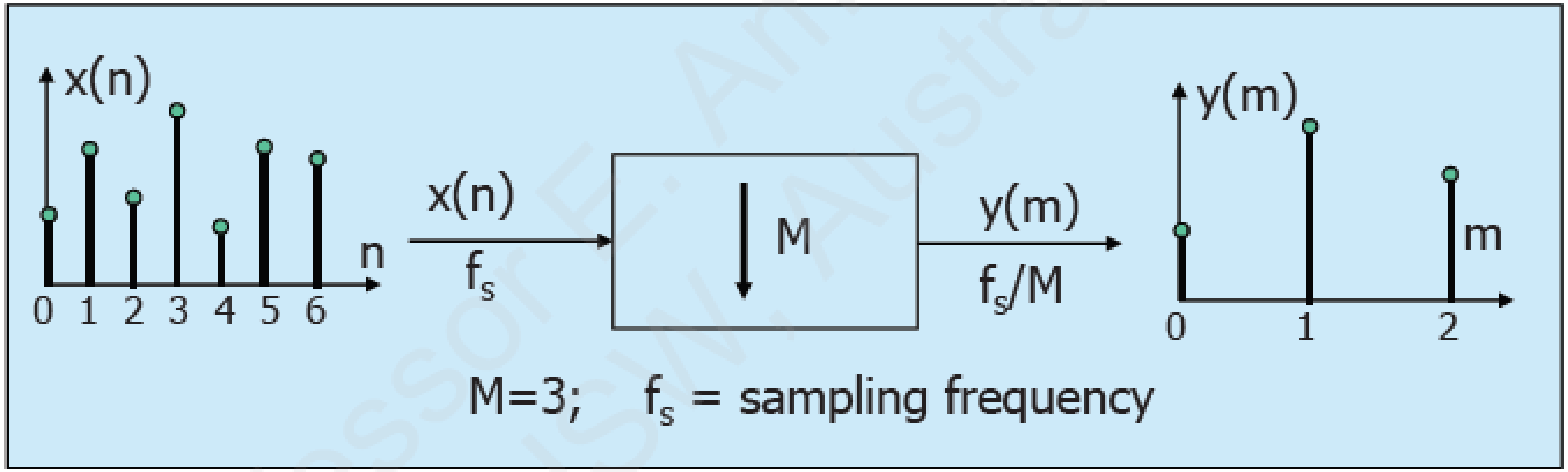
Multirate Digital Signal Processing

Chapter 4…Continue

Decimation: A sampling rate decreaser is shown below. We shall confine our attention to a decrease by an integer factor M (e.g. M =3)



The output signal y(m) is obtained by taking every Mth sample of the input signal. If M=3, we should just take every third sample of x(n) to form the desired signal y(m).

Example: $x[n] = \{1, 2, 4, 3, 5, -6, -8, 2, -3, 2\}$

↓ Down sample by 2

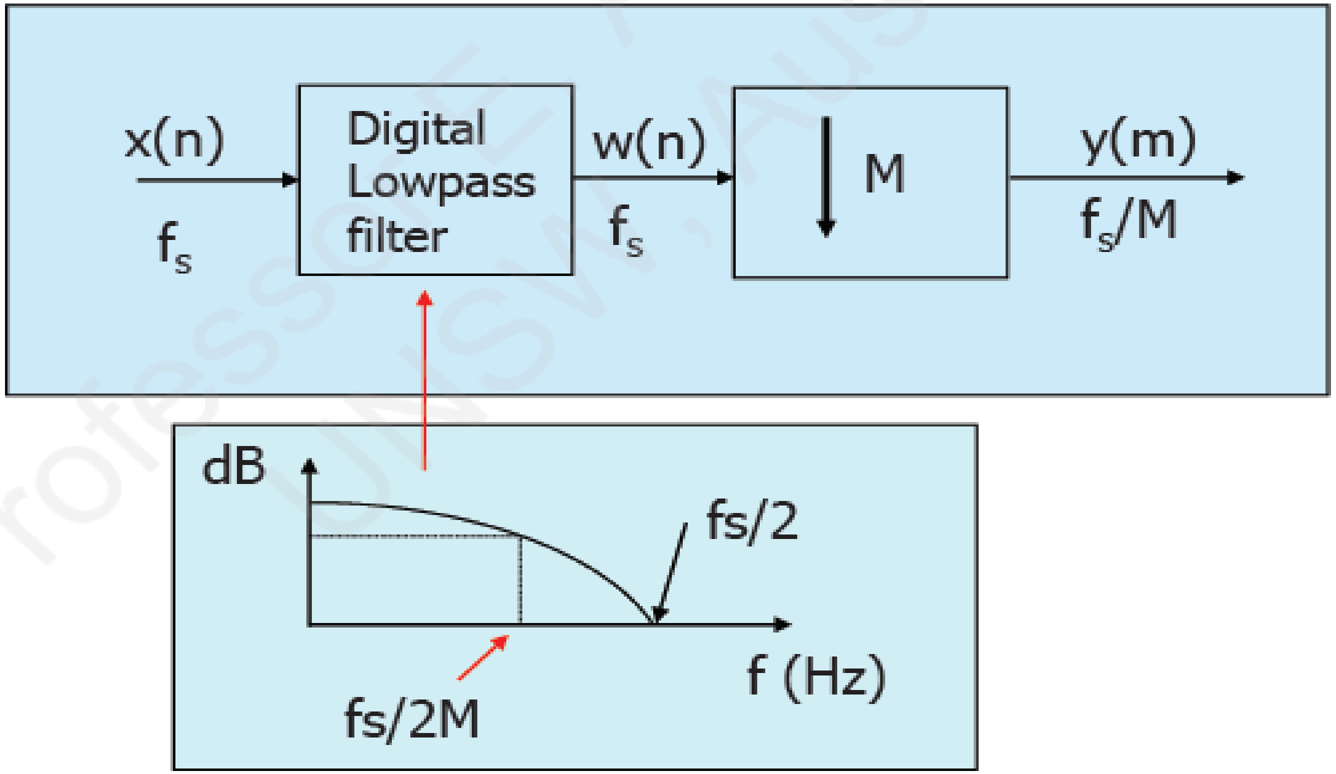
$y[m] = \{1,4,5,-8,-3\}$

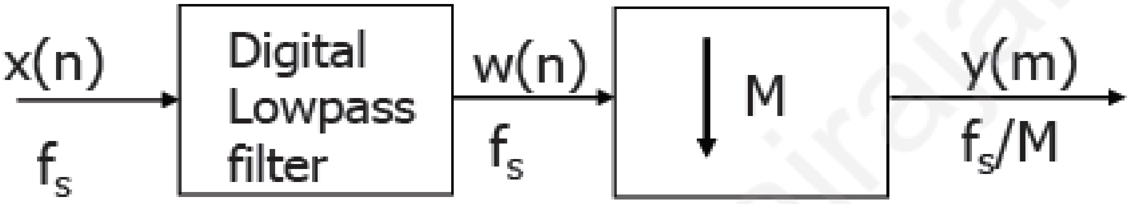
The output signal $y[n]$ is obtained by taking every M^{th} sample of the input signal. If $M = 3$, we should just take every third sample of $x[n]$ to form the desired signal $y[m]$.

