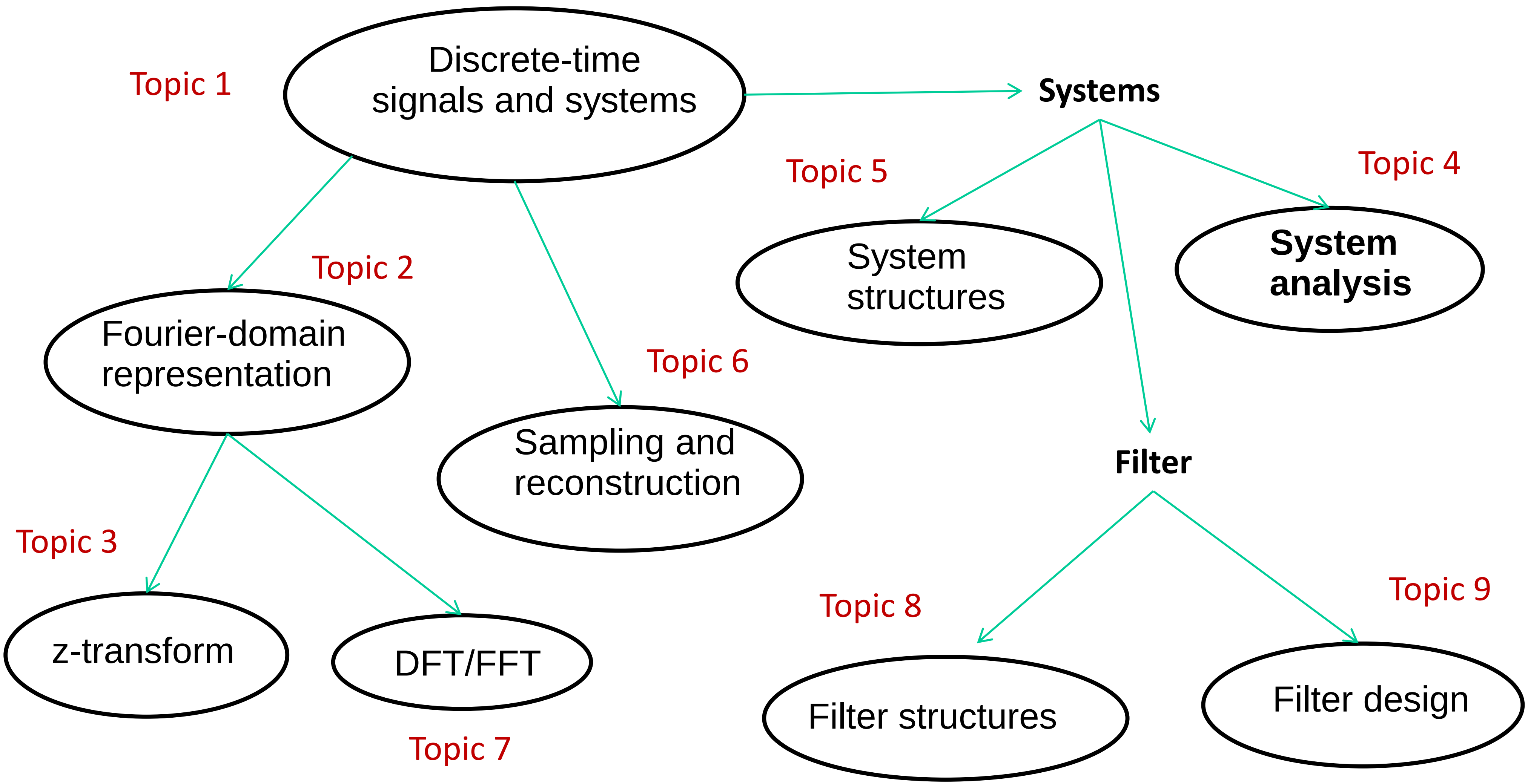
Course at a glance



System analysis

- Three domains
  - Time domain: impulse response, convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - Frequency domain: frequency response

