Sampling, Quantization and Encoding

Analogue to digital conversion Process

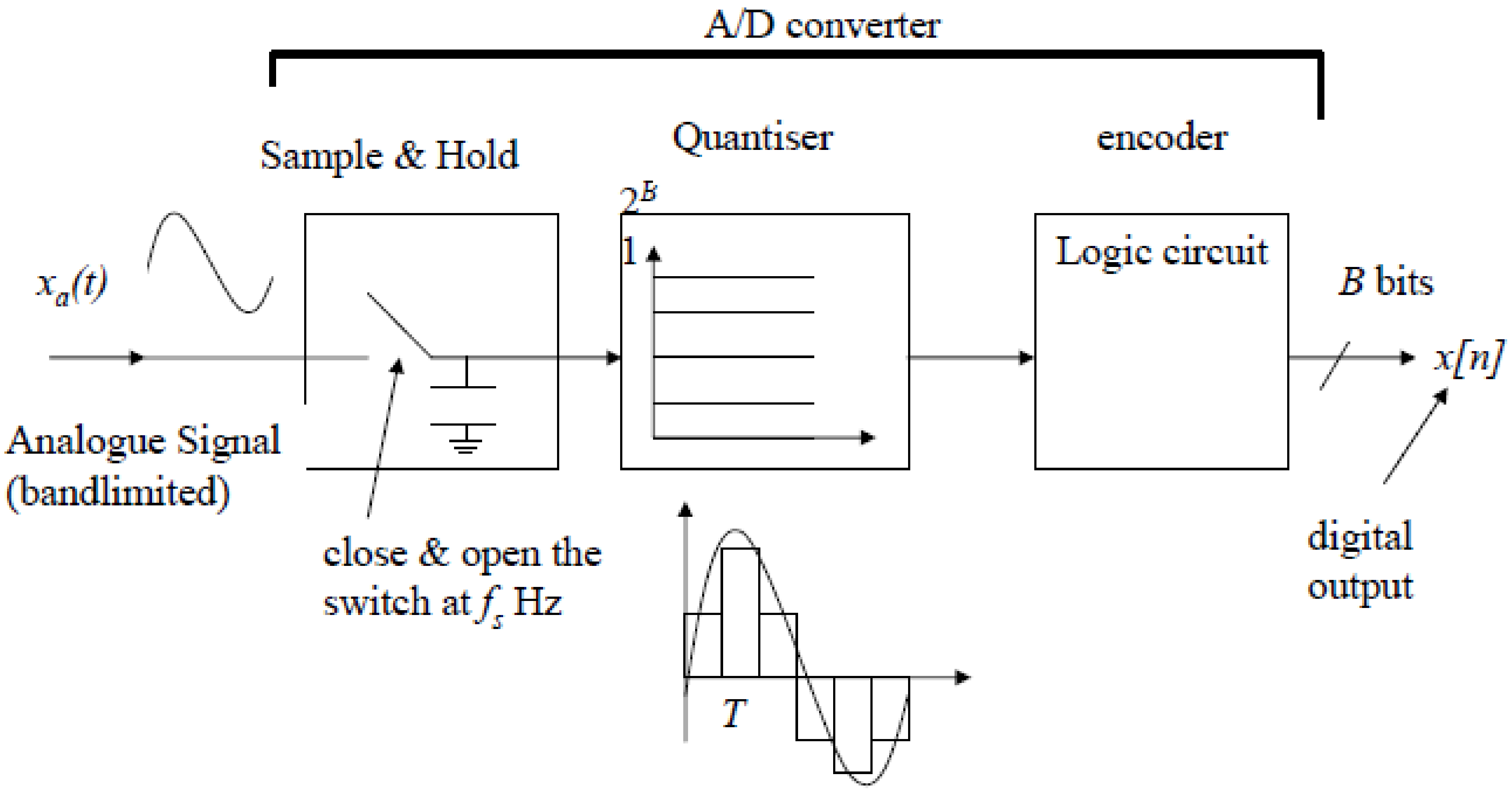
➔Before any DSP algorithm can be performed, the signal must be in a digital form. The A/D conversion process involves the following steps:

❑The signal (Band-limited) is first sampled, converting the analogue signal into a discrete-time signal