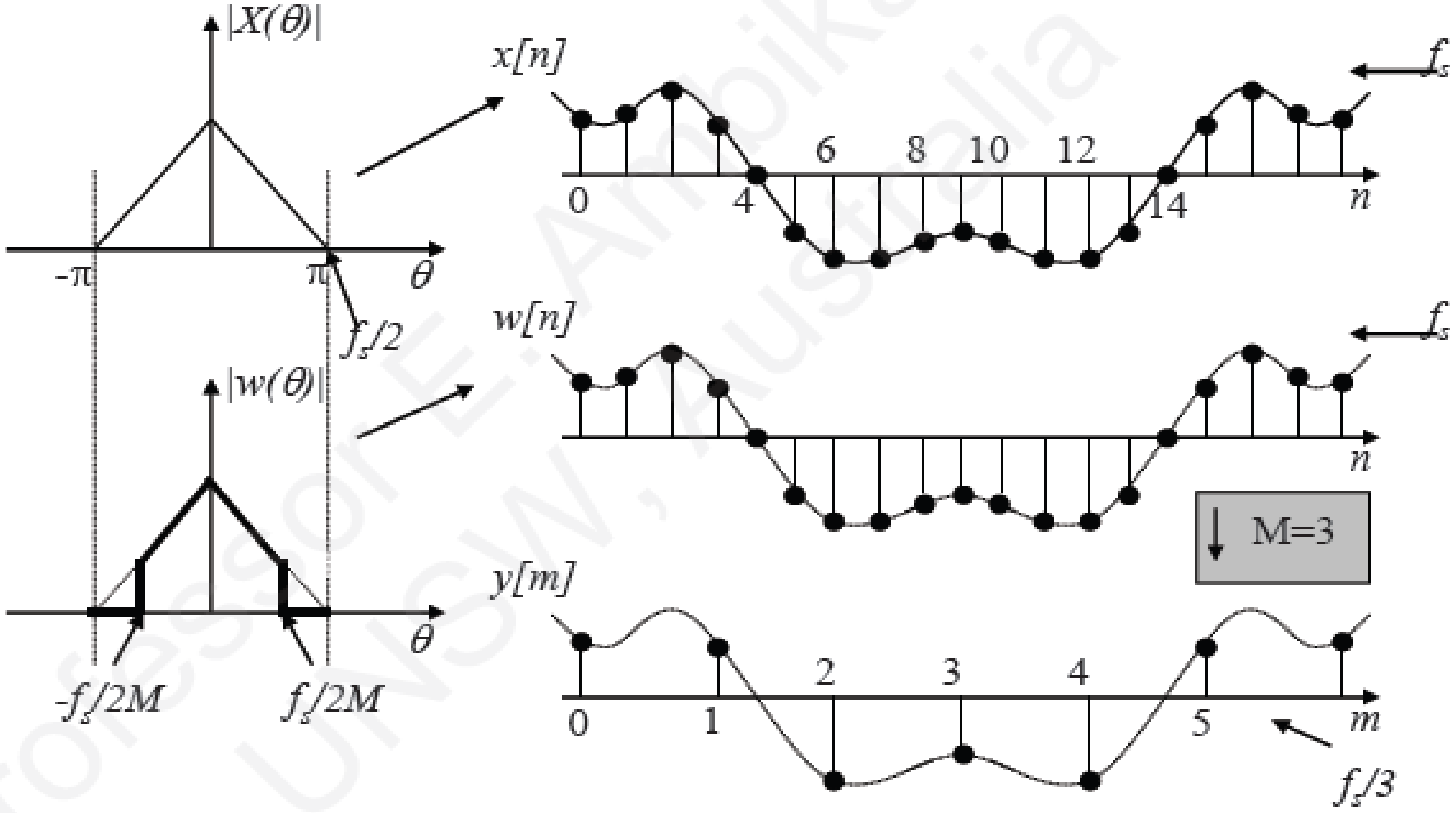
Obviously, it only makes sense to reduce the sampling rate if the information constant of the signal we wish to preserve is band limited to $\frac{f_s}{6}$; half the desired sampling rate since the spectral components above this frequency will be aliased into frequencies below $\frac{f_s}{6}$ according to the sampling rule.

The diagram below shows a block representations of a times M decimator.

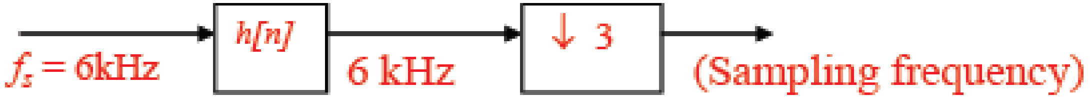
The signal x(n) is first passed through a lowpass filter that attenuates the band from {fs/2}/M to fs/2 to prevent aliasing.







Example:



$$\frac{f_s}{3} = 2 \; kHz$$

Note

- Such a processing requirement may arise, for example when a speech signal is over sampled at $f_s = 32$ kHz.

- Since we are interested only in a band of 0-4 kHz, sampling rate can be reduced to 8 kHz,

- so the first step in the decimation process has to be the digital filtering of the signal $x[n]$ is band limited to $\frac{f_s}{8}$

Should we use IIR or FIR for the lowpass filtering required?

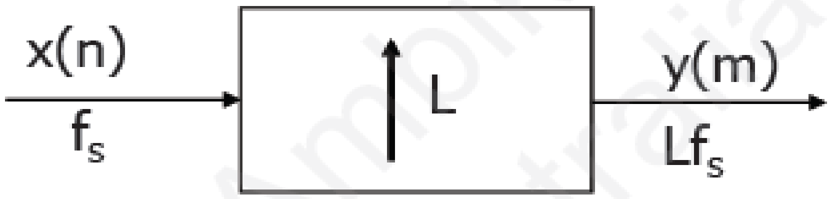
Using an IIR filter in this case has an obvious shortcoming. We cannot take advantage of the fact that we only have to compute every Nth output, since previous outputs are required to compute the Mth output. Thus no saving is realised.

On the other hand, using an FIR filter, in this case implies that we can do our computations at the rate of fs/M. Thus , using an FIR filter in the decimation process will lead to a significantly lower computation rate.

Another advantage of using an FIR filter is the fact that we can easily design linear phase filters and this is desirable in many applications.

Interpolation

- The process of interpolation involves a sampling rate increase



- The sequence x(n) was derived by sampling x(t) at a sampling rate f_s and we want to obtain a sequence y(n) that approximates as closely as possible the sequence that would have been obtained had we sampled x(t) at the rate Lf_s.

- Interpolation involves inserting between any samples x(n) and x(n-1) and additional L-1 samples.

Interpolation Examples

$x[n]=\{1, 2, 4, 3, -5, 6, -7, 2, 4, 3\}$

$\uparrow 2$

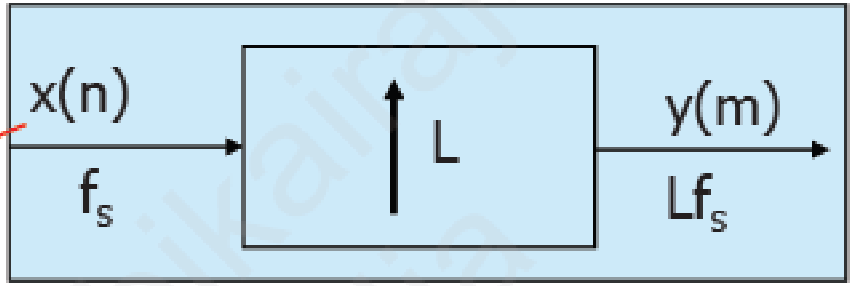
$y[m]=\{1,0,2,0,4,0,3,0,-5,0,6,0,-7,0,2,0,4,0,3,0\}$

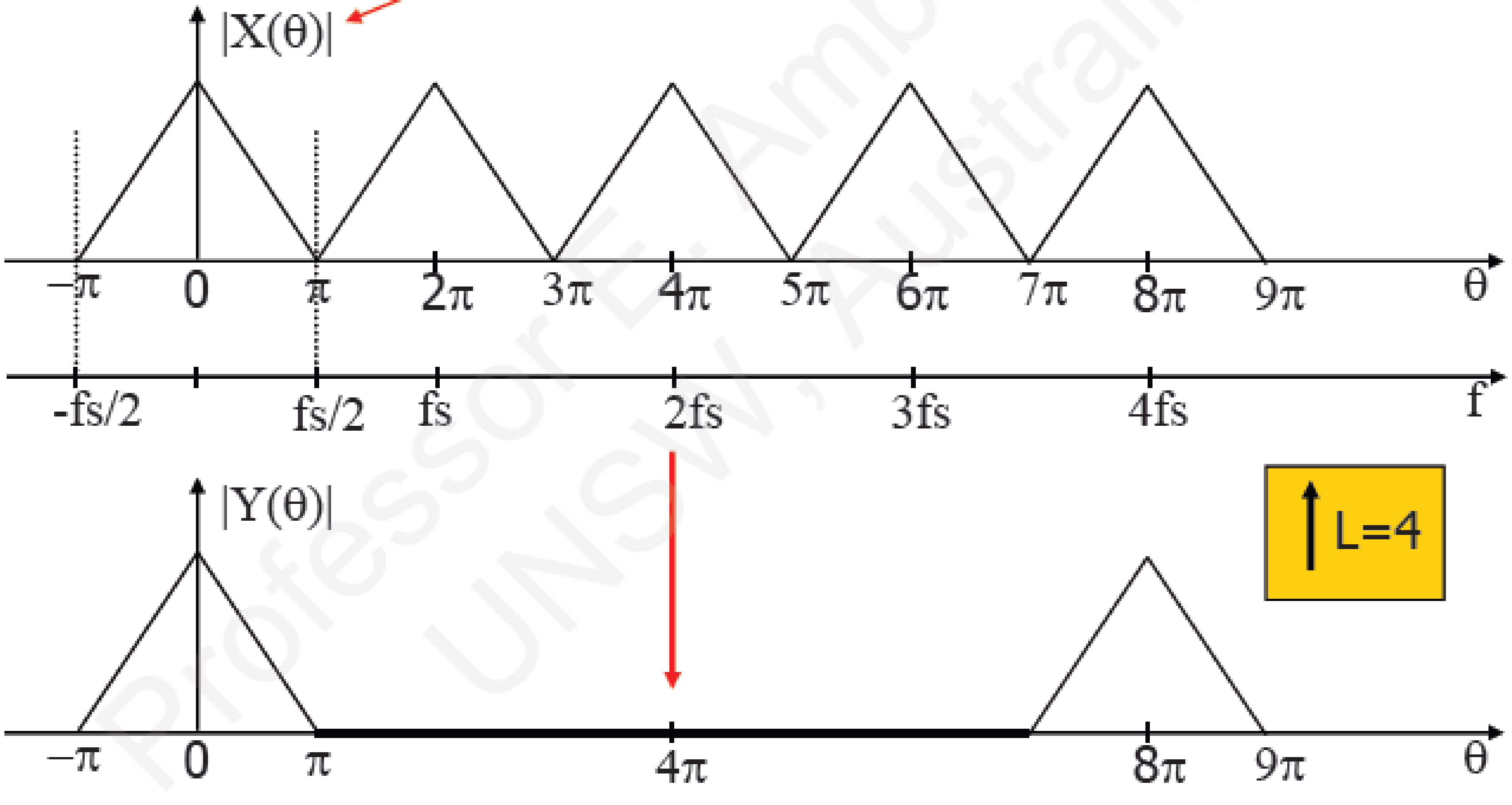
$x[n]=\{1, 2, 4, 3, -5, 6, -7, 2, 4, 3\}$