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

  - z-transform: system function

$$Y(z) = X(z)H(z)$$

- LTI system is completed characterized by …

Part I: Frequency response

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

Frequency response

- Relationship btw Fourier transforms of input and output

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

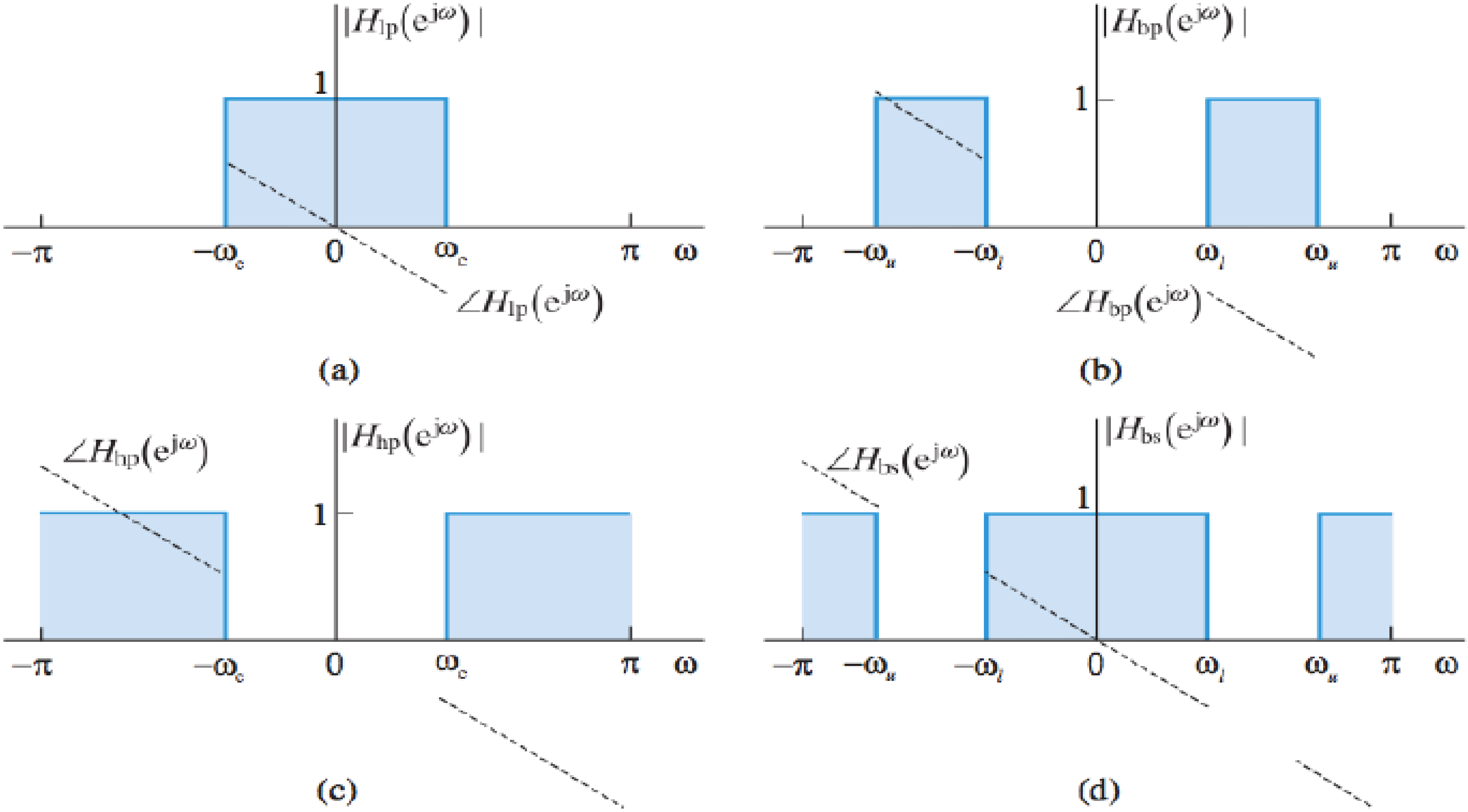
- In polar form
  - Magnitude → magnitude response, gain, distortion

$$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})|$$

  - Phase → phase response, phase shift, distortion

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

Ideal selective frequency filters



Ideal lowpass filter – an example

- Frequency response

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

  - Frequency selective filter
- Impulse response

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

  - Noncausal, cannot be implemented! $h[n] \overset{?}{=} 0, \quad n < 0$
  - How to make a noncausal system causal?