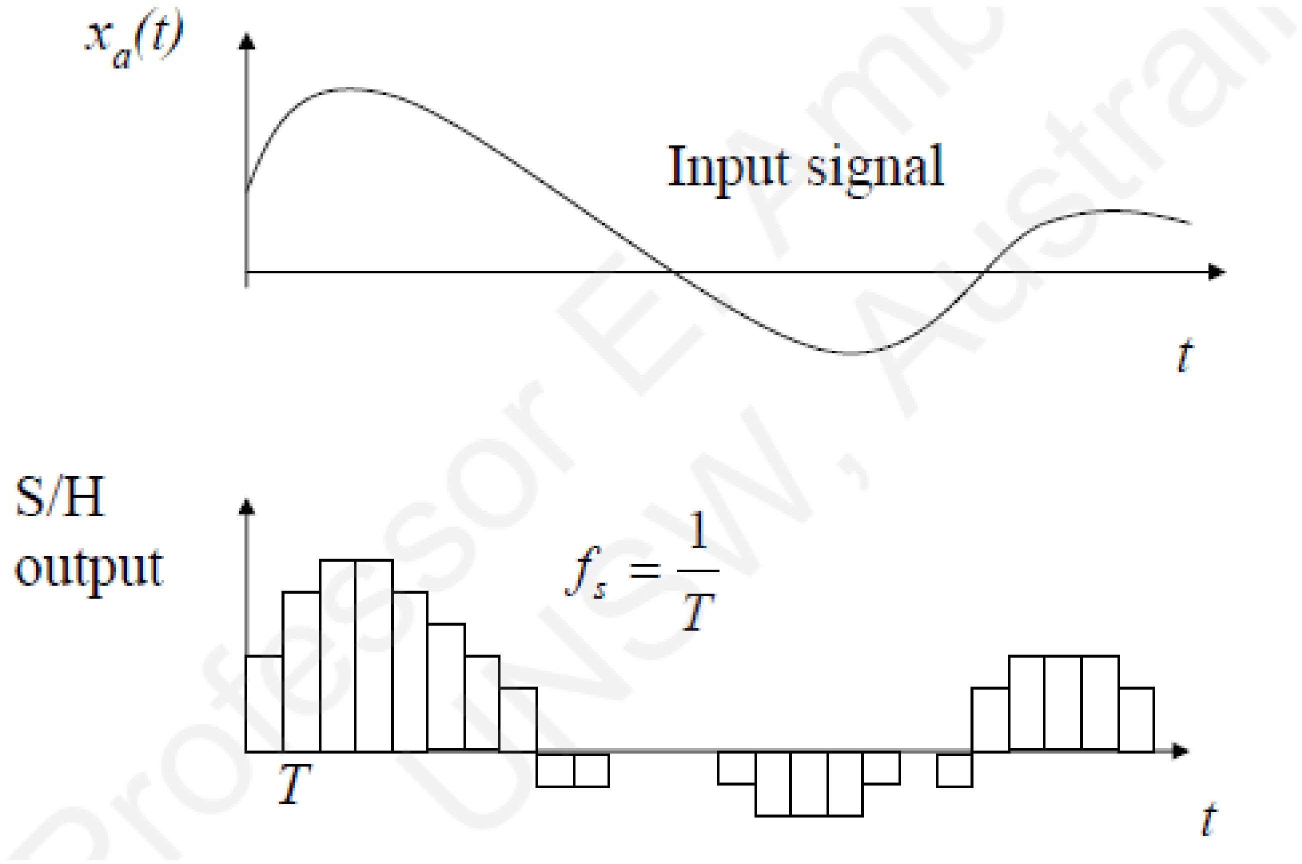
❑The amplitude of each sample is quantised into one of 2^B levels (where B is the number of bits used to represent a sample in the A/D converter)

❑The discrete amplitude levels are represented or encoded into distinct binary words each of length B bits

A practical representation of the A/D conversion process is shown in the Figure below:

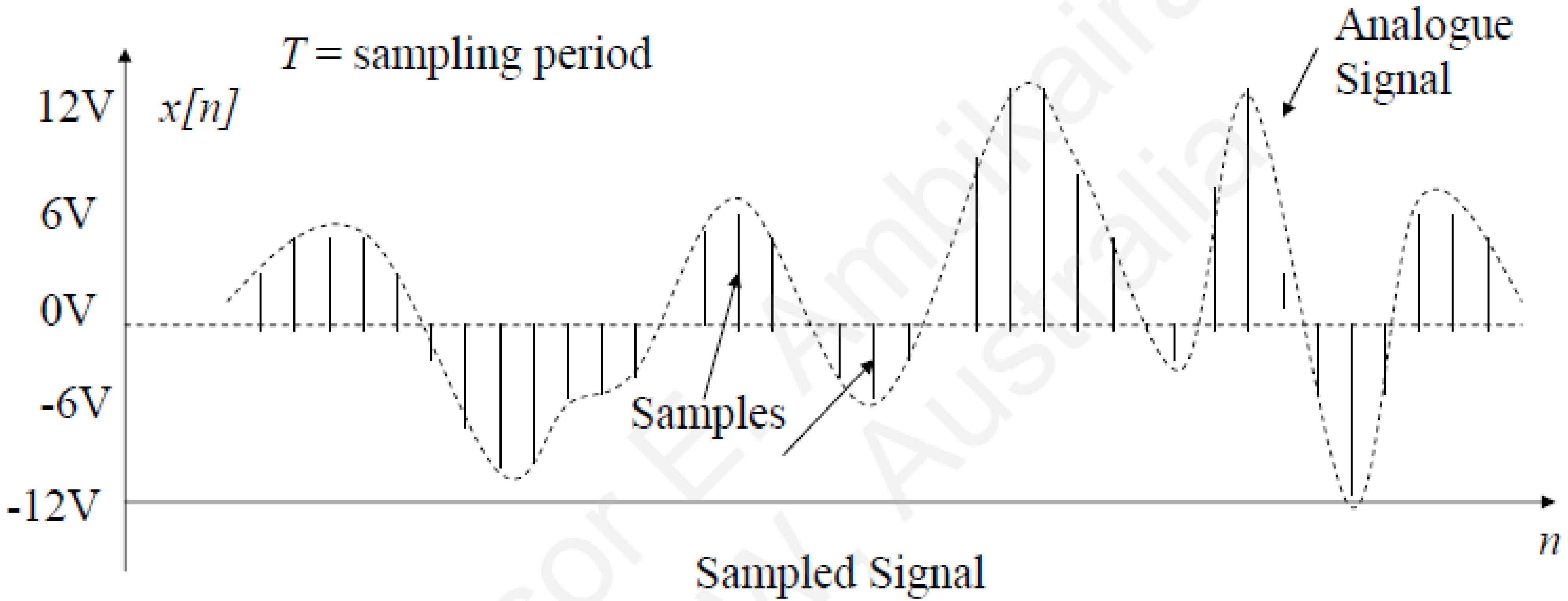


Sample and hold (S/H) takes a snapshot of the analogue signal every *T* sec and then holds that value constant for *T* secs until the next snapshot is obtained.



