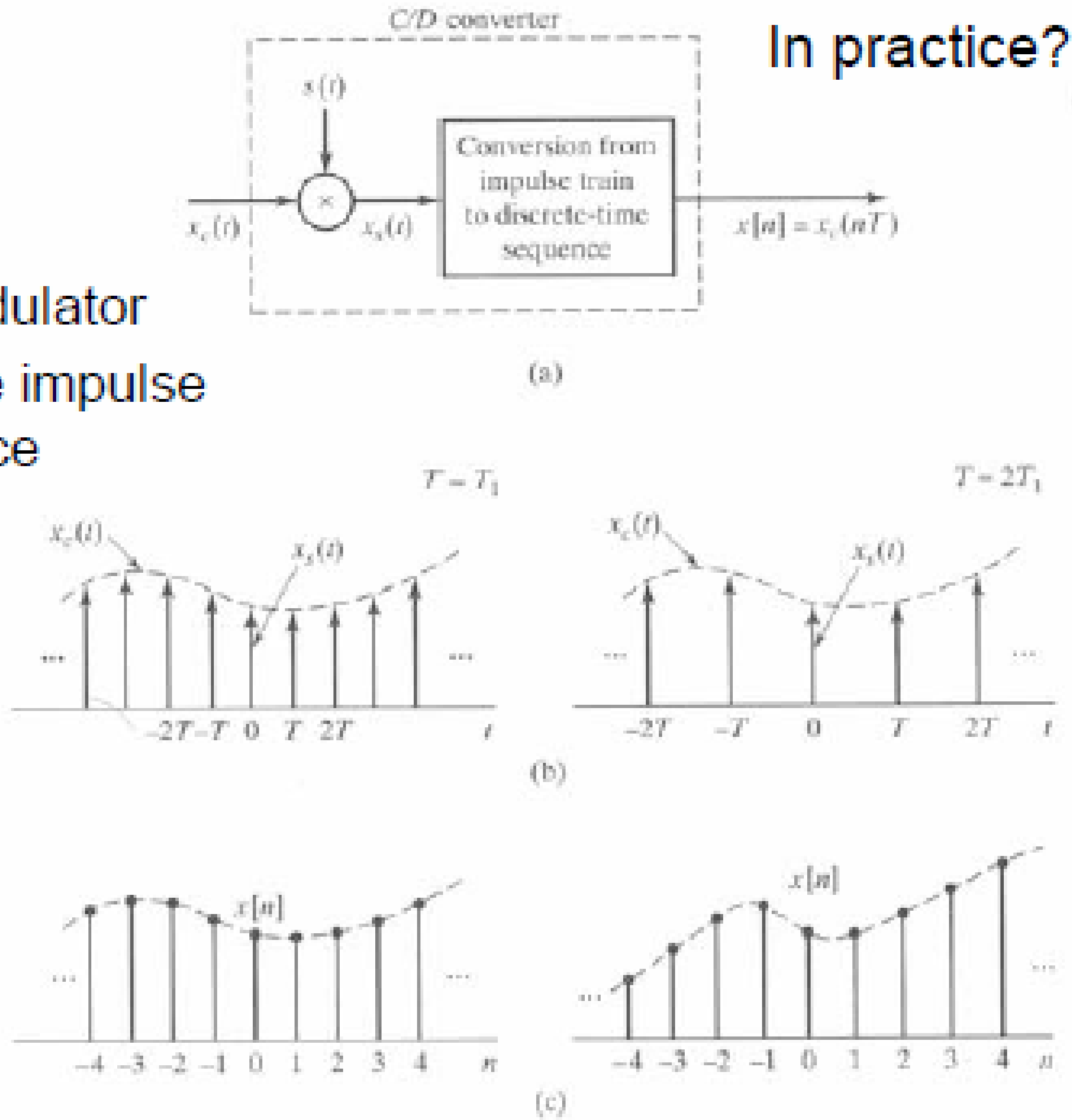


Figure 4.2 Sampling with a periodic impulse train followed by conversion to

Periodic sampling

- Tow-stage representation
  - Strictly a mathematical representation that is convenient for gaining insight into sampling in both the time and frequency domains.
  - Physical implementation is different.
  - $x_s(t)$ a continuous-time signal, an impulse train, zero except at nT
  - $x[n]$ a discrete-time sequence, time normalization, no explicit information about sampling rate
- Many-to-many → in general not invertible

Frequency-domain representation

- From $x_c(t)$ to $x_s(t)$

The Fourier transform of a periodic impulse train is a periodic impulse train.