Make noncausal system causal

- Cascading systems
  - Ideal delay $h[n] = \delta[n - n_d]$

$x[n] \longrightarrow$ Forward difference $h[n] = \delta[n+1] - \delta[n]$ $\longrightarrow$ One - sameple delay $h[n] = \delta[n-1]$ $\longrightarrow y[n]$

⇓

$x[n] \longrightarrow$ Backward difference $h[n] = \delta[n] - \delta[n-1]$ $\longrightarrow y[n]$

  - In general, any noncausal FIR system can be made cause by cascading it with a sufficiently long delay!
  - But ideal lowpass filter is an IIR system!

Phase distortion and delay

- Ideal delay system

$$h_{id}[n] = \delta[n - n_d]$$

Delay distortion

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega n_d, |\omega| < \pi$$

Linear phase distortion

- Ideal lowpass filter with linear phase

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \qquad -\infty < n < \infty$$

Ideal lowpass filter is always noncausal!

Group delay

It is a convenient measure of the linearity of the phase.

Defined as negative slope of the phase at specific frequency w_0

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}.$$

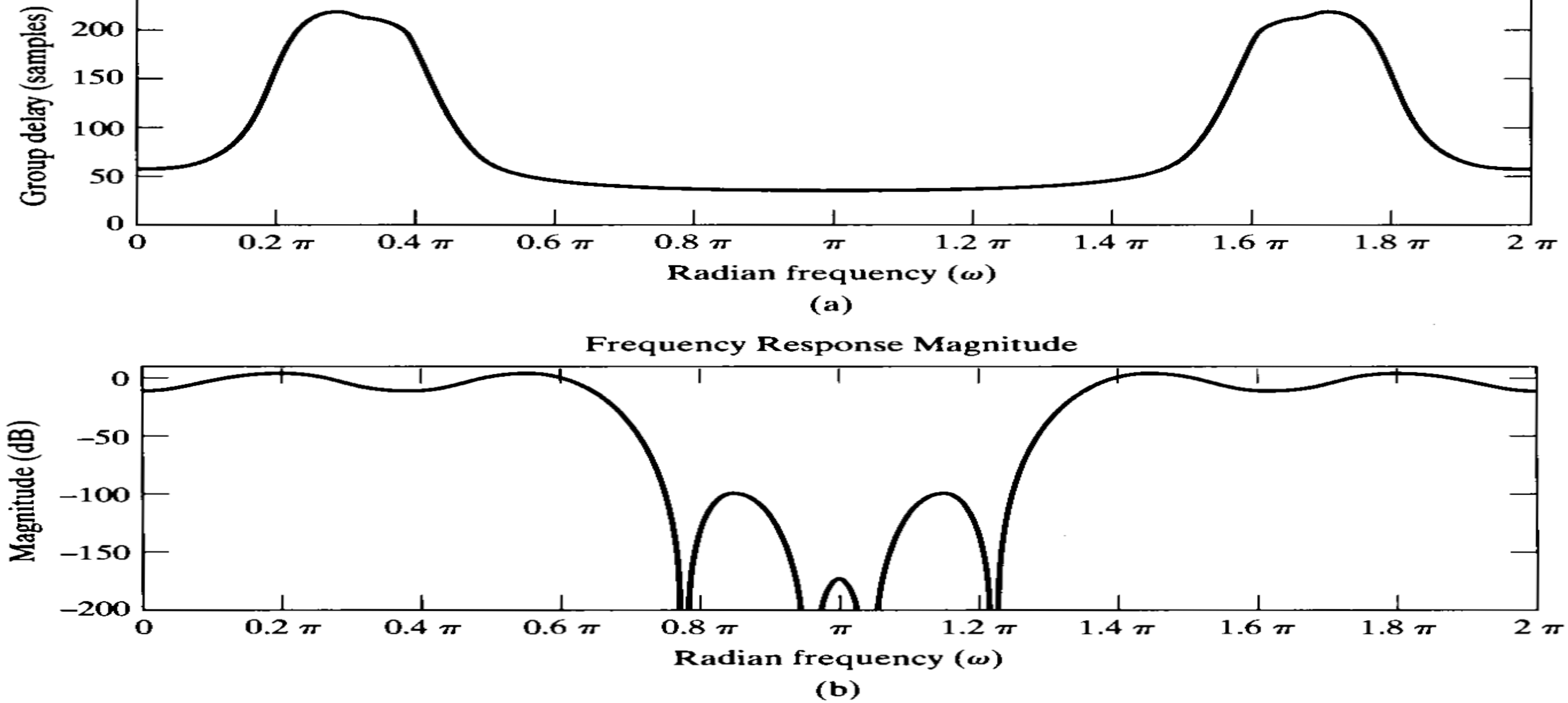
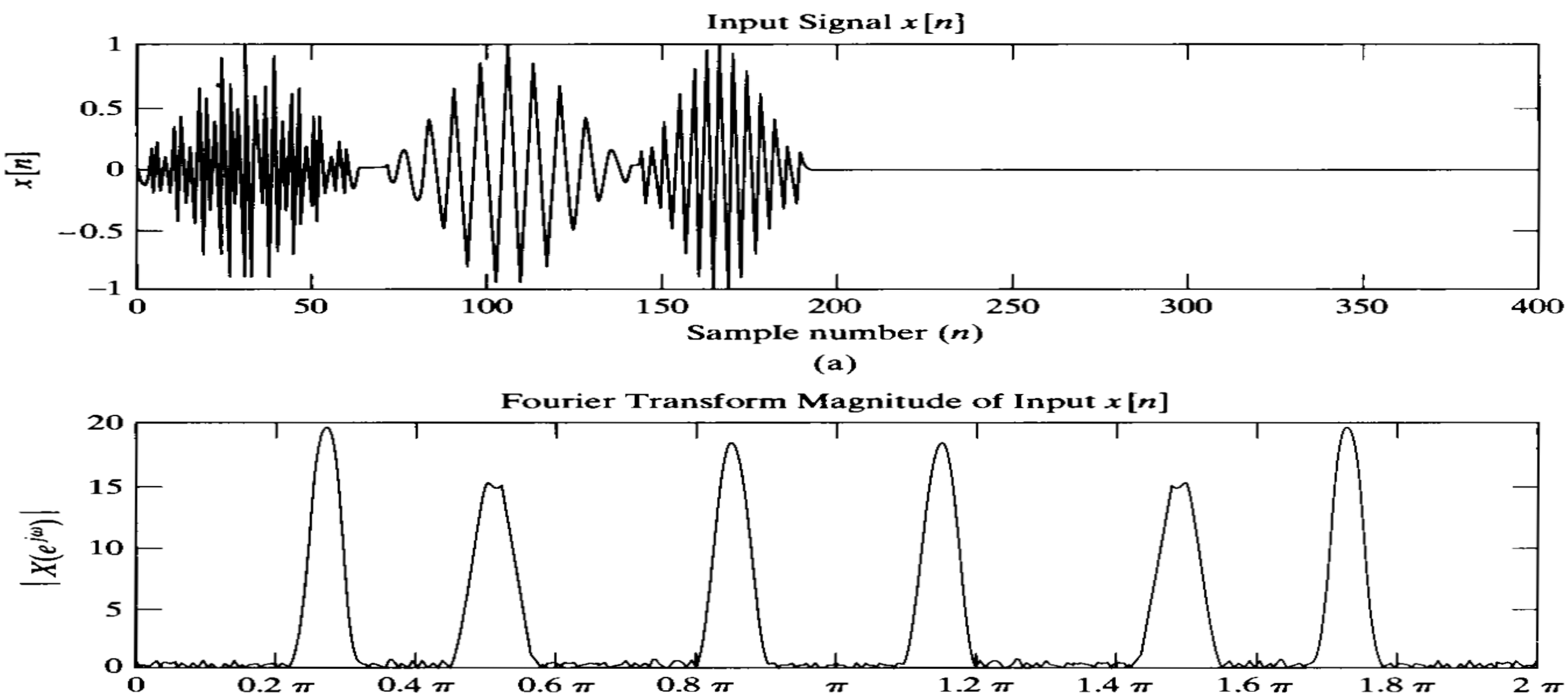


Figure 5.1 Frequency response magnitude and group delay for the filter in Example 5.1.



