Chapter – 5 –

Transfer Analysis of LTI systems

LTI sys.

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$\underset{\text{eigen function}}{e^{j\omega n}} \rightarrow \boxed{H(e^{j\omega})} \rightarrow \underbrace{H(e^{j\omega})}_{\substack{\text{Eigen Value.} \\ \text{(gain).}}} e^{j\omega n}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot \underbrace{H(e^{j\omega})}_{\text{freq. Response.}}$$

$$Y(z) = X(z) \cdot H(z)$$

$H(z)$ ↳ Transfer (system) function

Frequency Response of LTI systems:

$$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot \underline{|H(e^{j\omega})|} \rightarrow$$ Magnitude Response or 'gain'

$$\measuredangle Y(e^{j\omega}) = \measuredangle X(e^{j\omega}) + \underline{\measuredangle H(e^{j\omega})} \rightarrow$$ Phase Response or phase shift.

* Undesirable effects ⇒ Called magnitude or phase distortions.

Ideal Frequency Selective Filters.

* Ideal low-pass filter:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

$H_{LP}(e^{j\omega})$ is periodic with period of 2π.

$$\Rightarrow h_{LP}(n) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

* low-pass filter selects low freq. components and rejects high freq. comp.

High-pass filter:

$$H_{hp}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & \omega_c < |\omega| \leq \pi \end{cases}$$

since $H_{hp}(e^{j\omega}) = 1 - H_{LP}(e^{j\omega}) \Rightarrow h_{hp}(n) = \delta(n) - \dfrac{\sin \omega_c n}{\pi n}$

* Ideal high-pass filter selects high freq. components and rejects low freq. comp.

* The ideal low-pass filter is noncausal, and its impulse response $h_{lp}(n)$ extends from $-\infty$ to $+\infty$ $\Rightarrow$ Therefore, it is not possible to compute the output of either ideal low-pass or high-pass filters either recursively or nonrecursively, i.e. systems are not computationally realizable.

* Phase Response of ideal low-pass filter is specified to be zero. So, causal approximation to ideal frequency selective filters have nonzero phase response.

phase distortion and delay

Ideal delay system: $h_{id}(n) = \delta(n - n_d)$, $n_d \to$ Int.

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$|H_{id}(e^{j\omega})| = 1 , \quad \measuredangle H_{id}(e^{j\omega}) = -\omega n_d , \quad |\omega| < \pi .$$

* Design approximation to ideal filters and other LTI systems.
$\Rightarrow$ We accept linear-phase response rather than zero phase response. $\Rightarrow$ so ideal lowpass filter with linear phase.

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d} , & |\omega| < \omega_c \\ 0 , & \omega_c < |\omega| \leq \pi \end{cases}$$

$$\Rightarrow h_{lp}(n) = \frac{\sin(\omega_c(n-n_d))}{\pi(n-n_d)} , \quad -\infty < n < \infty$$

$$\measuredangle H_{lp}(e^{j\omega}) \simeq -\varphi_0 - \omega n_d \quad \text{(linear approximation)}.$$

* <u>Group delay</u>: measure of the linearity of the phase defined as negative slope of phase at specific frequency.

$$\tau(w) = -\frac{d}{dw}\{\measuredangle H(e^{jw})\}$$

<u>Example</u>: $h(n) = \delta(n-5)$ (ideal delay system).

$$H(z) = z^{-5}$$

$$H(e^{jw}) = H(z)\big|_{z=e^{jw}} = 1\, e^{-jw5}$$

$$|H(e^{jw})| = 1 \quad , \quad \measuredangle H(e^{jw}) = -5w$$

$$\tau(w) = -\frac{d}{dw}(-5w) = 5 \text{ samples}.$$

<u>Example 2</u>: $h(n) = \frac{1}{5}\{\underset{\substack{\uparrow\\ n=0}}{1}, 1, 1, 1, 1\}$ ⇐ 5-point Moving Averger.