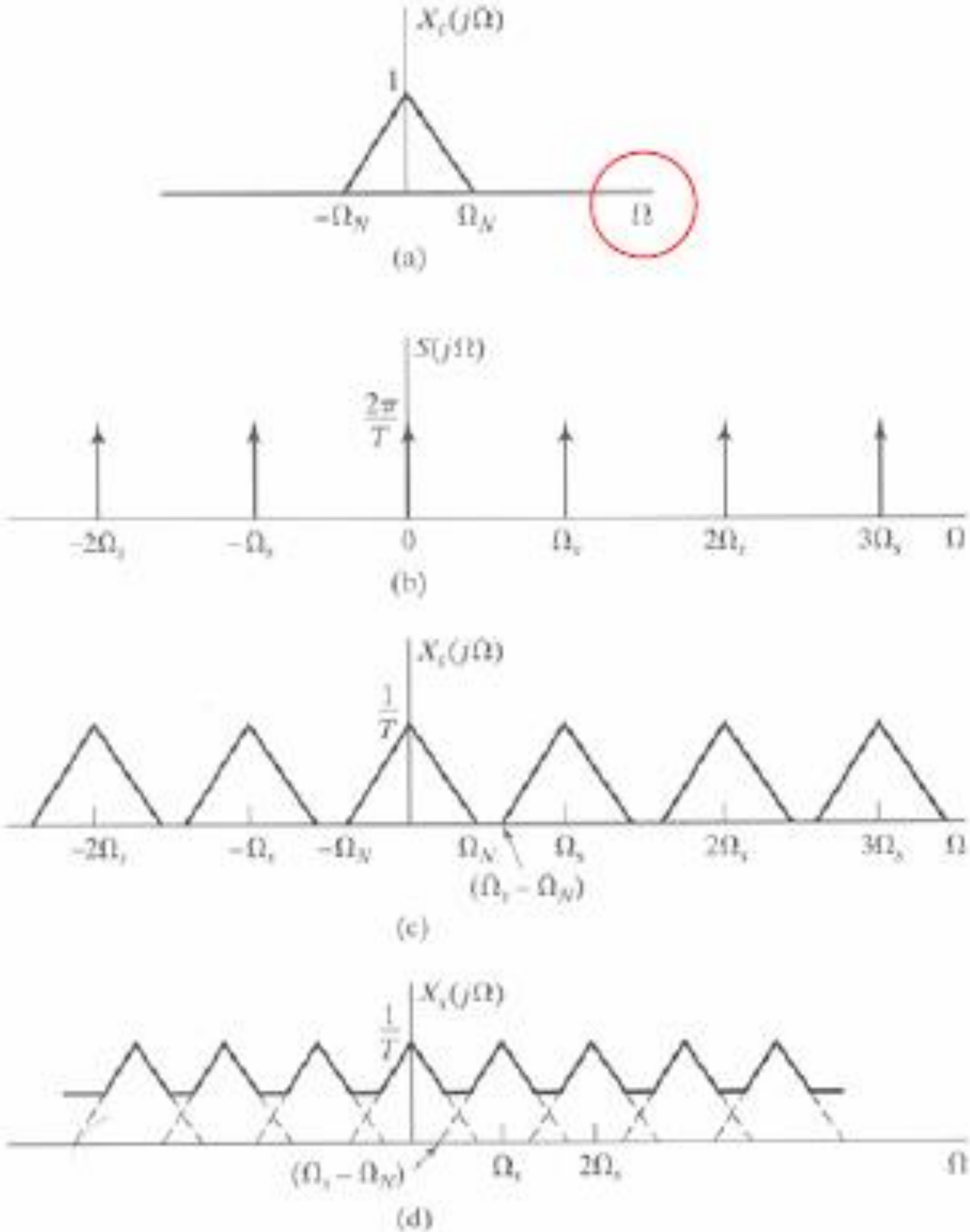
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \leftrightarrow \quad \underset{?}{S(j\Omega)} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$$

$$x_s(t) = x_c(t)s(t) \quad \leftrightarrow \quad X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega)$$

$$= x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \qquad = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega - k\Omega_s))$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

- The Fourier transform of $x_s(t)$ consists of periodic repetition of the Fourier transform of $x_c(t)$.



$$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega - k\Omega_s))$$

$$\Omega_s - \Omega_N > \Omega_N \text{ or } \Omega_s > 2\Omega_N$$

Figure 4.3 Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$.