$\uparrow 3$

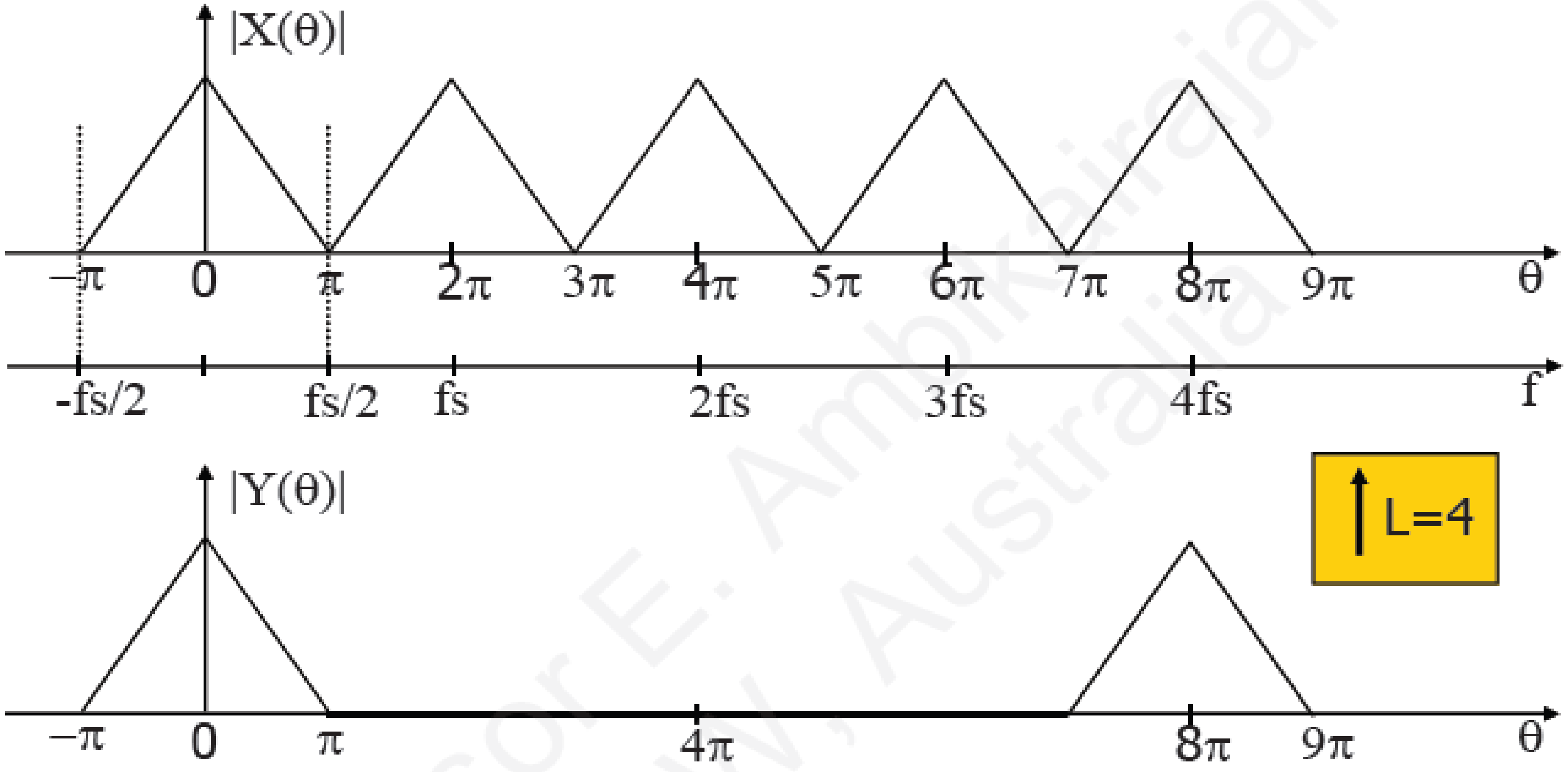
$y[m]=\{1,0,0,2,0,0,4,0,0,3,0,0,-5,0,0,6,0,0,-7,0,0,2,0,0,4,0,0,3,0,0\}$

Interpolation Example



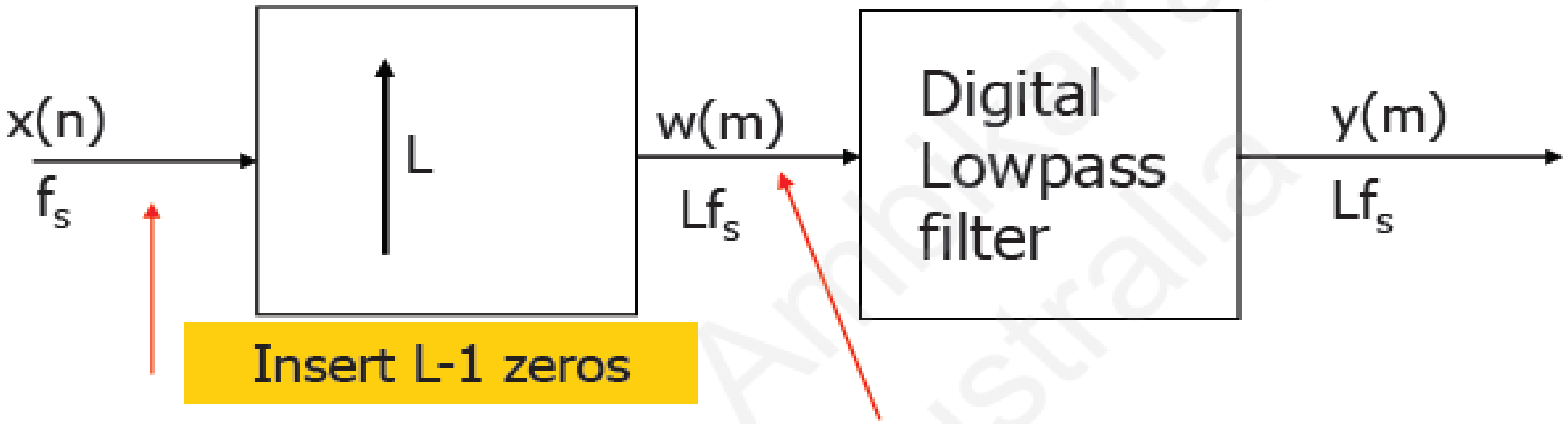


Sampling frequency of y(m) = $4f_s$; Signals must be band limited to $2f_s$



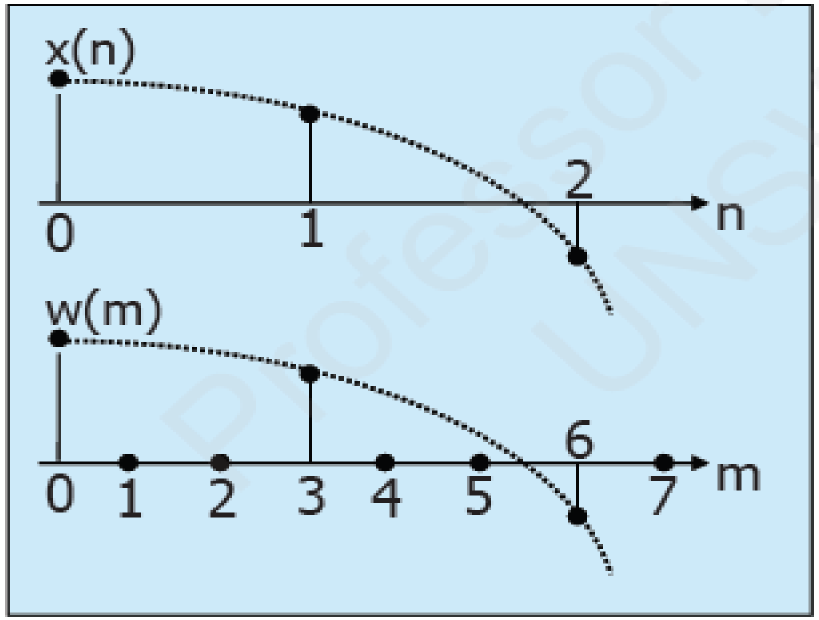
We observe that to go from X(θ) to Y(θ), we have to pass x(n) through a lowpass digitalfilter designed at the Lfs sampling rate that attenuates sufficiently any frequency components above fs/2.