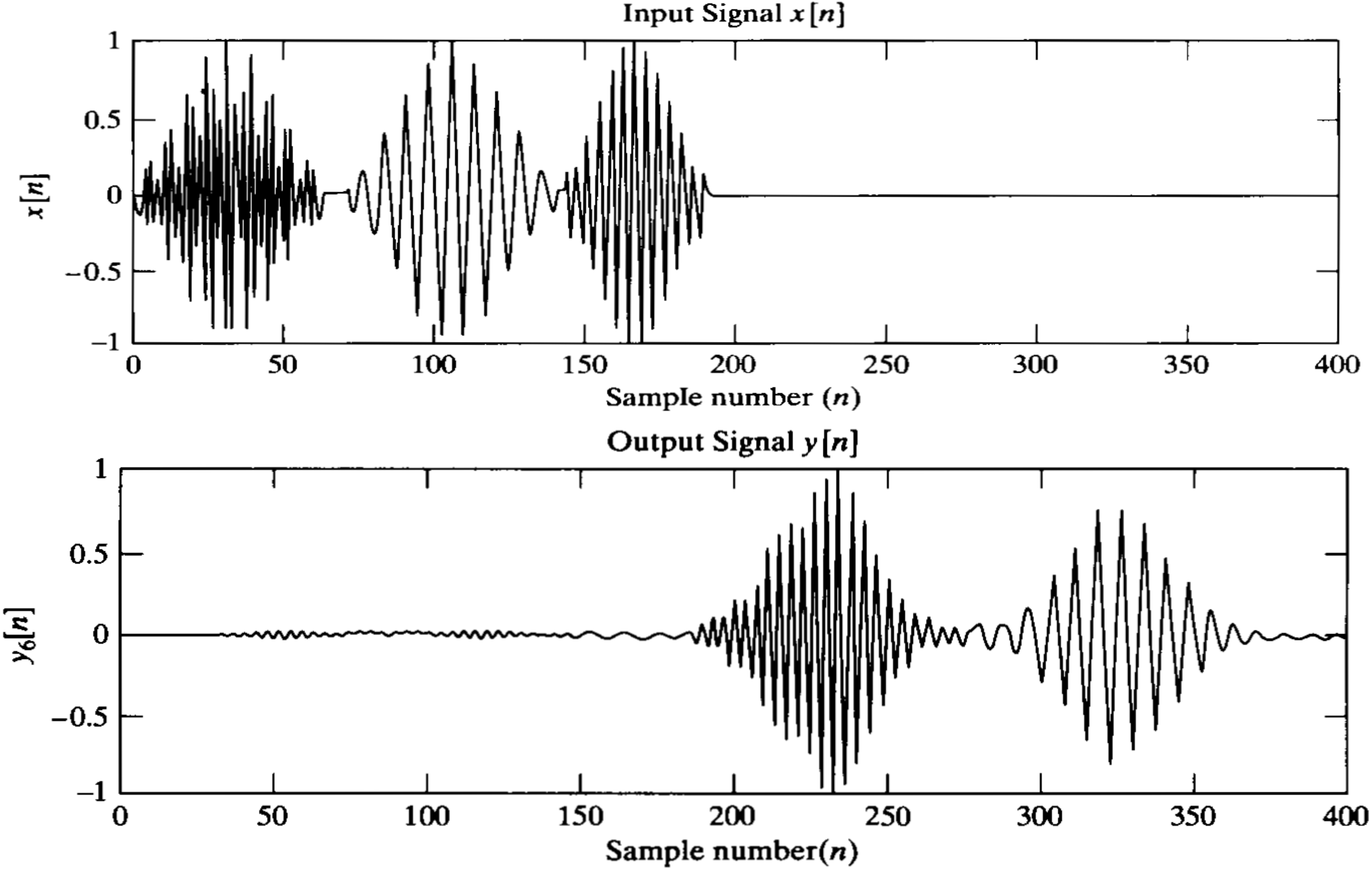


Figure 5.3 Output signal for Example 5.1.

Since the filter has considerable attenuation at $\omega = 0.85\pi$, the pulse at that frequency is not clearly present in the output. Also, since the group delay at $\omega = 0.25\pi$ is approximately 200 samples and at $\omega = 0.5\pi$ is approximately 50 samples, the second pulse in $x[n]$ will be delayed by about 200 samples and the third pulse by 50 samples, as we see is the case in Figure 5.3.