Recovery

$$X_r(j\Omega) = H_r(j\Omega)X_s(j\Omega)$$

Ideal lowpass filter with gain T and cutoff frequency Ω_c

$$\Omega_N < \Omega_c < (\Omega_s - \Omega_N)$$

$$X_r(j\Omega) = X_c(j\Omega)$$

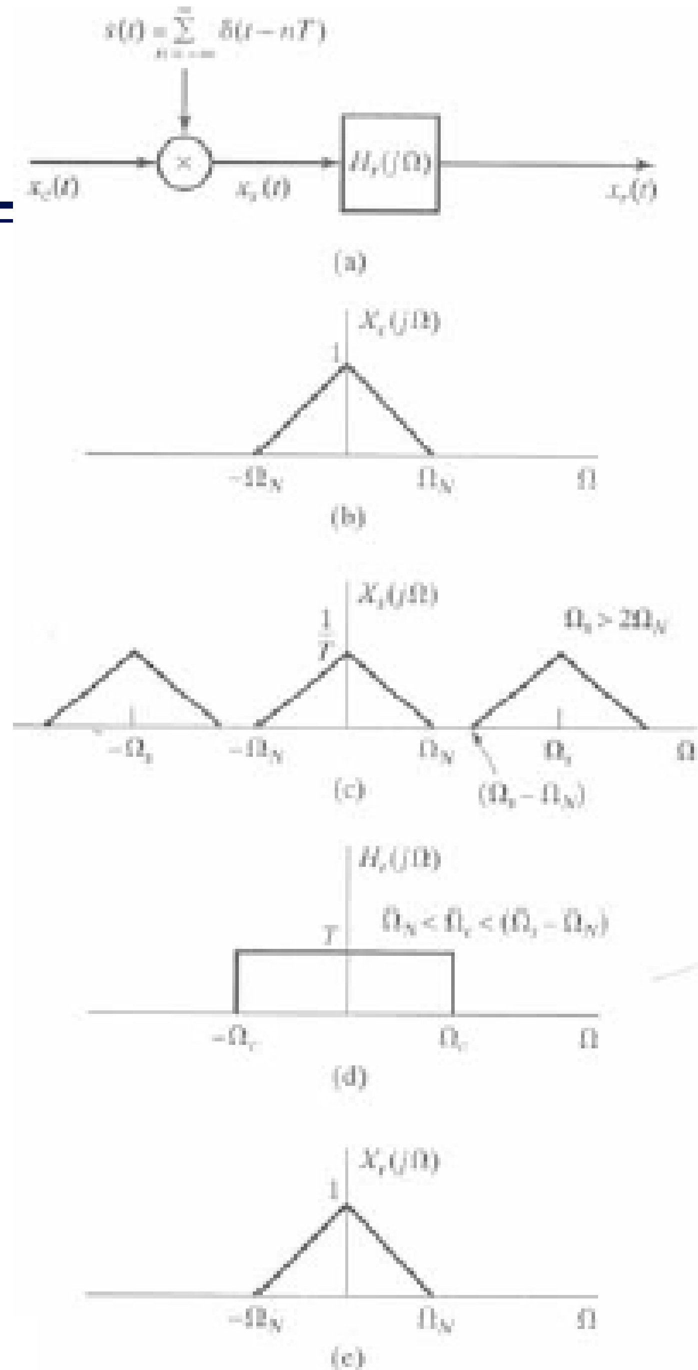
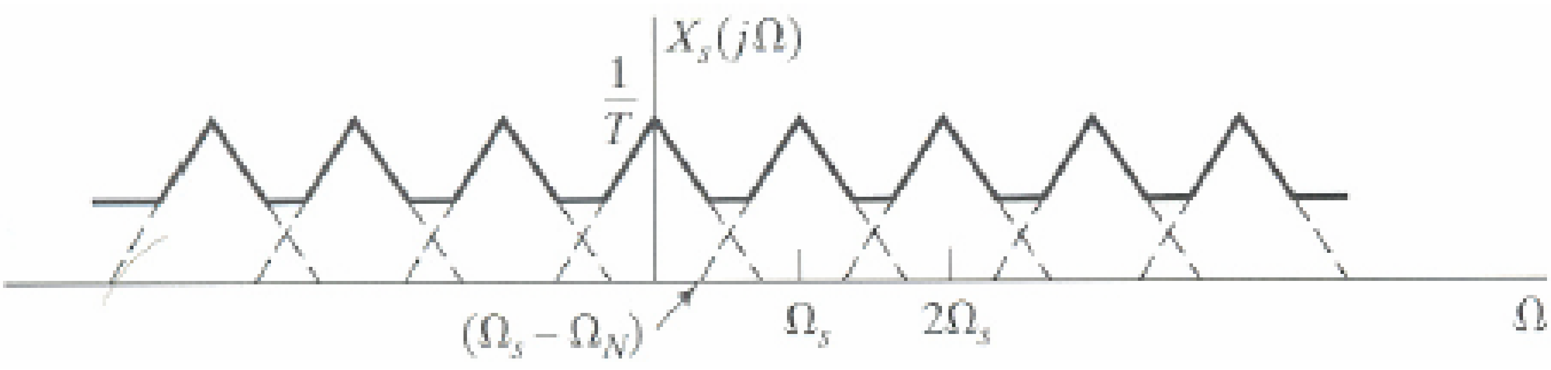