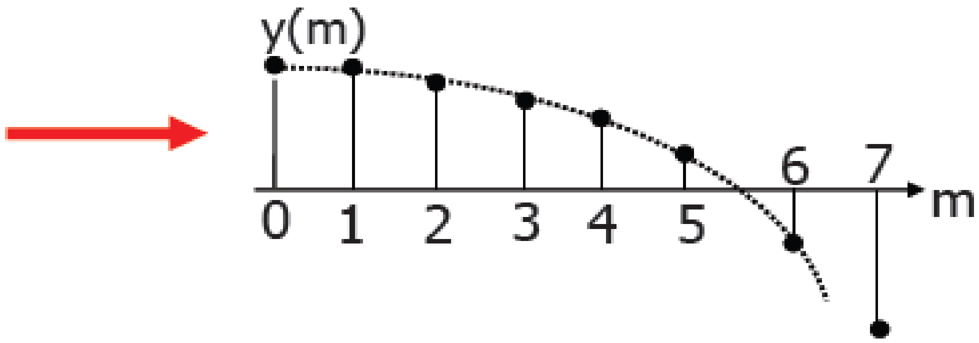
Interpolator



Example: x(n)={1,0.9,-0.5}; Let L = 3: w(m) = {1,0,0, 0.9,0,0,-0.5,0,0}

The lowpass filter joins all the samples of w(m) to produce a waveform as if x(n) had been sampled at Lfs





Interpolator ……

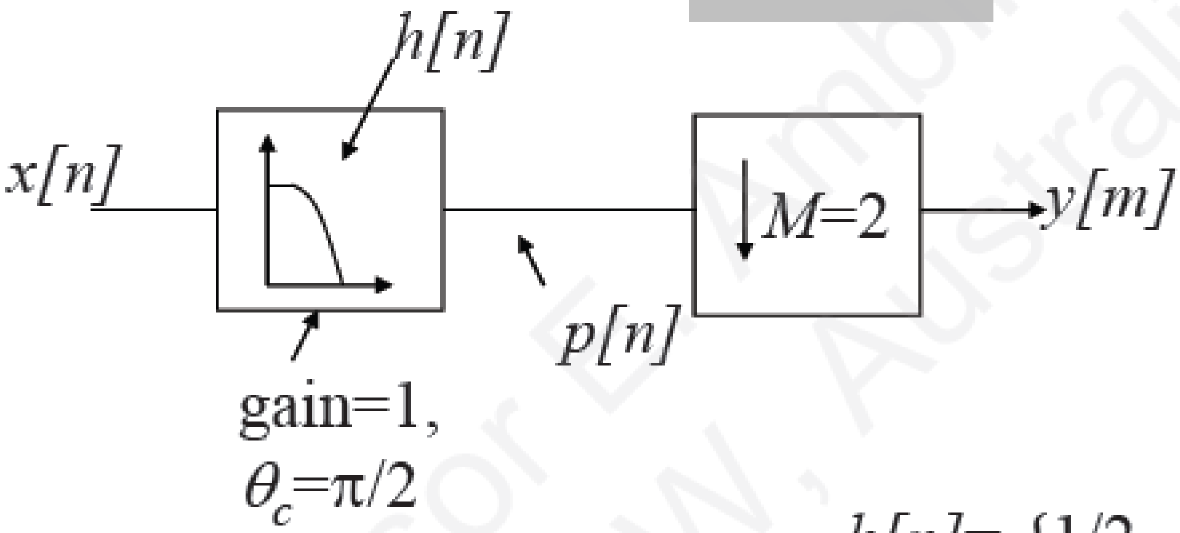
- We assume that behind each x(n) there are L-1 zero samples when we computing an output w(n)

- Note that for each sample of x(n) {see previous slide}, three output samples y(n) are obtained

- Obviously the same reasoning that led us to believe that FIR filters are preferable in the decimation process holds here also.

Example: Decimation of $x[n] = \{2, 6, 4, 2, 6, 8, 4, 2, 4, 4\}$

$\downarrow M{=}2$



$h[n] = \{1/2,\ 1/2\}$

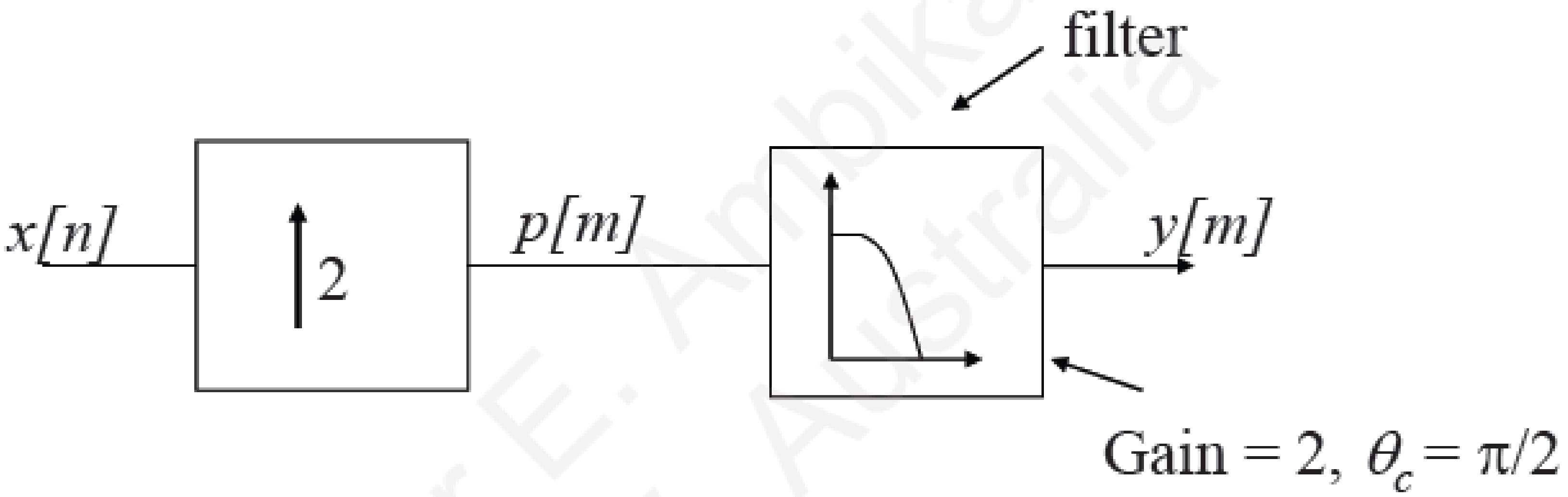
$h[0]=1/2,\ h[1]=1/2$

$$p[n] = x[n] * h[n] = \{4,5,3,4,7,6,3,3,4,2\}$$

$$\downarrow 2$$

$$y[n] = \{4,3,7,3,4\}$$

Example: Linear interpolation of $x[n] = \{1,3,5,3,7\}$

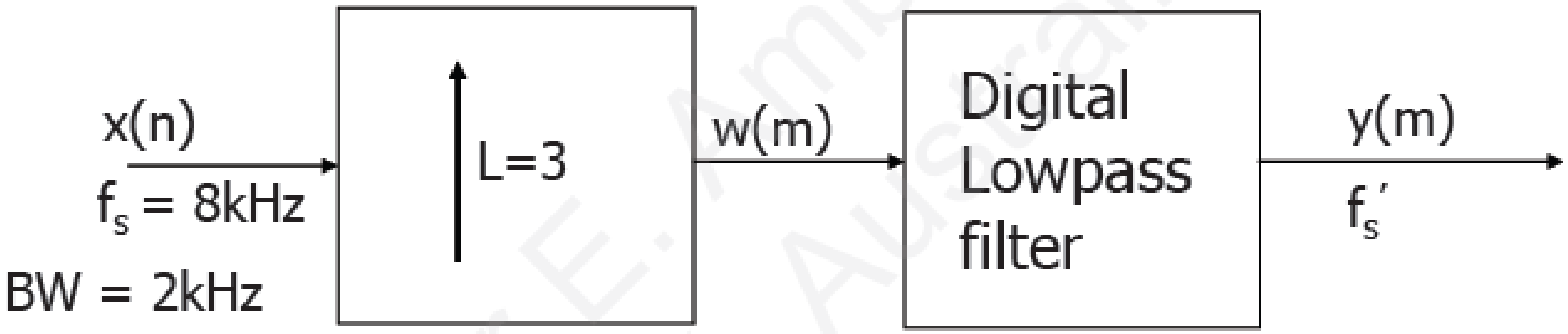


$x[n] = \{1,3,5,3,7\}$
$p[n] = \{1,0,3,0,5,0,3,0,7,0\}$ (insert zeros)

$$h[n] = \left\{\frac{1}{2}, 1, \frac{1}{2}\right\}$$

$y[n] = p[n] * h[n] = \{1,2,3,4,5,4,3,5,7,3.5\}$

Example



- What should be the cut-off frequency of the digital lowpass filter?