$$h(n) = \frac{1}{5}\left(\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)\right).$$

$$H(z) = \frac{1}{5}\left(z^{0} + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right)$$

$$H(e^{jw}) = \frac{1}{5}\left(e^{j0} + e^{-jw} + e^{-j2w} + e^{-j3w} + e^{-j4w}\right)$$

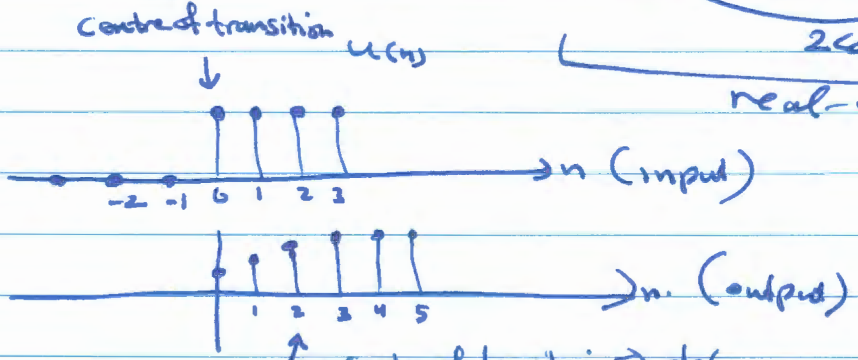
$$= \frac{1}{5}\underbrace{\left(e^{j2w} + e^{jw} + \underset{1}{e^{j0}} + e^{-jw} + e^{-j2w}\right)}_{\text{real-valued}} e^{-j2w}$$

$2\cos w$

$2\cos 2w$

$\Rightarrow \measuredangle H(e^{jw}) = -2w$

$\Rightarrow \tau(w) = 2$ samples.



Centre of transition ⇒ delay = 2 samples.

* see Example 5.1 (effects of attenuation and group delay).

System Functions for systems Characterized by LCCDE:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Another Assumption that system is Causal.

* LTI systems satisfy linear Const-Coeff. diference equation are best developed through Z-transform.

So, by applying Z-transform to both sides and using linearity and time-shifting properities, we obtain:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

* H(z) in factored form:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

* each of the factors $(1 - c_k z^{-1})$ in the numerator Contributes a Zero at $z = c_k$ and a Pole at $z = 0$.

similarly, $(1 - d_k z^{-1})$ in the denominator Contributes a Zero at $z = 0$ and a Pole at $z = d_k$.

Z-domain $\quad b_k z^{-k} \xleftrightarrow{\text{corresponds to}} b_k x(n-k)$ Time-domain

$\quad a_k z^{-k} \longleftrightarrow a_k y(n-k)$

Example: (second order system)

$$H(z) = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2} z^{-1})(1 + \frac{3}{4} z^{-1})}$$ Find Diff. equation?

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4} z^{-1} - \frac{3}{8} z^{-2}} = \frac{Y(z)}{X(z)}$$

Thus; $\left(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}\right)Y(z) = \left(1+2z^{-1}+z^{-2}\right)X(z)$

and Diff. eq. is:

$$y(n)+\frac{1}{4}y(n-1)-\frac{3}{8}y(n-2) = x(n)+2x(n-1)+x(n-2)$$

Stability and Causality:

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k) \Rightarrow H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Assumptions:
- linear
- Time-Invariant

but, no Assumption about Stability and Causality.

So, from difference equation we can get system function $(H(z))$ but not ROC.

* Many choices for ROC $\Rightarrow$ each choice of ROC leads to diferent impulse response, but all correspond to the same difference equation.

* If we assume system is Causal $\Rightarrow$ $h(n)$ must be right-sided therefore, ROC of $H(z)$ must be outside the outermost pole.