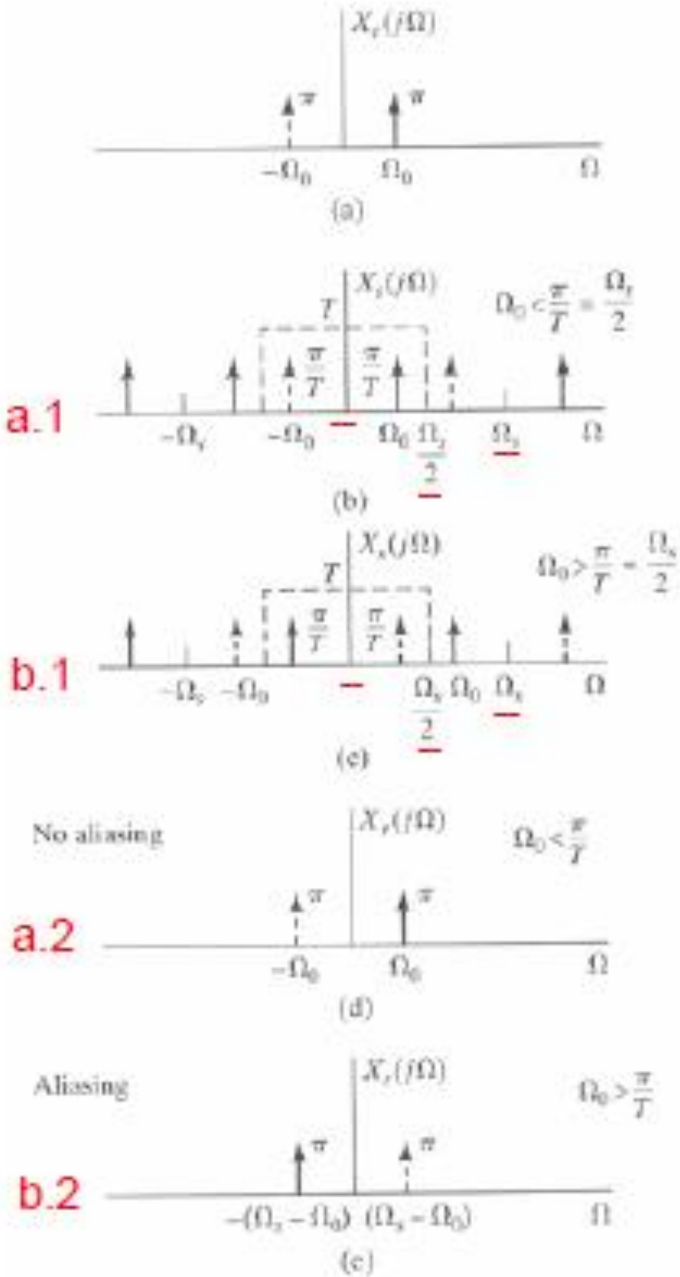
Figure 4.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.

Aliasing distortion

- Due to the overlap among the copies of $X_c(j\Omega)$, due to $\Omega_s \le 2\Omega_N$
- $X_c(j\Omega)$ not recoverable by lowpass filtering



Aliasing – an example



$$x_c(t) = \cos\Omega_0 t$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$x_r(t) = \cos\Omega_0 t$$

$$x_r(t) = \cos(\Omega_s - \Omega_0)t$$

Figure 4.5 The effect of aliasing in the sampling of a cosine signal.

Nyquist sampling theorem

Given bandlimited signal $x_c(t)$ with

$$X_c(j\Omega) = 0, \quad \text{for } |\Omega| \geq \Omega_N$$

Then $x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT), \qquad -\infty < n < \infty$$