- What should be the value of f_s'?

Sampling Rate Conversion by Non-Integer Factors

- In some applications, the need often arises to change the sampling rate by a non-integer factor
- An example is in digital audio applications where it may be necessary to transfer from one storage system to another, where both systems employ different rates

Example

- An example is transferring data from the compact disk system at a rate of 44.1 kHz to a digital audio tape (DAT) at 48 kHz
- This can be achieved by increasing the data rate of the CD by a factor of 48/44.1, a non-integer
- In practice, such a non-integer factor is represented by a rational number, that is a ration of two integers say L and M
- The sampling frequency change is then achieved by first interpolating the data by L and then decimating by M

Note: It is necessary that the interpolation process preceeds decimation,otherwise the decimation process would remove some of the desired frequency components

CD →DAT

44.1 kHz

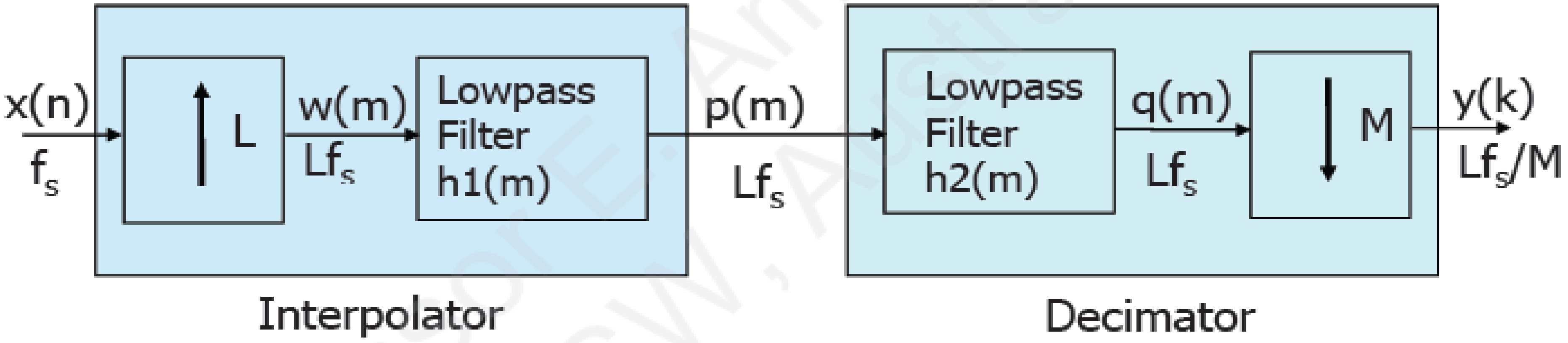
48 kHz

To change the sampling rate we require:

$$\frac{L}{M} = \frac{48000}{44100} = \frac{2^7.3.5^3}{2^2.3^2.5^2.7^2} = \frac{160}{147}$$

Therefore if we upsample by L=160 and then down sample by M=147, we achieve the desired sampling rate conversion.

Figure below shows that the sampling frequency change is achieved by first interpolating the data by L and then decimating by M.

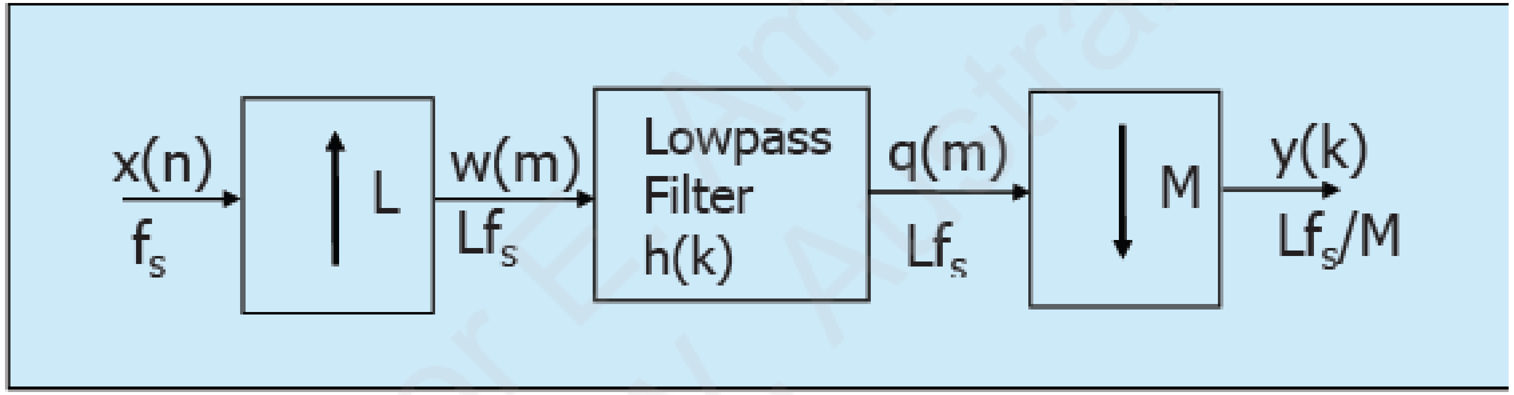


The two Digital Lowpas Filters, h1(m) and h2(m) can be combined into a single filter since they are in cascade and have a common sampling frequency

If $M > L$ the resulting operation is a decimation process by a non-integer

$M < L$ the resulting operation is an interpolation

Summary: Sampling Rate Conversion by Non-Integer Factors



The lowpass filter that we require is the one that has a cut-off frequency:

$$\theta_c = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$$

Example

Figure below shows sampling rate conversion by non-integer factors.

Calculate the values of L and M

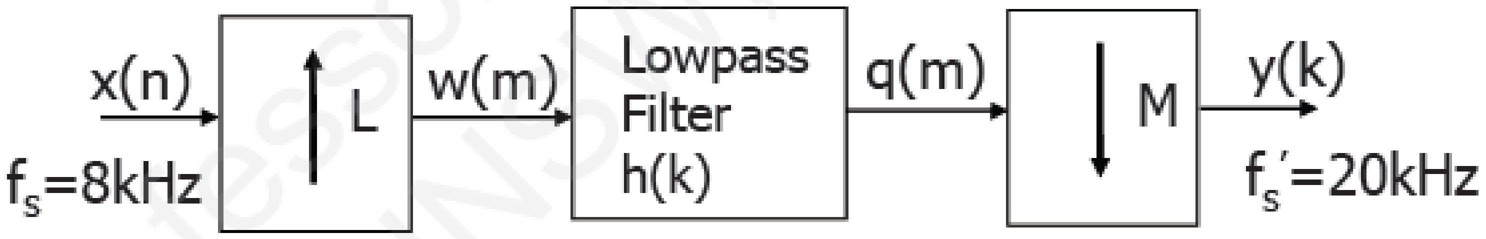
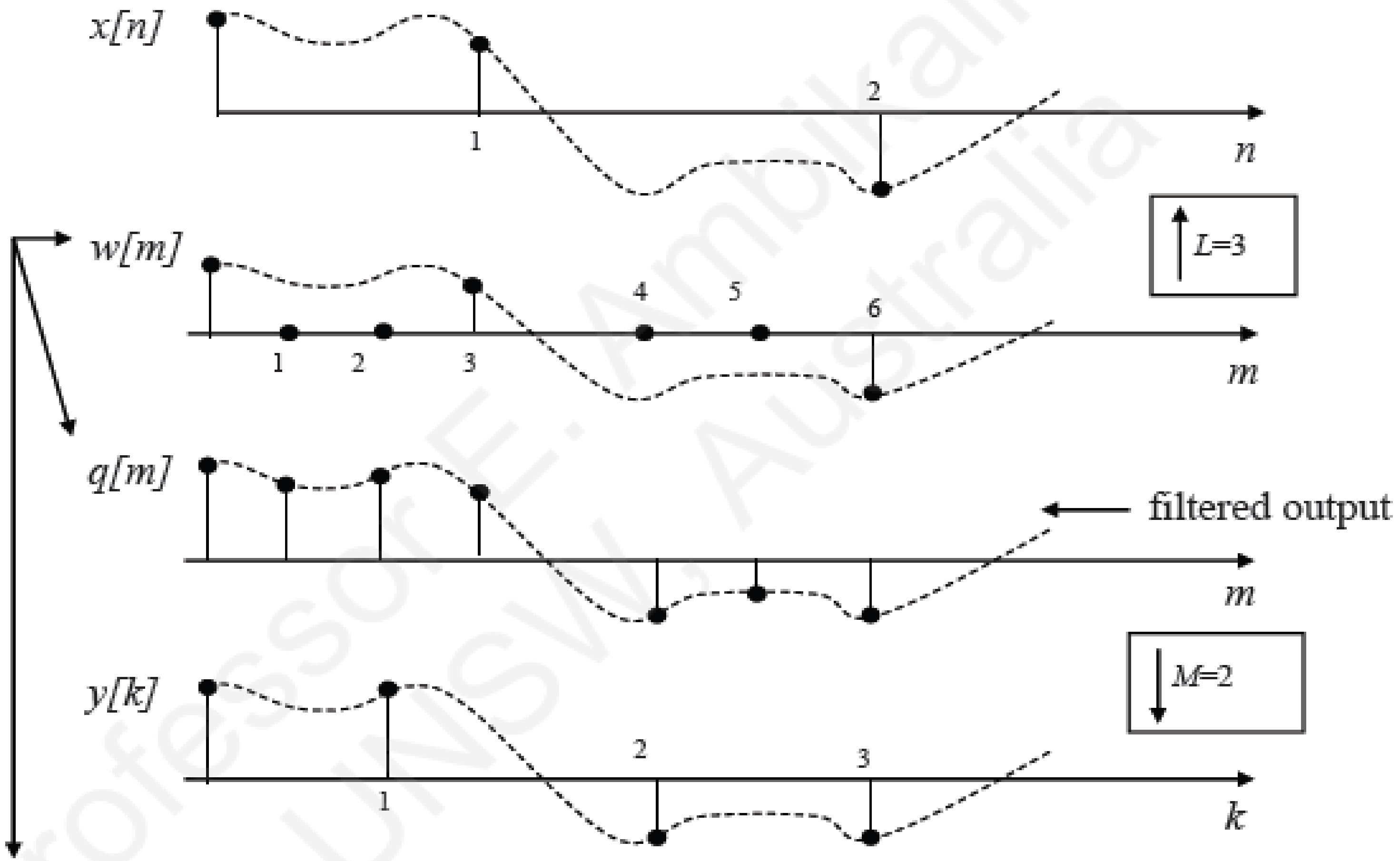
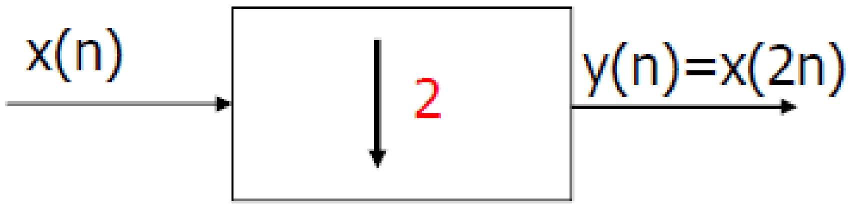