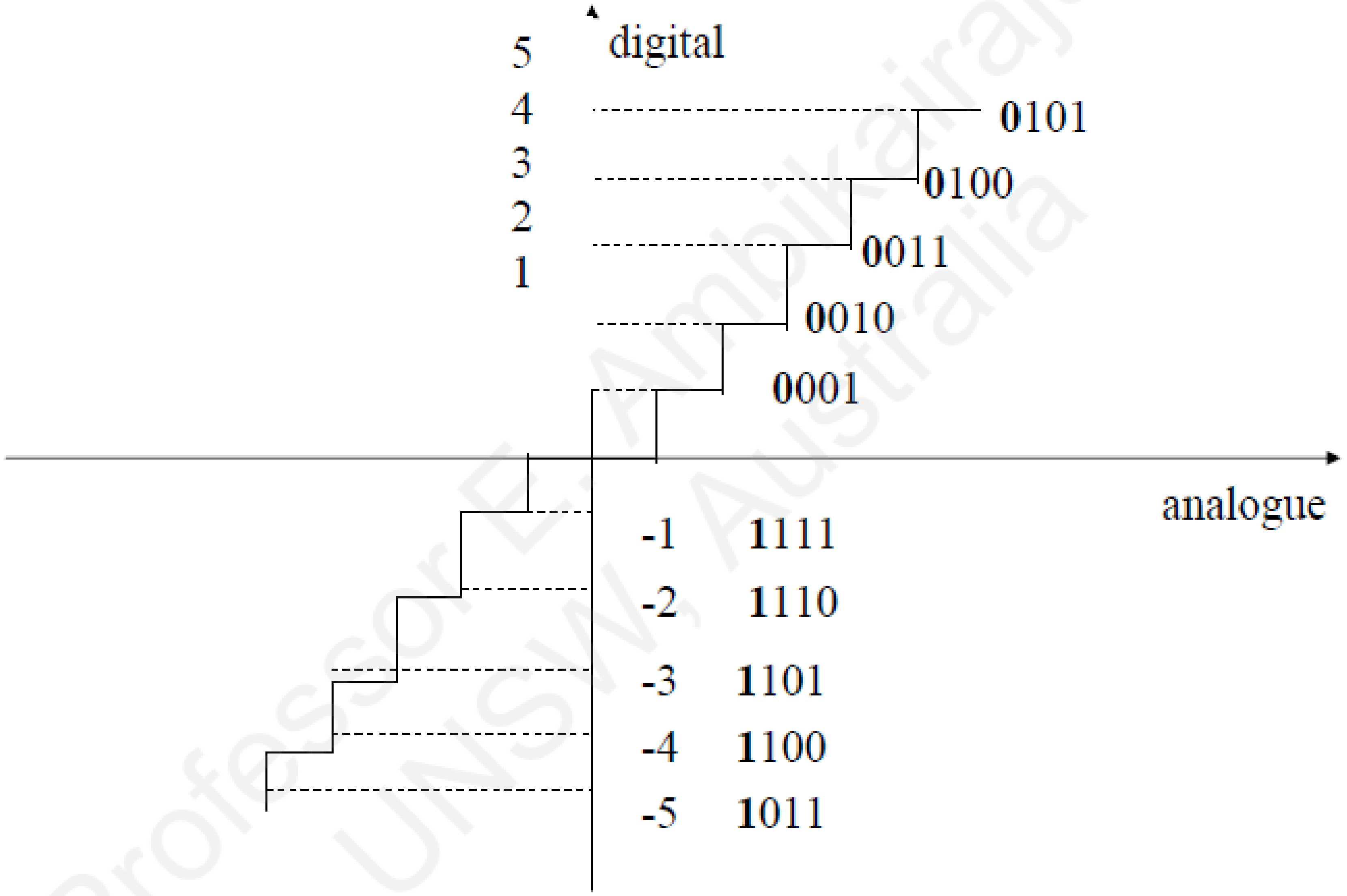
An example of "sample and hold" process to convert analogue signals into digital signals

Example :



An example of sampling analogue signals to discrete-time signals. The sampling period is T