System function of LCCDE systems

- Linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- z-transform format

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \left(\frac{b_0}{a_0}\right) \frac{\prod_{m=1}^{M} (1 - c_m z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$(1 - c_m z^{-1})$ in the numerator

a zero at $z = c_m$ a pole at $z = 0$

$(1 - d_k z^{-1})$ in the denominator

a zero at $z = 0$ a pole at $z = d_k$

Stability and causality

- Stable
  - h[n] absolutely summable
  - H(z) has a ROC including the unit circle
- Causal
  - h[n] right side sequence
  - H(z) has a ROC being outside the outermost pole

Inverse systems

- Many systems have inverses, specially systems with rational system functions

$$G(z) = H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

$$g[n] = h[n] * h_i[n] = \delta[n]$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{m=1}^{M}(1 - c_m z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{m=1}^{M}(1 - c_m z^{-1})}$$

  - Poles become zeros and vice versa.
  - ROC: must have overlap btw the two for the sake of G(z).

Example

$$H(z)=\frac{1-0.5z^{-1}}{1-0.9z^{-1}},|z|>0.9$$

$$H_i(z)=\frac{1-0.9z^{-1}}{1-0.5z^{-1}}=\frac{1}{1-0.5z^{-1}}-\frac{0.9z^{-1}}{1-0.5z^{-1}}$$

So, $|z|>0.5$

$$h_i[n]=(0.5)^n u[n]-0.9(0.5)^{n-1}u[n-1]$$

$$H(z)=\frac{z^{-1}-0.5}{1-0.9z^{-1}}, \qquad |z|>0.9.$$

$$H_i(z)=\frac{1-0.9z^{-1}}{z^{-1}-0.5}=\frac{-2+1.8z^{-1}}{1-2z^{-1}}.$$

The corresponding impulse response for an ROC $|z|<2$ is

$$h_{i1}[n]=2(2)^n u[-n-1]-1.8(2)^{n-1}u[-n]$$

and, for an ROC $|z|>2$, is

$$h_{i2}[n]=-2(2)^n u[n]+1.8(2)^{n-1}u[n-1].$$

We see that $h_{i1}[n]$ is stable and noncausal, while $h_{i2}[n]$ is unstable and causal.

Inverse systems ..cont.

If H(Z) is causal system with zeros at $C_k, k = 1, \ldots M$

Then the inverse system will be causal if and only if we associate ROC:

$$|z| > \max_k |c_k|$$

With $H_i(Z)$

If we require inverse system to be stable, then its ROC must include unit circle, which means that:

$$\max_k |c_k| < 1$$

i.e. all zeros of H(z) must be inside unit circle

Minimum-phase systems

- Magnitude does not uniquely characterize the system
  - Stable and causal → poles inside unit circle, no restriction on zeros
  - Zeros are also inside unit circle → inverse system is also stable and causal (in many situations, we need inverse systems!)
  - → such systems are called minimum-phase systems (explanation to follow): are stable and causal and have stable and causal inverses

All-Pass System

A system with frequency response magnitude constant

Important uses such as compensating for phase distortion

Simple all-pass system

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$H_{ap}\left(e^{j\omega}\right) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\,\frac{1 - a^*e^{j\omega}}{1 - ae^{-j\omega}}$$

Magnitude response constant

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1} - d_k}{1 - d_k z^{-1}}\prod_{k=1}^{M_c}\frac{\left(z^{-1} - c_k^*\right)\left(z^{-1} - c_k\right)}{\left(1 - c_k z^{-1}\right)\left(1 - c_k^* z^{-1}\right)}$$

All-Pass system … cont.