If

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$$

Ω_N is called Nyquist frequency

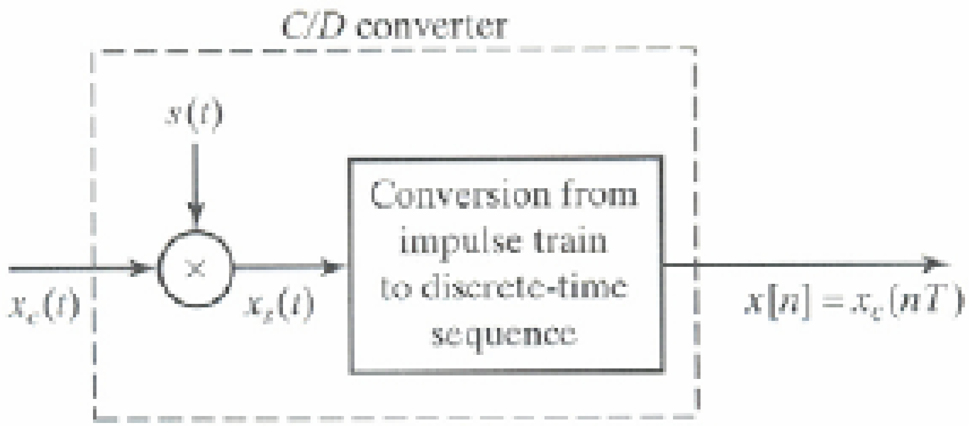
$2\Omega_N$ is called Nyquist rate

Fourier transform of x[n]

From $x_s(t)$ to $x[n]$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$



From $X_s(j\Omega)$ to $X(e^{j\omega})$

By taking continuous-time Fourier transform of $x_s(t)$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn} \qquad \left(X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \right)$$

By taking discrete-time Fourier transform of x[n]

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T})$$

Fourier transform of x[n]

$$X_s(j\Omega) = X(e^{j\omega})\,|_{\omega=\Omega T} = X(e^{j\Omega T})$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)$$

$X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$ with $\omega = \Omega T$

$x_s(t)$ retains a spacing between samples equal to the sampling period T while $x[n]$ always has unity space.

Sampling and reconstruction of Sin Signal

$$x_c(t) = \cos(4000\pi t) \rightarrow \Omega_0 = 4000\pi$$

$$T = 1/6000 \rightarrow \Omega_s = 2\pi/T = 12000\pi \quad \therefore \text{ no aliasing}$$

$$x[n] = x_c(nT) = \cos(4000\pi nT) = \cos((2\pi/3)n) = \cos(\omega_0 n)$$

$$x_c(t) \leftrightarrow X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi) \qquad X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T} = X_s(j\omega/T) \text{ with normalized frequency } \omega = \Omega T$$

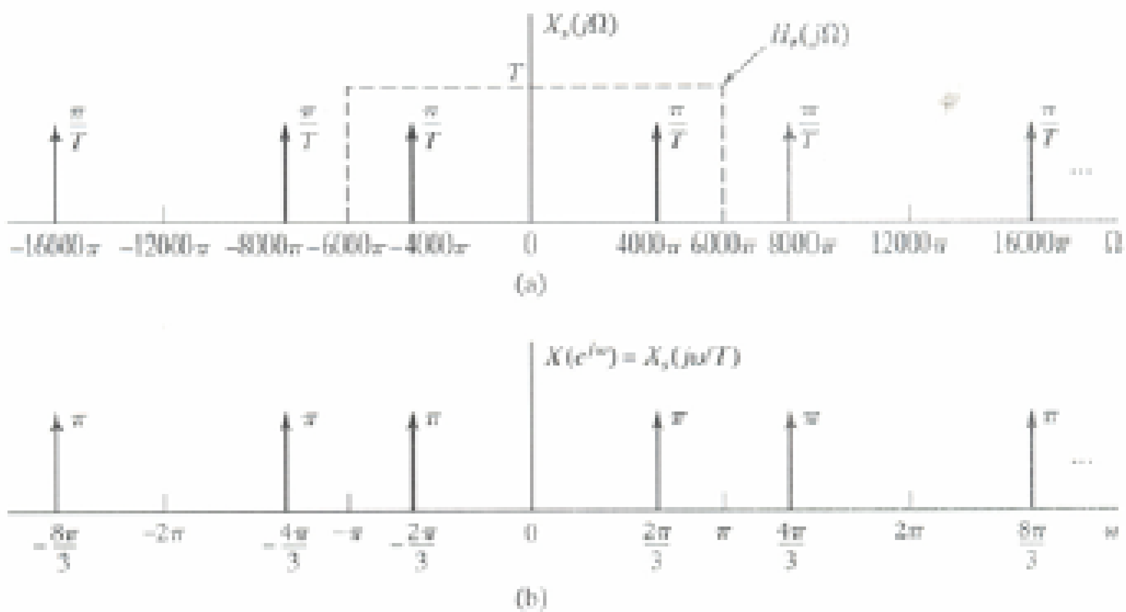


Figure 4.6 Continuous-time (a) and discrete-time (b) Fourier transforms for sampled cosine signal with frequency $\Omega_0 = 4000\pi$ and sampling period $T = 1/6000$.