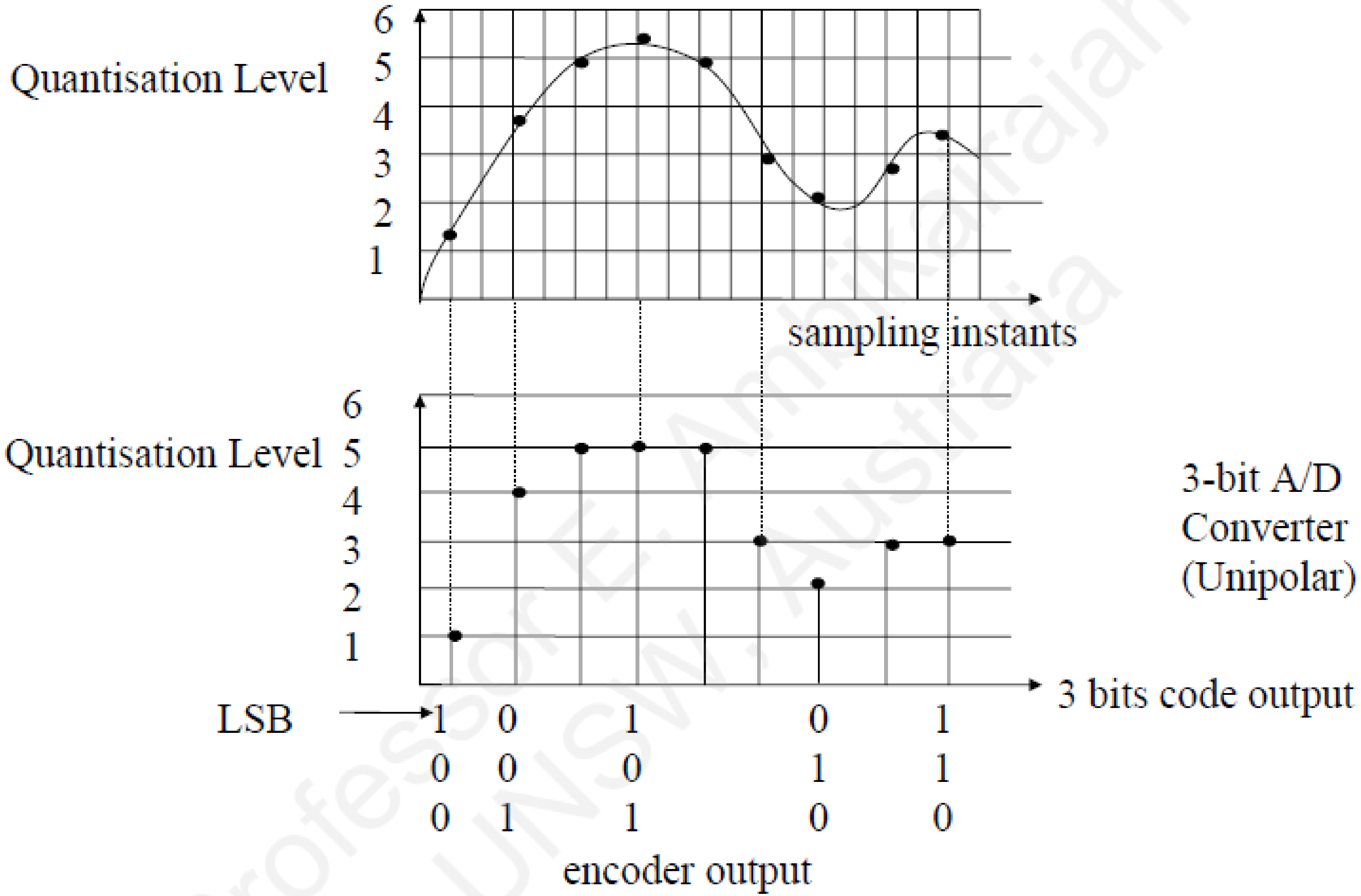
Example: 4-bit (B = 4) A/D converter (<u>bipolar</u>)



Input-output characteristic of 4-bit quantiser (linear) (two's complement notation)

Quantisation and encoding

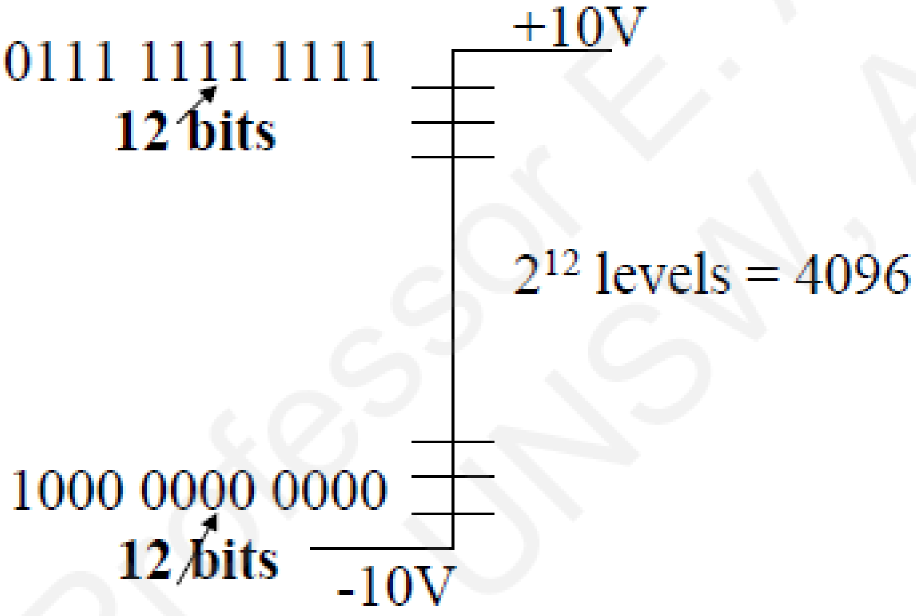
➔ Before conversion to digital, the analogue sample is assigned one of 2^B values.
This process, termed quantization, introduces an error, which cannot be removed.



Quantisation of discrete-time signals.