Illustration of Interpolation by a factor 3/2



The sample rate is first increased by 3, by inserting two zero-value samples for each sample of *x[n]* and low-pass filtered to yield *q[m]*. The filtered data is then reduced by a factor of 2 by retaining only one sample for every two samples of *q[m]*.

Decimation by 2



Aliasing term

$$Y(z) = \frac{1}{2}[X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})]$$

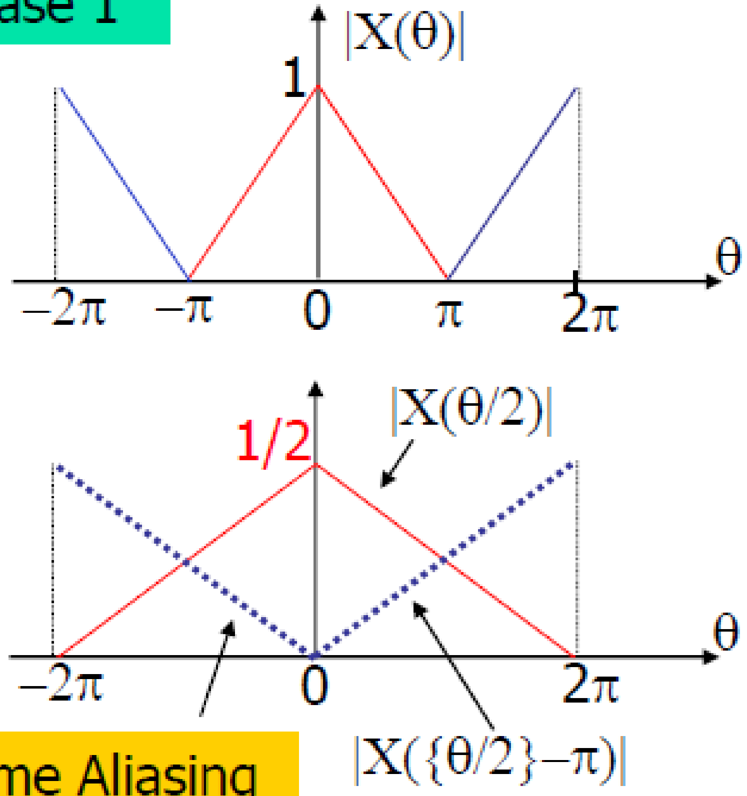
$$Y(\theta) = \frac{1}{2}[X(e^{\frac{j\theta}{2}}) + X(-e^{\frac{j\theta}{2}})]$$

$$= \frac{1}{2}[X(e^{\frac{j\theta}{2}}) + X(e^{j(\frac{\theta}{2}-\pi)})]$$

$$= \frac{1}{2}[X(\frac{\theta}{2}) + X(\frac{\theta}{2}-\pi)]$$

Aliasing term

Case 1

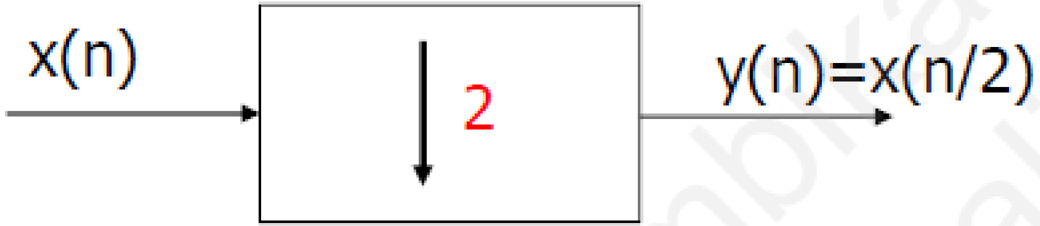


Extreme Aliasing

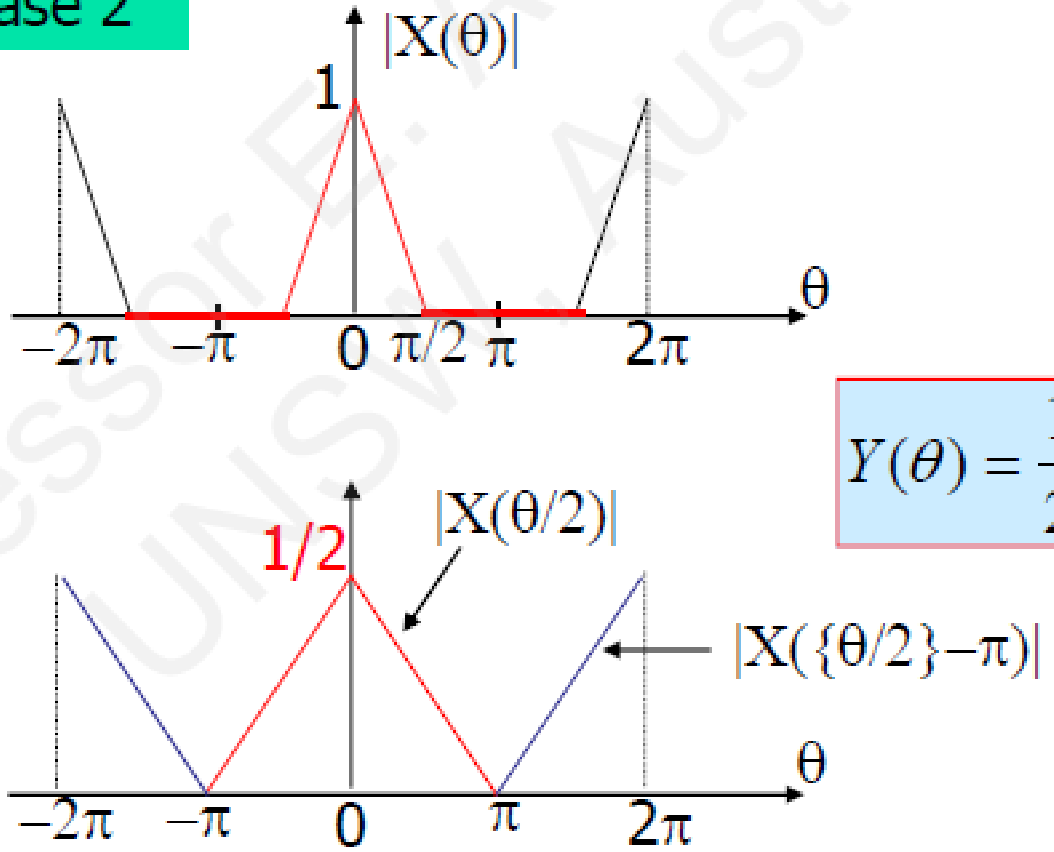
Stretch X(θ) by a factor 2 to obtain X(θ/2)