How about

$$x_c(t) = \cos(16000\pi t)$$

Requirement for reconstruction

- On the basis of the sampling theorem, samples represent the signal exactly when:
  - Bandlimited signal
  - Enough sampling frequency
  - + knowledge of the sampling period → recover the signal

Reconstruction steps

(1) Given x[n] and T, the impulse train is

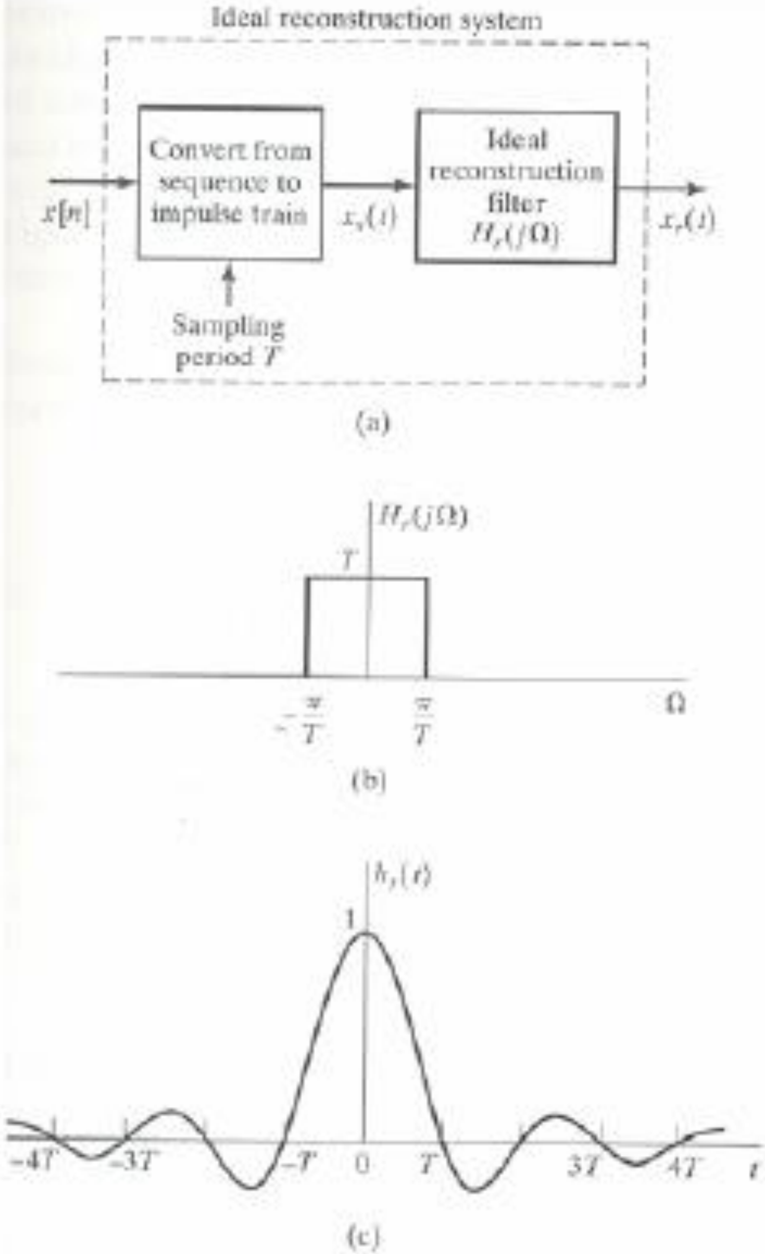
$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

i.e. the nth sample is associated with the impulse at t=nT.

(2) The impulse train is filtered by an ideal lowpass CT filter with impulse response $h_r(t) \leftrightarrow H_r(j\Omega)$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n)h_r(t-nT)$$