- A 12 bit A/D converter (bipolar) with an input voltage range of ±10V will have a least significant bit (LSB) of

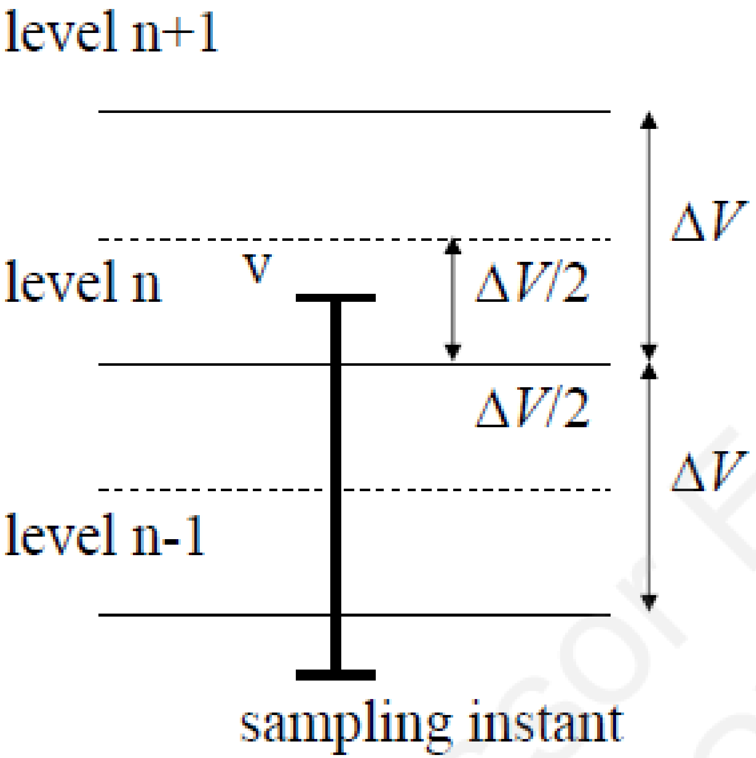
$$\frac{20V}{2^{12}-1} mV = 4.9mV \text{ (resolution)}$$



Resolution (step-size)

$$\Delta V = \frac{20V}{2^{12}-1} = 4.9mV$$

Note:



Quantisation error = $\frac{\Delta V}{2}$ (one half of an LSB) = 4.9 mV / 2 = 2.45 mV

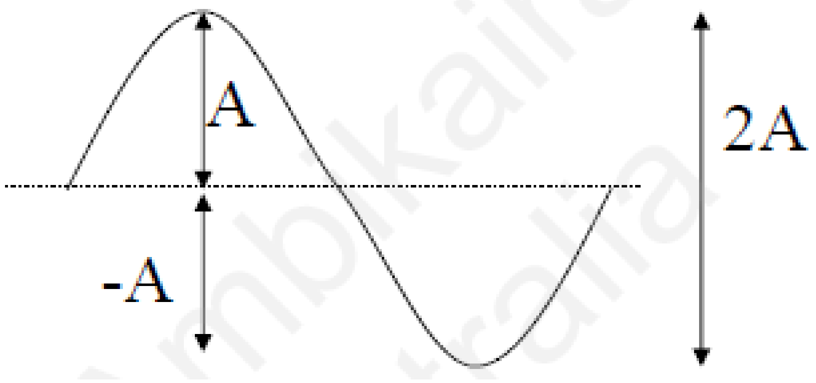
- For an A/D converter with Binary digits the number of quantisation level is 2^B, and the interval between levels, that is the quantisation step size (ΔV) – resolution is given by

$$\Delta V = \frac{V}{2^B - 1} \approx \frac{V}{2^B}$$

- V-full scale range of the A/D converter with bipolar signal inputs. The maximum quantisation error, for the case where the values are rounded up or down

- For a sine wave input of amplitude A, the quantisation step size becomes

$$\Delta V \approx \frac{2A}{2^B}$$



- The quantisation error (e) for each sample, is normally assumed to be random and uniformly distributed in the interval with zero mean

- In this case, the quantisation noise power or variance is given by

e = actual amplitude - quantised amplitude

$$\sigma_e^2 = \int_{-\frac{\Delta V}{2}}^{\frac{\Delta V}{2}} e^2 P(e)\,de = \frac{1}{\Delta V}\int_{-\frac{\Delta V}{2}}^{\frac{\Delta V}{2}} e^2\,de$$

constant = $\frac{1}{\Delta V}$

Hence,

$$\sigma_e^2 = \frac{\Delta V^2}{12}$$ for uniform quantisation

(Note : Uniform quantisation - all steps are of equal size)

- For the sine wave input, the average signal power is $\frac{A^2}{2}$, ie. $\left(\frac{A}{\sqrt{2}}\right)^2$ rms value
- The signal-to-quantisation noise power ratio (SQNR) in decibles

$$SQNR = 10\log\left(\frac{\frac{A^2}{2}}{\frac{\Delta V^2}{12}}\right) = 10\log\left(\frac{\frac{A^2}{2}}{\frac{\left(2A/2^B\right)^2}{12}}\right)$$

$$= 10\log\left(\frac{3\times 2^{2B}}{2}\right)$$

SQNR = 6.02B + 1.76 dB

- The SQNR increases with the number of bits, B. In many DSP applications, an A/D converter resolution between 12 and 16 bits is adequate