The spectrum is stretched by down sampling

Decimation by 2



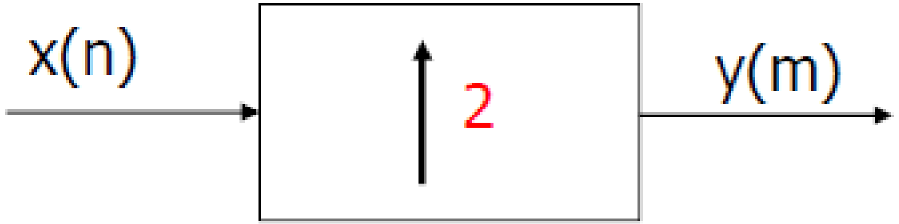
Case 2

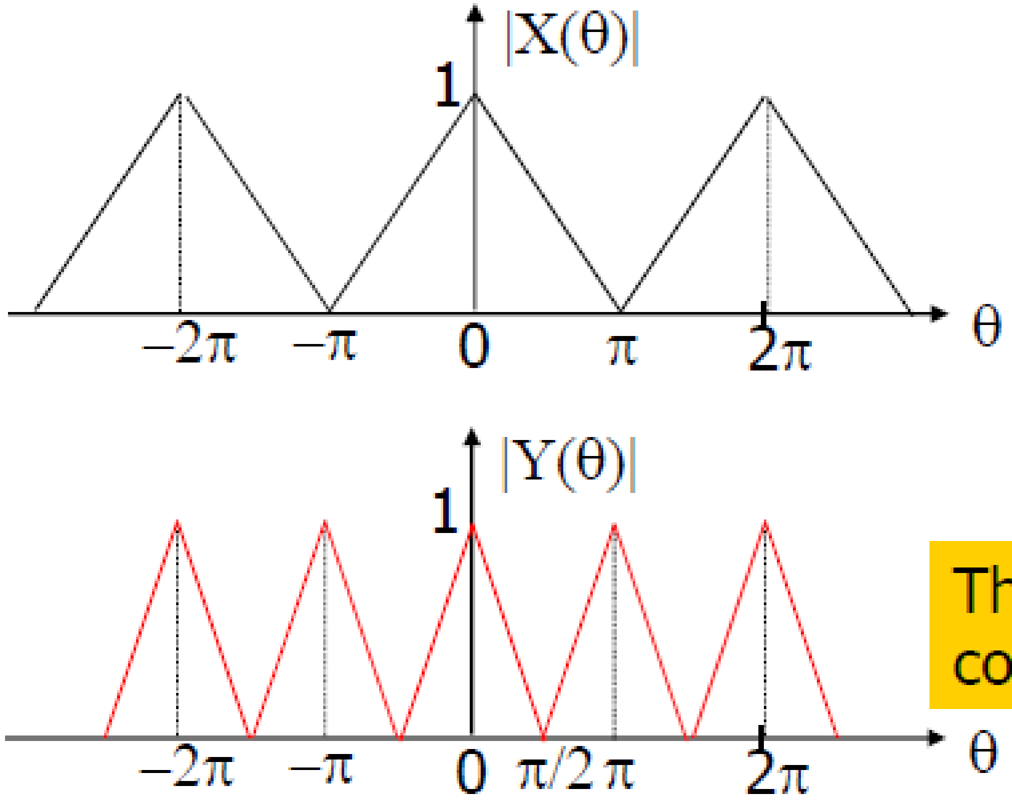


$$Y(\theta) = \frac{1}{2}\left[X\left(\frac{\theta}{2}\right) + X\left(\frac{\theta}{2} - \pi\right)\right]$$

No Aliasing

Interpolation by 2





$$Y(z) = [X(z^2)]$$

$$Y(\theta) = [X(e^{j2\theta})]$$

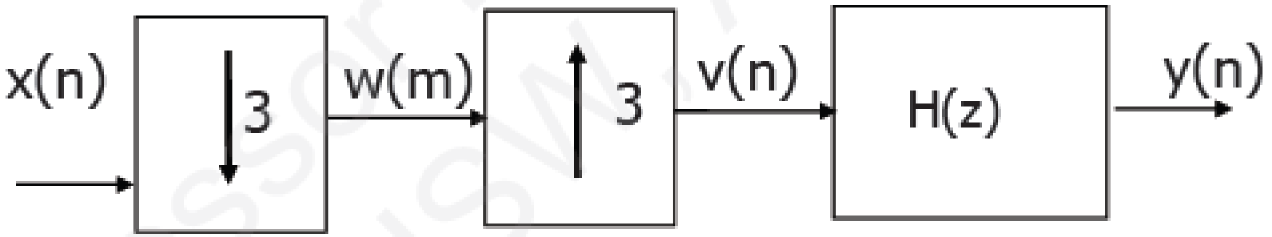
$$= [X(2\theta)]$$

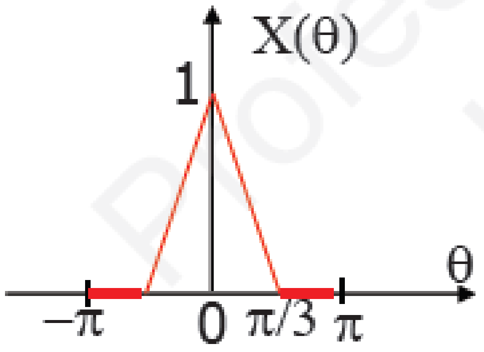
The upsampled spectrum has compressed images of X(θ)

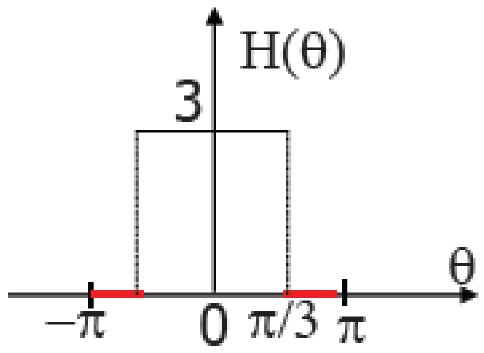
The spectrum is compresseed by upsampling

Example

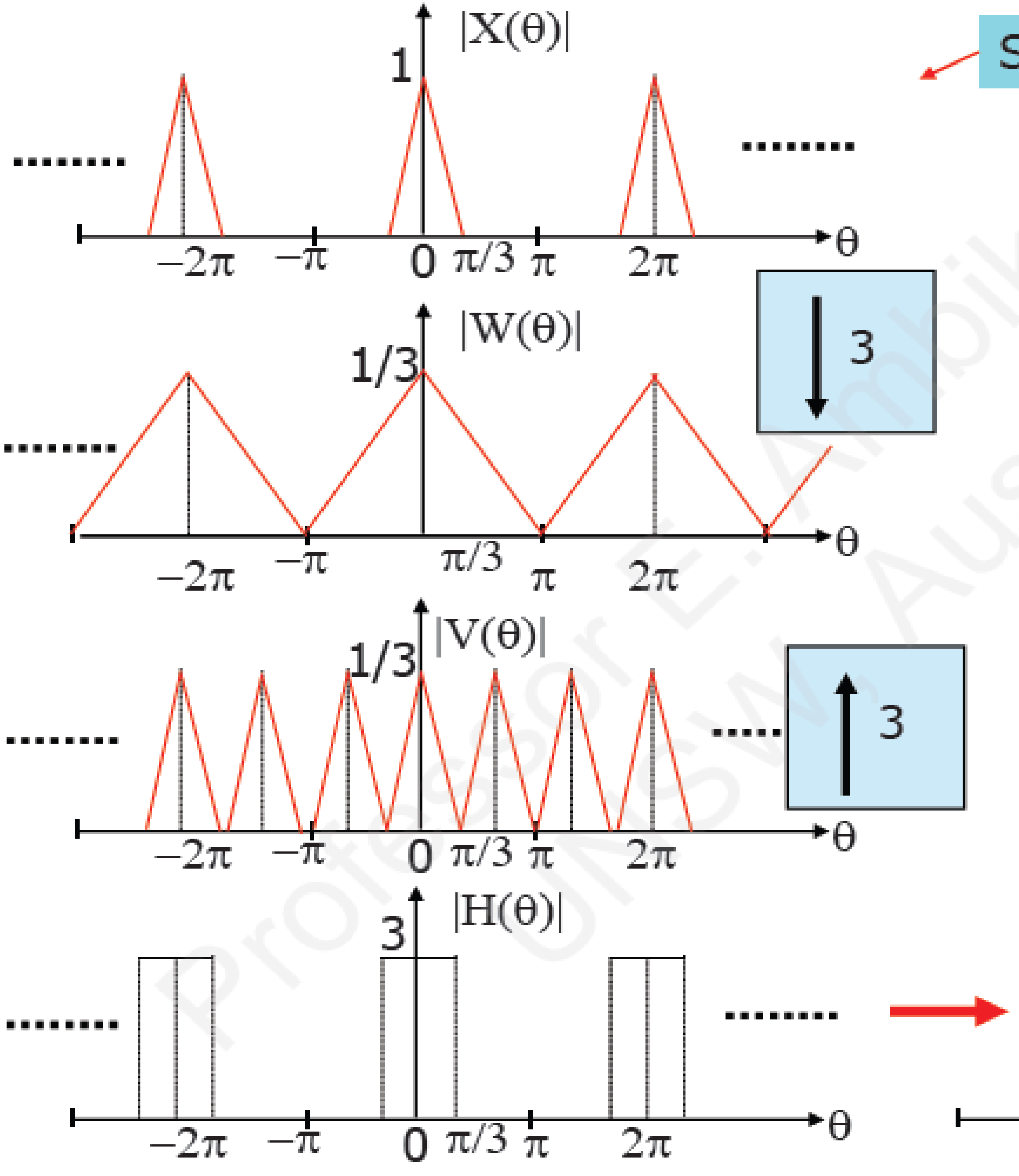
An input signal x(n) with spectrum X(θ) is shown below. The input signal is applied to the system shown below. Sketch |X(θ)|, |W(θ)|,|V(θ)|, |Y(θ)| against θ.