* If we assume system is Stable $\Rightarrow$ $h(n)$ must be absolutly summable i.e.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$ which is identical to:

Identical

$$\sum_{n=-\infty}^{\infty} |h(n)\, z^{-n}| < \infty$$

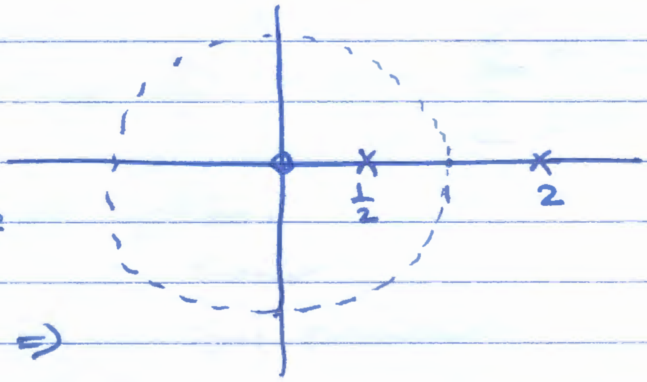
* for $|z| = 1$, condition for stability is equivelant to that ROC of $H(z)$ include unit circle.

Example: Determine ROC

$$y(n) - \frac{5}{2}y(n-1) + y(n-2) = x(n)$$

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

* There are 3 choices for ROC:
- If system is assumed to be Causal $\Rightarrow$ ROC $|z| > 2$
in this case, system is <u>unstable</u>



- If we assume system is stable $\Rightarrow$
$\Rightarrow$ ROC $\frac{1}{2} < |z| < 2$

- Third choice of ROC $|z| < \frac{1}{2}$ $\longrightarrow$ not stable, noncausal.

* for Causal and stable system $\Rightarrow$ ROC outside outermost pole and include unit circle $\Rightarrow$ <u>This requires all poles must be inside unit circle.</u>

<u>Inverse Systems</u>: $H_i(z)$

$$h(n) * h_i(n) = \delta(n)$$

$$H(z)\,H_i(z) = 1$$

$$\Rightarrow H_i(z) = \frac{1}{H(z)} \quad \text{and} \quad H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

* Equivalently, log-magnitude and phase of the inverse system are negatives of the corresponding function for the original system.

* Not all systems have an inverse, e.g. ideal lowpass filter doesn't $\Rightarrow$ There is no way to recover frequency components above cutoff frequency (ω_c) that are set to zero by the filter.

* Rational systems functions have inverse:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-k})}{\prod_{k=1}^{N} (1 - d_k z^{-k})}$$

with zeros at $z = c_k$ and poles at $z = d_k$, In addition to possible zeros and poles at $z = 0$ and $z = \infty$, Then

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N} (1 - d_k z^{-k})}{\prod_{k=1}^{N} (1 - c_k z^{-k})}$$

* poles of $H_i(z)$ are zeros for $H(z)$ and vice versa.

* What ROC associate with $H_i(z)$?
  Answer from convolution theorem.

$$h(n) * h_i(n) = \delta(n)$$

So, $H(z)$ and $H_i(z)$ must overlap. If $H(z)$ is Causal, ROC is

$$|z| > \max_k |d_k|$$

* Thus, any ROC for $H_i(z)$ that overlaps with ROC of $H(z)$ is a valid ROC for $H_i(z)$.

Example: First-order system.

$$H(z) = \frac{1 - 0.5 z^{-1}}{1 - 0.9 z^{-1}}, \quad |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9 z^{-1}}{1 - 0.5 z^{-1}}$$

* Since $H_i(z)$ has only one pole, there are only two possibilities for ROC and the only choice that overlaps with $|z| > 0.9$ is $|z| > 0.5$

$\Rightarrow$ impulse response of the inverse system is:

$$h(n) = (0.5)^n u(n) - 0.9 (0.5)^{n-1} u(n-1)$$

In this case, inverse system is both stable and causal.

Example:

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}}, \quad |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}$$

$\Rightarrow$ there are two possible ROC's : $|z| < 2$ and $|z| > 2$
both ROC's overlap with $|z| > 0.9$ $\Rightarrow$ So, both are valid inverse system.