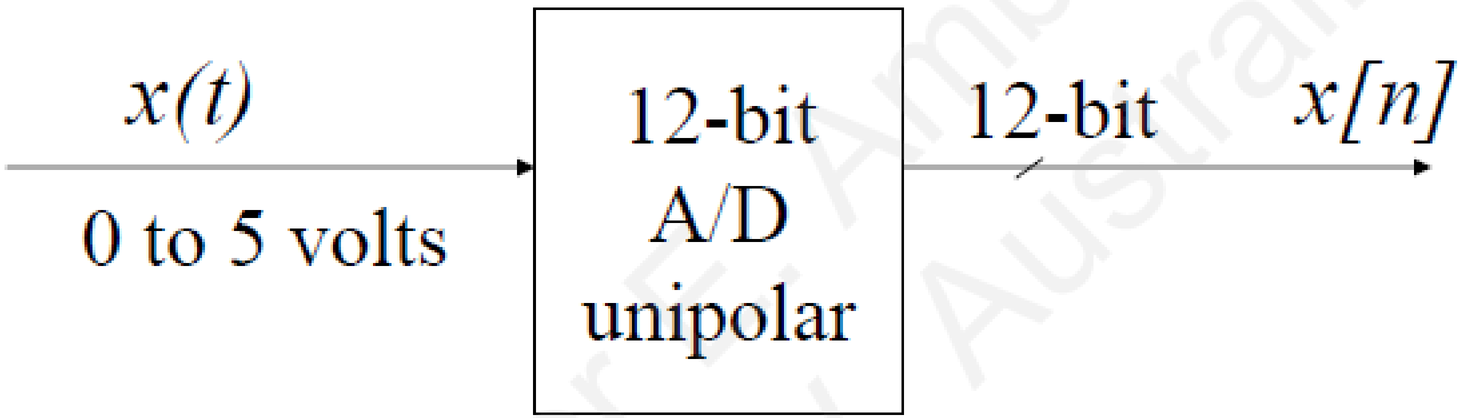
Number of Bits	Levels	SQNR
3	8	18.7 dB
4	16	25.3 dB
5	32	31.6 dB
6	64	37.7 dB
7	128	43.8 dB

Thus, the signal-to-quantisation noise ratio increases approximately 6dB for each bit.

Summary: Analogue to Digital Converter

Example:

Sampling frequency



Step size or Resolution $= \dfrac{5}{2^{12}-1}\,volts$

Conversion time (say) $= 35\ \mu s$

- In practice, the A/D is proceeded by a sample and hold (S/H) which freezes the signal during conversion.

- Two parameters related to S/H:

  aperture time $\approx$ 25 ns (for example)
  acquisition time $\approx$ 2 μs (for example)

- Then the maximum frequency that can be converted becomes

$$f_s\big|_{max} = \frac{1}{(35+2+0.025)10^{-6}} = 27kHz$$

- maximum sampling frequency for the above A/D converter.

Example:

Consider the analogue signal

$$x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$$

What is the **Nyquist** rate for this signal?

The frequencies present in the signal above are

$$f_1 = 25Hz;\ f_2 = 150\ Hz;\ f_3 = 50\ Hz$$

Hence, $f_{max} = 150\ Hz$

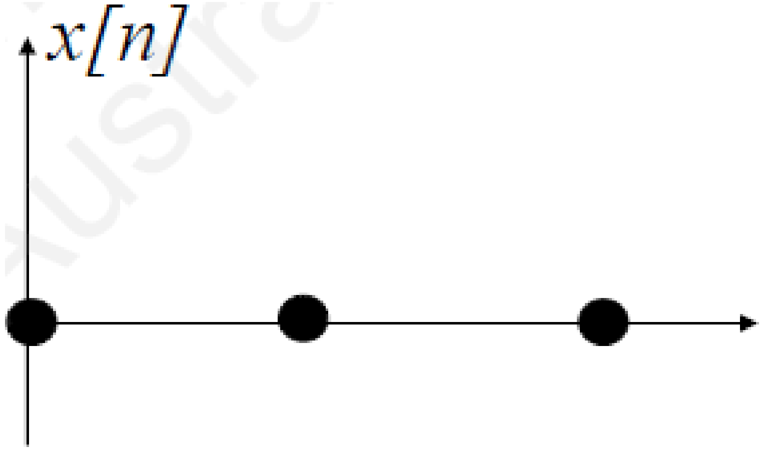
$$f_{sampling} > 2\,f_{max} = 300\ Hz$$

The Nyquist rate is $f_N = 2\,f_{max} = 300\ Hz.$

Note: Consider *x(t) = 10 sin 300πt*

fs ≥ 2 × f = 300 Hz

$$x[n] = 10\sin(300\pi nT) = 10\sin\left(\frac{300\pi}{f_s}n\right) = 10\sin(\pi n)$$



- We are sampling the analogue sinusoid at its zero-crossing points and hence we miss the signal completely. The situation will not occur if the sinusoid is offset by some phase (here).

- In such case we have

$x(t) = 10\sin(300\pi t + \phi)$ and $T = \frac{1}{f_s}$, where $f_s = 300Hz.$

$$
\begin{aligned}
x[n] &= 10\sin(\pi n + \phi) \\
&= 10[\sin(\pi n)\cos(\phi) + \cos(\pi n)\sin(\phi)] \\
&= 10\cos(\pi n)\sin(\phi)
\end{aligned}
$$

for n = 0,1,2,..

Since cos(π*n*) = (-1)*n* , *x* [*n*]= $(-1)^n 10 \sin(\phi)$

If θ ≠ 0 or θ ≠ π, the samples of the sinusoid taken at the Nyquist rate are not all zero

Note: $x(t) = A\cos(2\pi f_0 t)$ is a continuous-time sinusoidal sign

$$x(n) = A\cos\left(2\pi \frac{f_0}{f_s}\right)n$$

$$-\frac{f_s}{2} \leq f_0 \leq \frac{f_s}{2}$$

Example :

- Consider the analogue signal

$$x(t) = 3\cos(2000\pi t) + 5\sin(6000\pi t) + 10\cos(12000\pi t)$$

(a) What is the Nyquist rate for this signal?