$$X_r(j\Omega) = H_r(j\Omega)X(e^{j\Omega T})$$

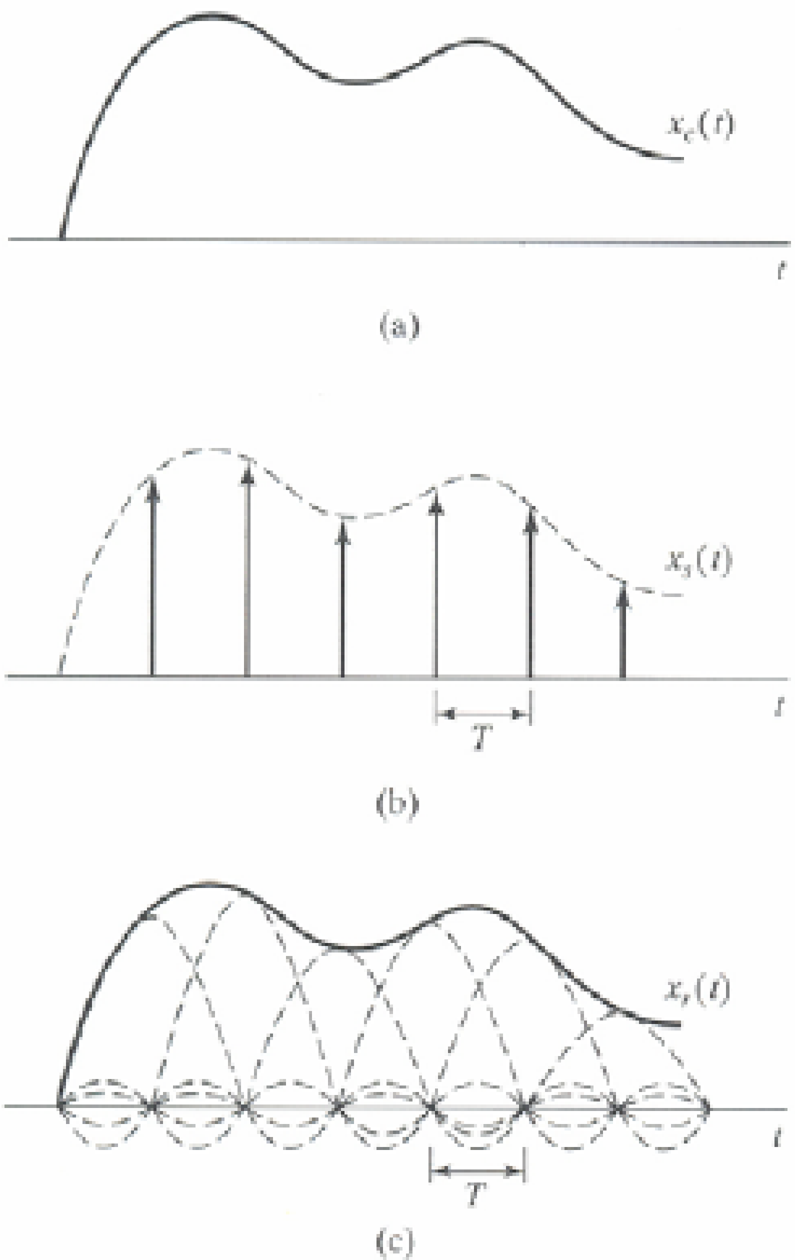


Commonly choose cutoff frequncy as

$$\Omega_c = \Omega_s / 2 = \pi / T$$

$$h_r(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

Figure 4.8 (a) Block diagram of an ideal bandlimited signal reconstruction system. (b) Frequency response of an ideal reconstruction filter. (c) Impulse response of an ideal reconstruction filter.



CT signal

Modulated impulse train

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

Figure 4.9 Ideal bandlimited interpolation.