* For ROC $|z| < 2 \Rightarrow h_{i_1}(n) = 2(2)^n u(-n-1) - 1.8(2)^{n-1} u(-n)$
* for ROC $|z| > 2 \Rightarrow h_{i_2}(n) = -2(2)^n u(n) + 1.8(2)^{n-1} u(n-1)$

$h_{i_1}(n) \longrightarrow$ stable and noncausal.
$h_{i_2}(n) \longrightarrow$ Causal and nonstable.

* In general, if $H(z)$ is Causal with zero's at C_k, $k = 1, \ldots, M$ then, its $H_i(z)$ will be Causal if and only if we associate ROC

$$|z| > \max_k |C_k| \quad \text{with } H_i(z).$$

$\Rightarrow$ if we also require that the inverse system be stable, then the ROC of $H_i(z)$ must include the unit circle $\Rightarrow$ it must be true that

$$\max_k |C_k| < 1$$

i.e. all zeros of $H(z)$ must be inside unit circle.
$\Rightarrow$ Therefore LTI system is stable and causal and also has a stable and Causal inverse system if and only if, both poles and zeros of $H(z)$ are inside unit circle. Such systems are refered to as minimum-phase systems.

Impulse Response for Rational System Functions:

* any rational function of z^{-1} with only first-order poles can be expressed in the form:

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \quad \text{.......} \quad (*)$$

Where, the terms in the first sum would be obtained by long division of the denominator into numerator and would be present only if $M \geq N$. Coeff. A_k in the second sum are obtained using

$$A_k = (1-d_k z^{-1}) H(z) \Big|_{z=d_k} \quad \text{(residues).}$$

* If $H(z)$ has multiple-order pole, its partial fraction expansion would have the form:

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1-d_i z^{-1})^m}$$

* If the system is assumed to be Causal $\Rightarrow$ ROC outside all poles of equation $(*)$ above. and

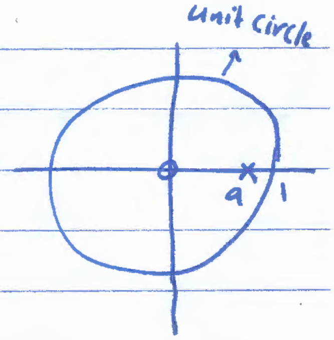
$$h(n) = \underbrace{\sum_{r=0}^{M-N} B_r \delta(n-r)}_{\text{only if } M \geq N} + \sum_{k=1}^{N} A_k d_k^n u(n)$$

Two Classes of LTI system:

Δ IIR: at least one nonzero pole of $H(z)$ is not cancelled by a zero. $\Rightarrow$ at least one term of $A_k (d_k)^n u(n)$, and so $h(n)$ will not be of finite length (i.e. will not be zero outside of finite interval), such systems are called Infinite Impulse Response (IIR) Systems.

Example: First-order IIR system

$$y(n) - a\,y(n-1) = x(n) \Rightarrow H(z) = \frac{1}{1-az^{-1}}$$



RoC $|z| > |a|$, condition for stability is $|a| < 1$ and inverse z-transform of $H(z)$ is $h(n) = a^n u(n)$.

$\boxed{2}$ **FIR:** $H(z)$ has no poles except at $z=0$, i.e. $N=0$
Thus, a partial fraction expansion is not possible. and $H(z)$ a polynomial in z^{-1}

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \quad (\text{assume } a_0 = 1)$$

$$\Rightarrow h(n) = \sum_{k=0}^{M} b_k \delta(n-k) = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

In this case, the impulse response is finite in length, i.e. it is zero outside a finite interval ($h(n) = 0$, $n \notin [0, M]$)
$\Rightarrow$ These systems are called Finite Impulse Response (FIR)

* For FIR systems, difference equation is identical to convolution sum, i.e.