The frequencies existing in the analogue signal are:

f1 = 1 kHz; f2 = 3 kHz; f3 = 6 kHz

Thus *fmax = 6 kHz* and according to the sampling theorem,

fs > 2 *f*max = 12 kHz

The Nyguist rate is = 12 kHz

(b) Assume now that we sample this signal *x(t)* using a sampling rate *fs* = 5 *KHz* (samples/sec). What is the discrete-time signal obtained after sampling?

$$f_s = 5000Hz \Rightarrow \frac{f_s}{2} = 2500$$

x(t) = 3 cos (2π × 1000t) + 5 sin(2π × 3000t) + 10 cos(2π×6000t)

$$x[n] = 3\cos\left(2\pi\frac{1000}{5000}n\right) + 5\sin\left(2\pi\frac{3000}{5000}n\right) + 10\cos\left(\frac{6000}{5000}n\right)$$

$$= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(2\pi\frac{3}{5}n\right) + 10\cos\left(2\pi\left(\frac{6}{5}\right)n\right)$$

$$= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(2\pi\left(1-\frac{2}{5}\right)n\right) + 10\cos\left(2\pi\left(1+\frac{1}{5}\right)n\right)$$

$$= 3\cos\left(2\pi\left(\frac{1}{5}\right)n\right) + 5\sin\left(-2\pi\frac{2}{5}n\right) + 10\cos\left(2\pi\left(\frac{1}{5}\right)n\right)$$

$$x[n] = 13\cos\left(2\pi\left(\frac{1}{5}\right)n\right) - 5\sin\left(2\pi\left(\frac{2}{5}\right)n\right)$$

(c) What is the analogue signal *y(t)* we can reconstruct from the samples if we use ideal interpolation.

➔Since only frequency components at 1 kHz and 2 kHz are present in the sampled signal, the analogue signal we can recover is,

$y(t)=13\cos(2000\pi t)-5\sin(4000\pi t)$

which is obviously different from the original signal *x(t)*. The distortion of the original analogue signal was caused by the aliasing effect, due to the low sampling rate used.

Example:

An analogue signal *x(t)* =sin(480π*t*)+3sin(720π*t*) is sampled 600 times per second.

(a) Determine the Nyguist sampling rate for *x(t)*

(b) Determine the folding frequency (or half the sampling frequency)

(c) What are the frequencies, in radians, in the resulting discrete time signal *x[n]*?

(d) If *x[n]* is passed through an ideal D/A converter what is the reconstructed signal *y(t)*

(a) $x(t) = \sin(2\pi\, 240t) + 3\sin(2\pi\, 360t)$

$f1$ = 240 Hz $f2$ = 360 Hz

∴ $f$max = 360 Hz ∴ FNyquist = 2 × $f$max = 720 Hz

(b) fs = 600 Hz ∴$f$fold or $fs/2$ = 300 Hz

(c)

$$x[n] = x(t)\big|_{t=nT} = \sin\left(2\pi\frac{240}{600}n\right) + 3\sin\left(2\pi\frac{360}{600}n\right)$$

$$= \sin\left(\frac{4\pi}{5}n\right) + 3\sin\left(\frac{6\pi}{5}n\right)$$

$$= \sin\left(\frac{4\pi}{5}n\right) + 3\sin\left(\left(2\pi - \frac{4\pi}{5}\right)n\right)$$

$$= \sin\left(\frac{4\pi}{5}n\right) - 3\sin\left(\frac{4\pi}{5}n\right)$$

$$= -2\sin\left(\frac{4\pi}{5}n\right)$$

(d)

$$y(n) = -2\sin\left(2\pi \frac{240}{600} n\right) \qquad t = nT = \frac{n}{f_s} = \frac{n}{600}$$

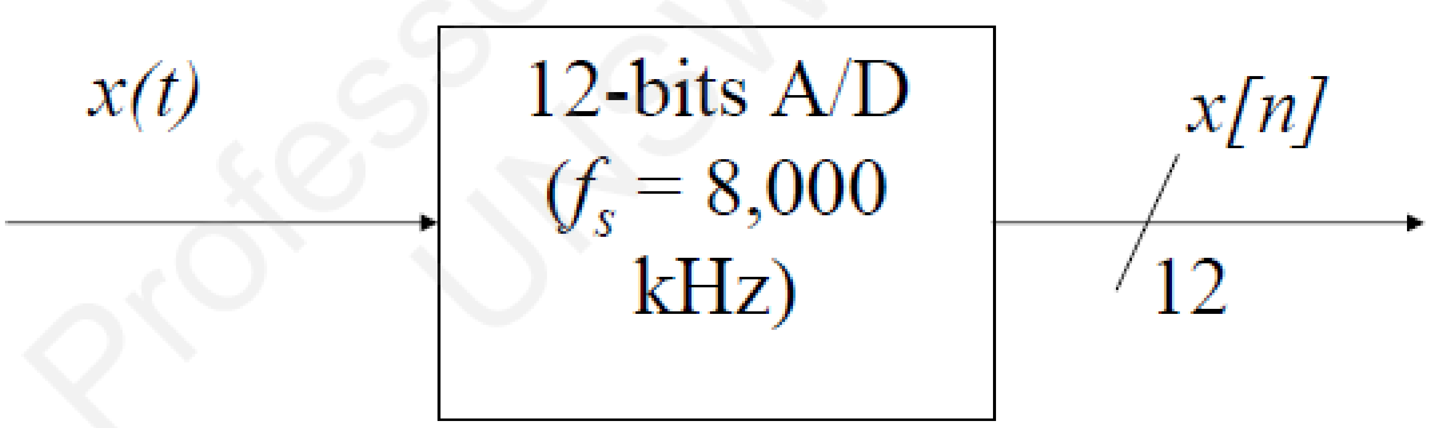
$$y(t) = -2\sin(480\pi t)$$

Note :