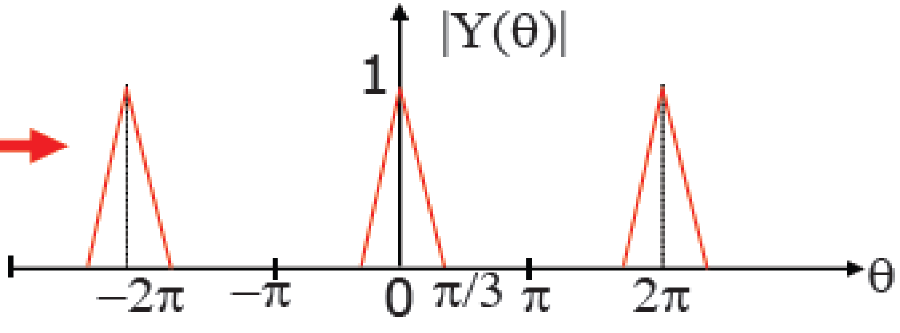


Solution



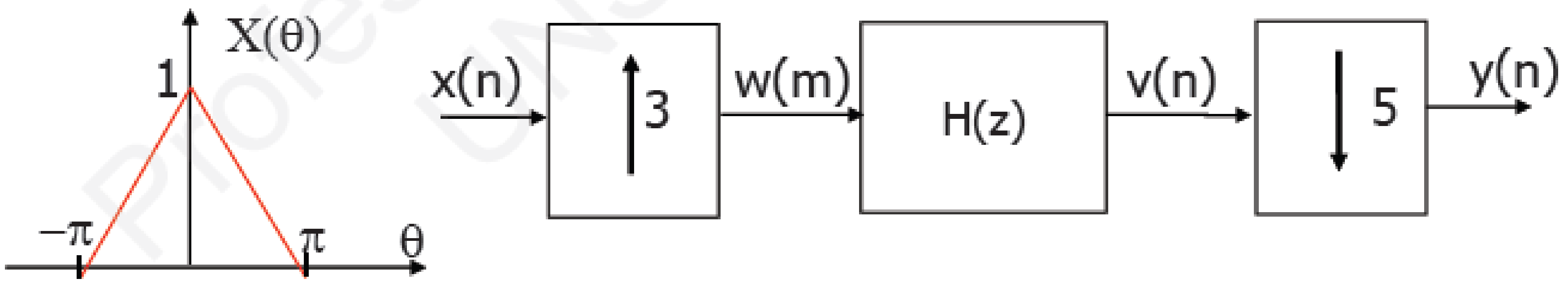
Note: By using a lowpass filter H(θ), we can eliminate the images and extract the original spectrum X(θ).

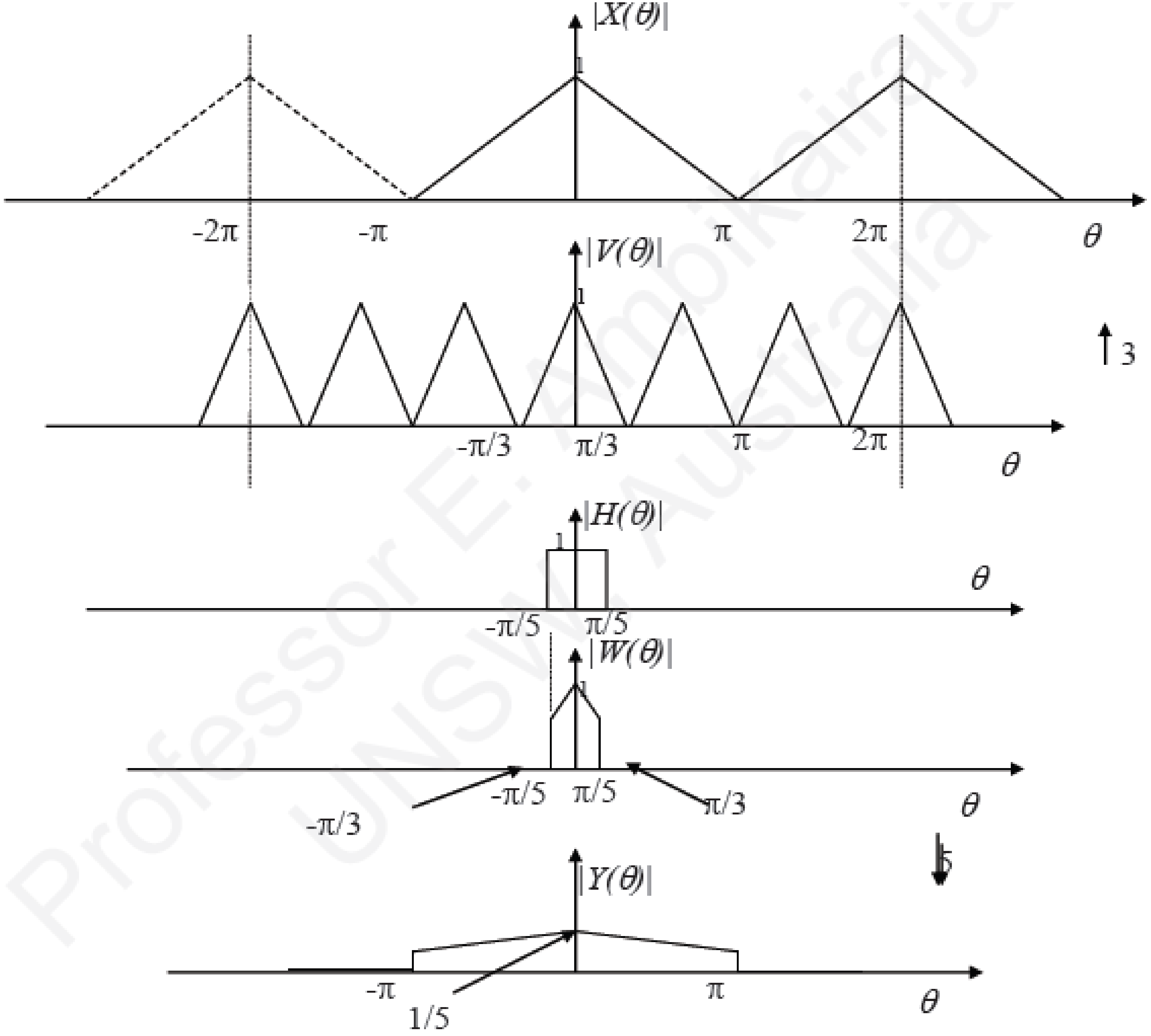
V(θ) is a compressed version of W(θ)



Exercise

A signal x(n) has a spectrum X(θ) as shown below. The signal is applied to the system shown below. The ideal lowpass filter H(z) has a gain factor of 1 in the passband and a cut-off frequency θ=π/5. Sketch |X(θ)|, |W(θ)|, |V(θ)|, |Y(θ)| against θ.





|X(θ)|
1
-2π
-π
π
2π
θ
|V(θ)|
1
-π/3
π/3
π
2π
θ
↑3
|H(θ)|
1
-π/5
π/5
θ
|W(θ)|
1
-π/5
π/5
-π/3
π/3
θ
↓5
|Y(θ)|
-π
π
1/5
θ