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

Example:

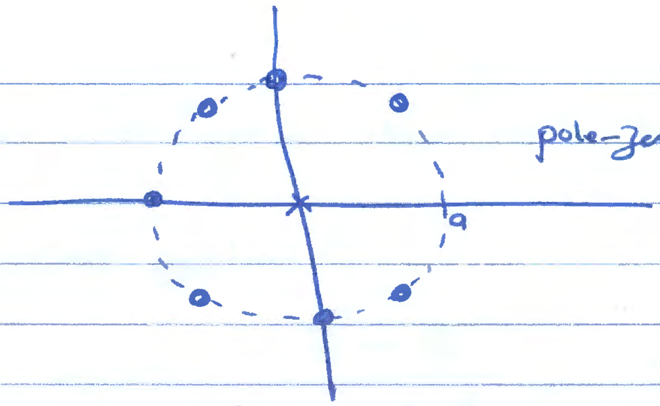
$$h(n) = \begin{cases} a^n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M+1}}{1 - az^{-1}} \quad \text{-------} \ (*)$$

Since zeros of the numinator are at:

$$z_k = a\,e^{j2\pi k/(M+1)}, \quad k = 0, 1, \ldots, M$$

When, a is assumed real and positive, the pole at $z=a$ is cancelled by a zero at z_0 ($k=0$).



pole-zero plot for $M=7$.

* The diff. equation satisfied by input and output of LTI system is discrete convolution

$$y(n) = \sum_{k=0}^{M} a^k x(n-k)$$

Also equation ⊛ above suggests I/O also satisfy the diff. eq.

$$y(n) - a\,y(n-1) = x(n) - a^{M+1} x(n-M-1)$$

Two diff-eq. result from two equivalent form of $H(z)$ in eq. ⊛.

All-Pass Systems:

$$|H_{ap}(e^{j\omega})| = 1$$

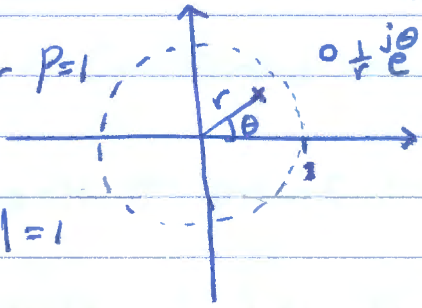
All-Pass $\Longleftrightarrow$ poles and zeros are in conjugate reciprocal pairs.

$$H_{ap}(z) = \prod_{i=1}^{P} \frac{z^{-1} - c_i^*}{1 - c_i z^{-1}}$$

Poles: $c_i = r e^{j\theta}$

$\frac{1}{c_i^*} = \frac{1}{r} e^{j\theta}$

* To show $|H_{ap}(e^{j\omega})| = 1$, consider $P=1$



$\circ\ \frac{1}{r} e^{j\theta}$

$$|H_{ap}(e^{j\omega})| = \left| \frac{e^{-j\omega} - c^*}{1 - c\,e^{-j\omega}} \right|$$

$|e^{-j\omega}| = 1$

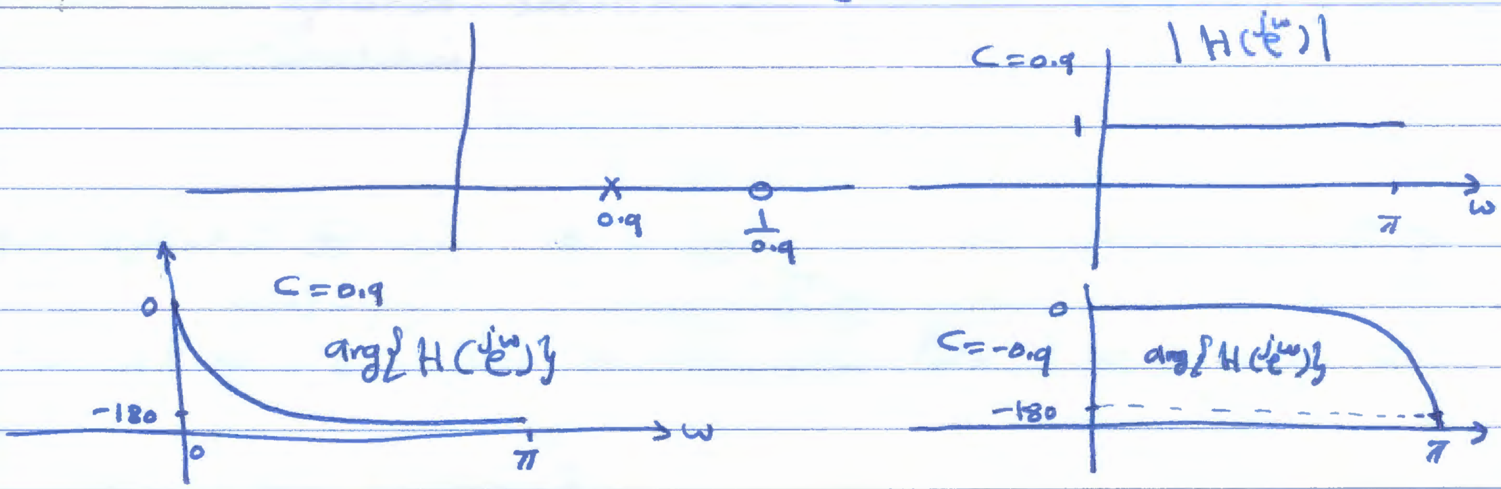
$$= \left| \frac{e^{-j\omega}(1 - c^* e^{j\omega})}{1 - c\,e^{-j\omega}} \right|$$