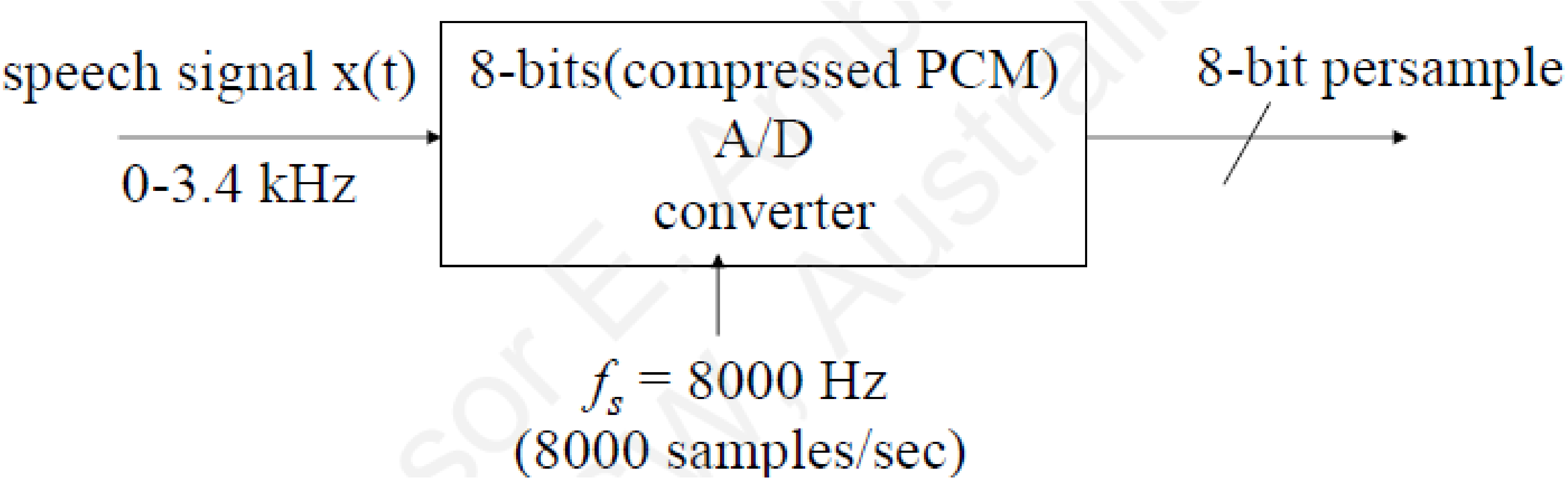
(a) bit rate = *fs* × no of bits

= 8000 samples/sec × 12

=96,000 samples/sec

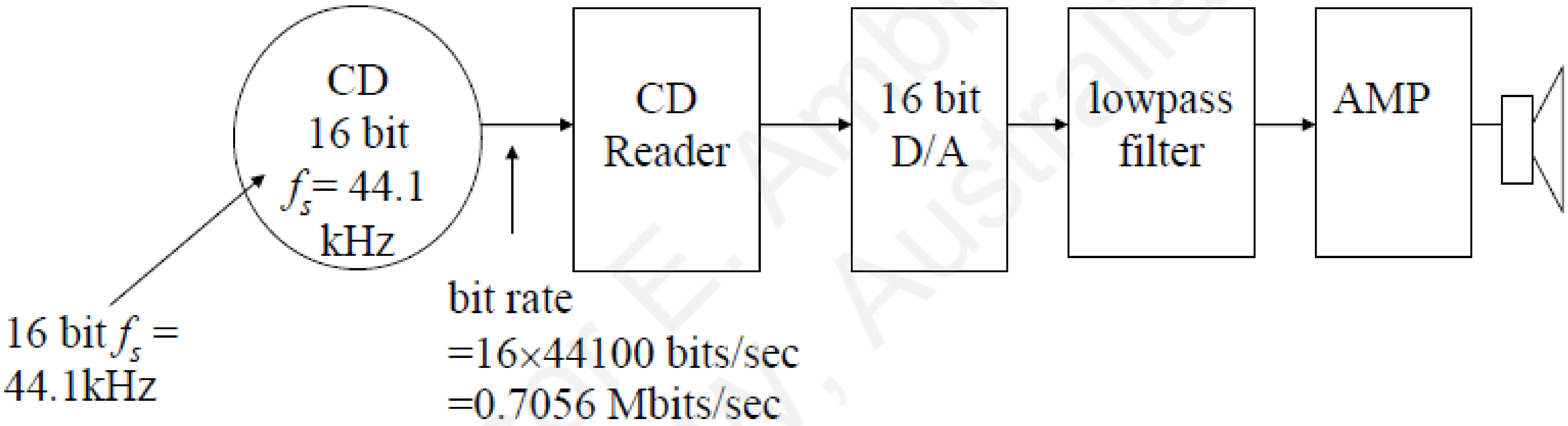


(b) In the case of PCM, speech signals are filtered to remove effectively all frequency components above 3.4 kHz and the sampling rate is 8000 samples per sec



Bit rate (bits per second)
= sampling frequency × bits/sample
= 8000 samples/second × 8-bits/sample
= 64,000 bits/sec

(c)


CD
16 bit
f_s= 44.1
kHz
CD
Reader
16 bit
D/A
lowpass
filter
AMP
16 bit f_s =
44.1kHz
bit rate
=16×44100 bits/sec
=0.7056 Mbits/sec