Ideal discrete-to-continuous-time converter

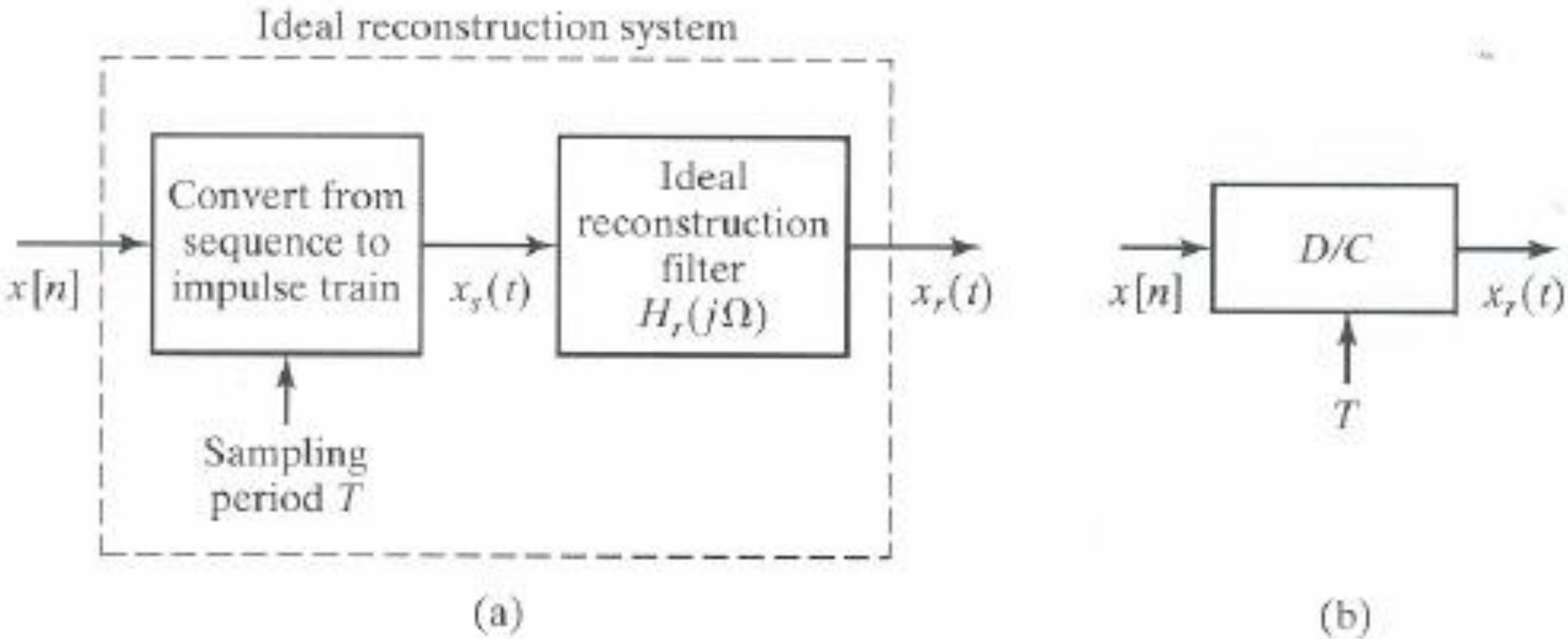


Figure 4.10 (a) Ideal bandlimited signal reconstruction. (b) Equivalent representation as an ideal D/C converter.

Applications

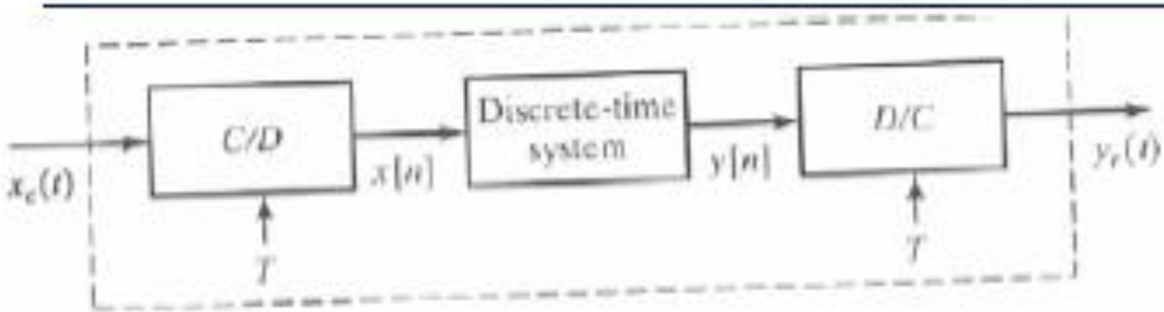


Figure 4.11 Discrete-time processing of continuous-time signals.

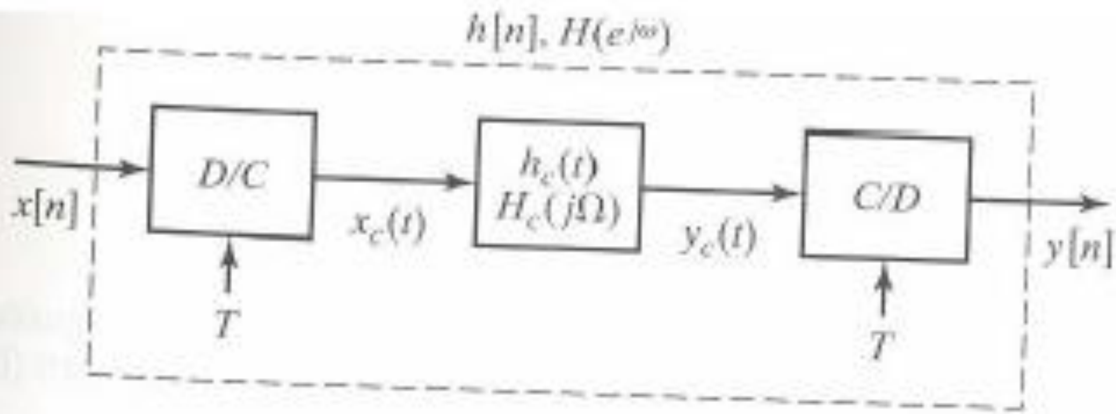


Figure 4.16 Continuous-time processing of discrete-time signals.

Decimation and Interpolation

- See lecture notes and textbook