$$= \left| \frac{1 - c^* e^{j\omega}}{\underbrace{1 - c\,e^{-j\omega}}_{b}} \right| = \frac{|b^*|}{|b|} = 1$$

* All-pass systems $\longrightarrow$ form as product of $\dfrac{z^{-1}-c^*}{1-cz^{-1}}$

<u>Example:</u> All-pass system

$$H(z) = \frac{z^{-1}-c^*}{1-cz^{-1}}, \quad c = 0.9$$



* Any arbitrary system function can be factorized as a cascade of minimum-phase and all-pass sub-systems unless it hasn't poles at unit circle ($|z| = 1$).

Minimum-phase and All-pass decomposition

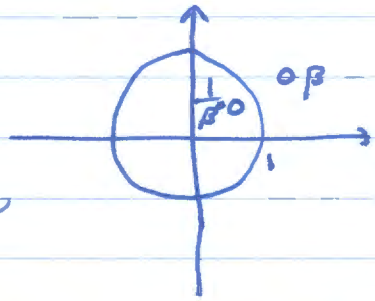
To Factor $H(z) = H_{min}(z) \cdot H_{ap}(z)$

1. Take zeros that lie outside $|z|=1$ (unit-circle) and move to $H_{ap}(z)$.
2. Add poles to $H_{ap}(z)$ in conjugate reciprocal locations of zeros.
3. Put zeros $H_{min}(z)$ to cancel poles added to $H_{ap}(z)$.

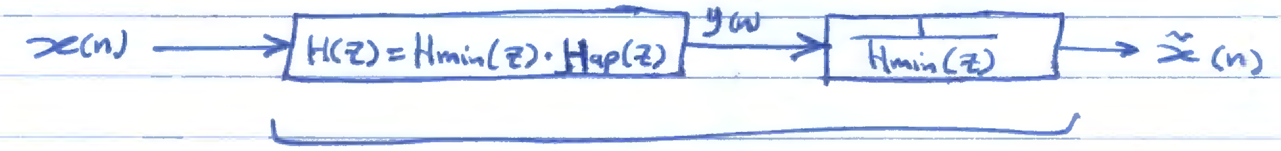
Example: suppose $H(z) = H_1(z)(1-\beta z^{-1})$; $|\beta| > 1$, $H_1(z)$ min. phase

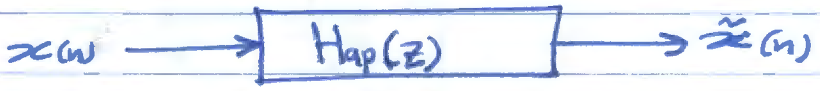
① $H(z) = H_1(z)(-\beta)(z^{-1} - \frac{1}{\beta})$ → zero $z=\beta$ in $H(z)$ reflected inside $|z|=1$.