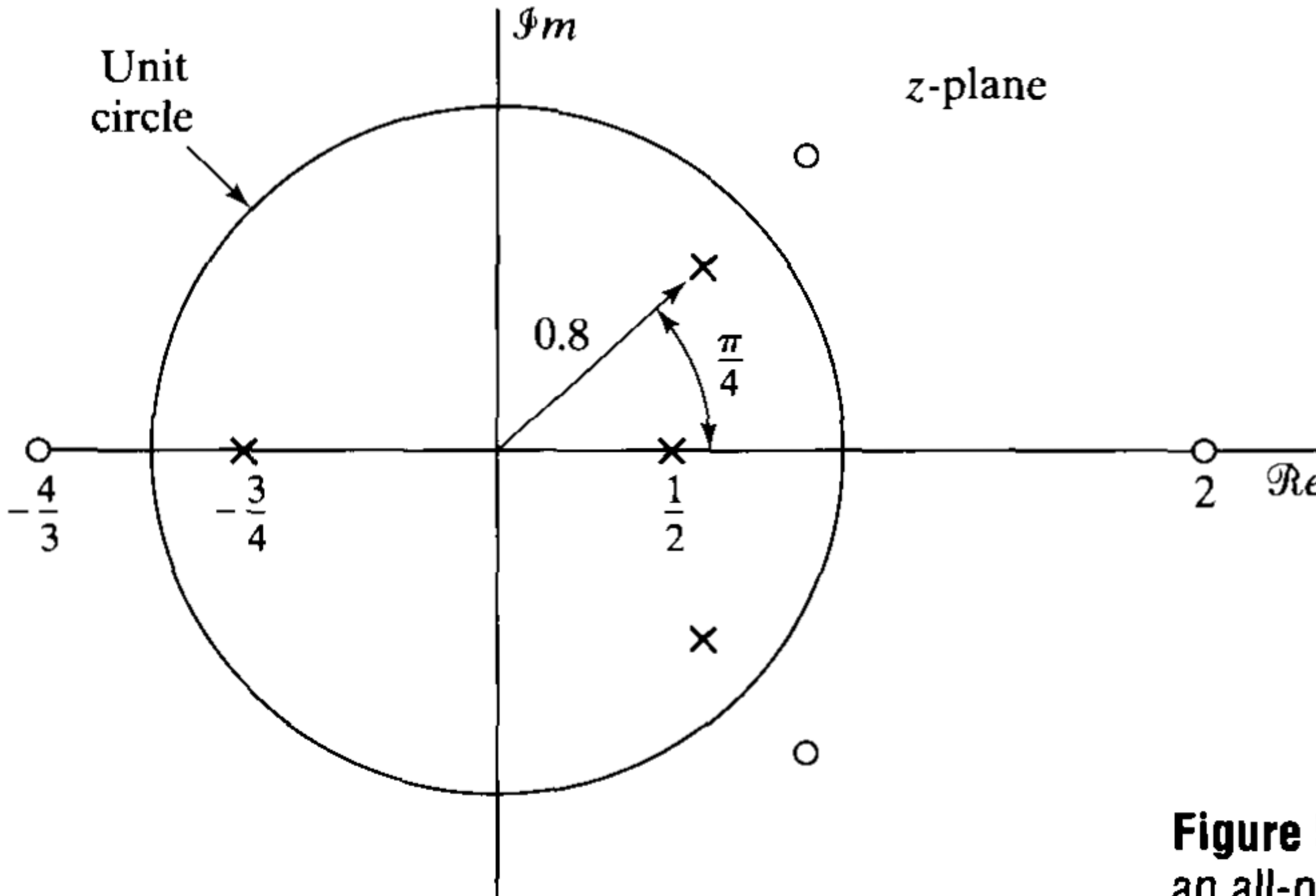


Figure 5.21 Typical pole–zero plot for an all-pass system.

Minimum phase and allpass decomposition

$$\mathbf{H(z) = H_{min}(z)\ H_{ap}(z)}$$

1. take zeros that lie outside unit-circle to $H_{ap}(z)$.
2. Add poles to $H_{ap}(z)$ in conjugate reciprocal locations of zeros.
3. Put zeros $H_{min}(z)$ to cancel poles added to $H_{ap}(z)$

Example

$$H(z)=\frac{1+5z^{-1}}{1+\frac{1}{2}z^{-1}} = H_{min}(z)\,H_{ap}(z).$$

Find $H_{min}(z)$ and $H_{ap}(z)$?

Matlab

-freqz(b,a) => plot both magnitude and phase response

-abs() => compute the magnitude

-angle() => compute the phase

-grpdelay(b,a,N) => compute and plot group delay

Example:

$$H(z) = \frac{1 + 1.655z^{-1} + 1.655z^{-2} + z^{-3}}{1 - 1.57z^{-1} + 1.264z^{-2} - 0.4z^{-3}}$$