② $$H(z) = \underbrace{H_1(z)(-\beta)\left(1-\frac{1}{\beta^*}z^{-1}\right)}_{H_{min}(z)} \cdot \underbrace{\frac{z^{-1}-\frac{1}{\beta}}{1-\frac{1}{\beta^*}z^{-1}}}_{H_{ap}(z)}$$



* The min. phase portion of any system has a stable, causal inverse system.

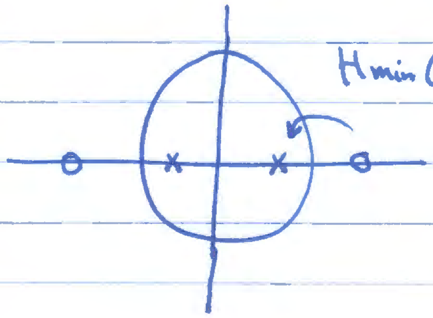


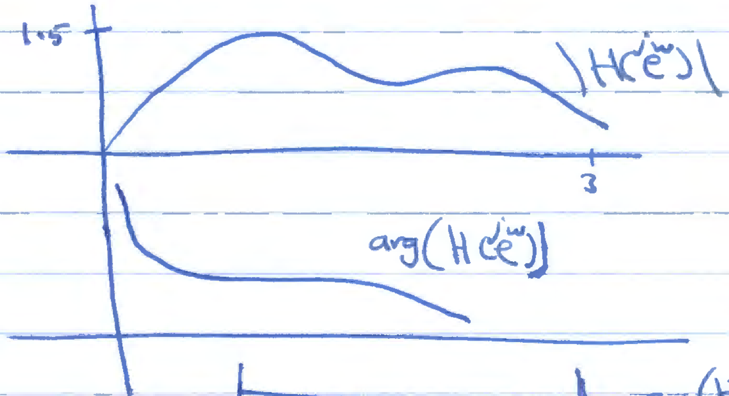


- No Magnitude distortion
- only phase distortion

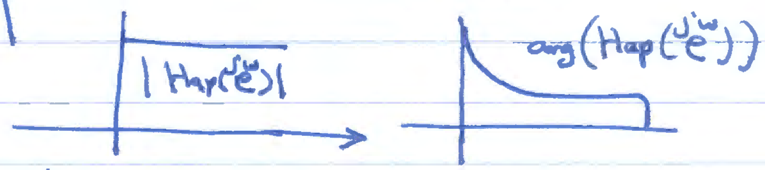
Example: decompose $H(z)$ into min-phase and All-pass

$$H(z) = \left(1-\frac{1}{0.9}z^{-1}\right)\left(1+\frac{1}{0.9}z^{-1}\right)\left(1-j0.7z^{-1}\right)\left(1+j0.7z^{-1}\right)$$





$$H_{ap}(z) = \frac{(z^{-1}-0.9)(z^{-1}+0.9)}{(1-0.9z^{-1})(1+0.9z^{-1})}$$



$$H_{min}(z) = \frac{1}{-0.8}(1-0.9z^{-1})(1+0.9z^{-1})(1-j0.7z^{-1})(1+j0.7z^{-1})$$

Example: $H(z) = \dfrac{1+5z^{-1}}{1+\frac{1}{2}z^{-1}}$, decompose $H(z)$ into H_{min} and H_{ap}?

$$H(z) = 5\,\frac{z^{-1}+\frac{1}{5}}{1+\frac{1}{2}z^{-1}}$$

$$= 5\,\frac{z^{-1}+\frac{1}{5}}{1+\frac{1}{2}z^{-1}} \cdot \frac{1+\frac{1}{5}z^{-1}}{1+\frac{1}{5}z^{-1}}$$

$$= \underbrace{5\,\frac{1+\frac{1}{5}z^{-1}}{1+\frac{1}{2}z^{-1}}}_{H_{min}(z)} \cdot \underbrace{\frac{z^{-1}+\frac{1}{5}}{1+\frac{1}{5}z^{-1}}}_{H_{ap}(z)}.$$