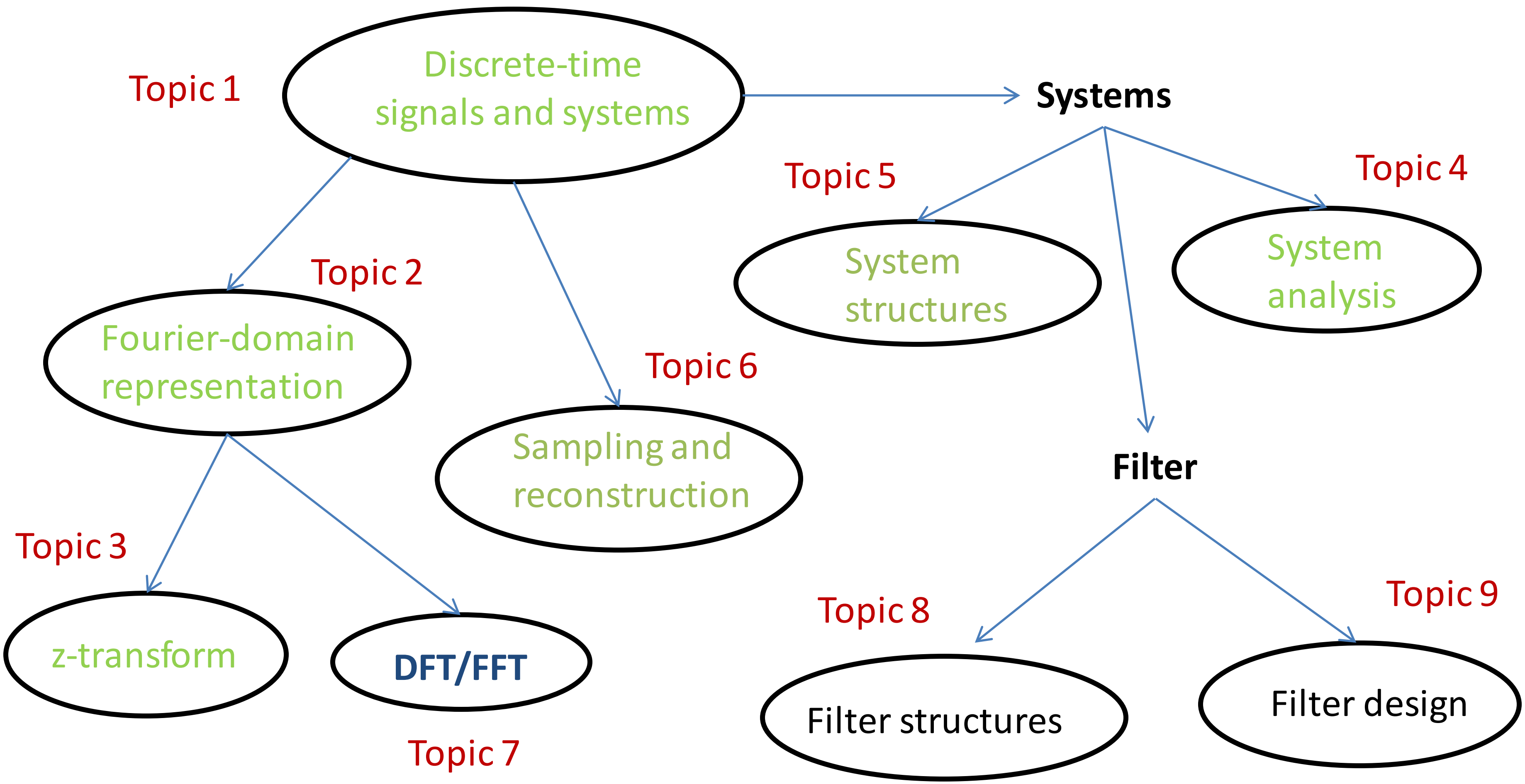
Discrete Fourier Transform (DFT)

Chapter 8 in the textbook

Course at a glance



The discrete-time Fourier transform (DTFT)

- The DTFT is useful for the theoretical analysis of signals and systems.
- But, according to its definition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

computation of DTFT by computer has several problems:

  - The summation over n is infinite
  - The independent variable w is continuous

The discrete Fourier transform (DFT)

- In many cases, only finite duration is of concern
  - The signal itself is finite duration
  - Only a segment is of interest at a time
  - Signal is periodic and thus only finite unique values
- For finite duration sequences, an alternative Fourier representation is DFT
  - The summation over n is finite
  - DFT itself is a sequence, rather than a function of a continuous variable
  - Therefore, DFT is computable and important for the implementation of DSP systems
  - DFT corresponds to samples of the Fourier transform

Part I: The discrete Fourier series

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

The discrete Fourier series

- A periodic sequence with period N

$$\tilde{x}[n] = \tilde{x}[n + rN]$$

- Periodic sequence can be represented by a Fourier series, i.e. a sum of complex exponential sequences with frequencies being integer multiples of the fundamental frequency $(2\pi / N)$ associated with the $\tilde{x}[n]$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k} \tilde{X}[k] e^{j(2\pi/N)kn}$$

The frequency of the periodic sequence.

  - Only N unique harmonically related complex exponentials since

$$e^{j(2\pi/N)(k+mN)n} = e^{j(2\pi/N)kn} e^{j2\pi mn} = e^{j(2\pi/N)kn}$$

  - so $$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

The Fourier series coefficients

- The coefficients

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)kn}$$

- The sequence is periodic with period N

$$\tilde{X}[k+N] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)(k+N)n} = \tilde{X}[k]$$

- For convenience, define $W_N = e^{-j(2\pi/N)}$

Synthesis equation $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$

Analysis equation $\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn}$

Very similar equations
→ duality

Periodic Convolution

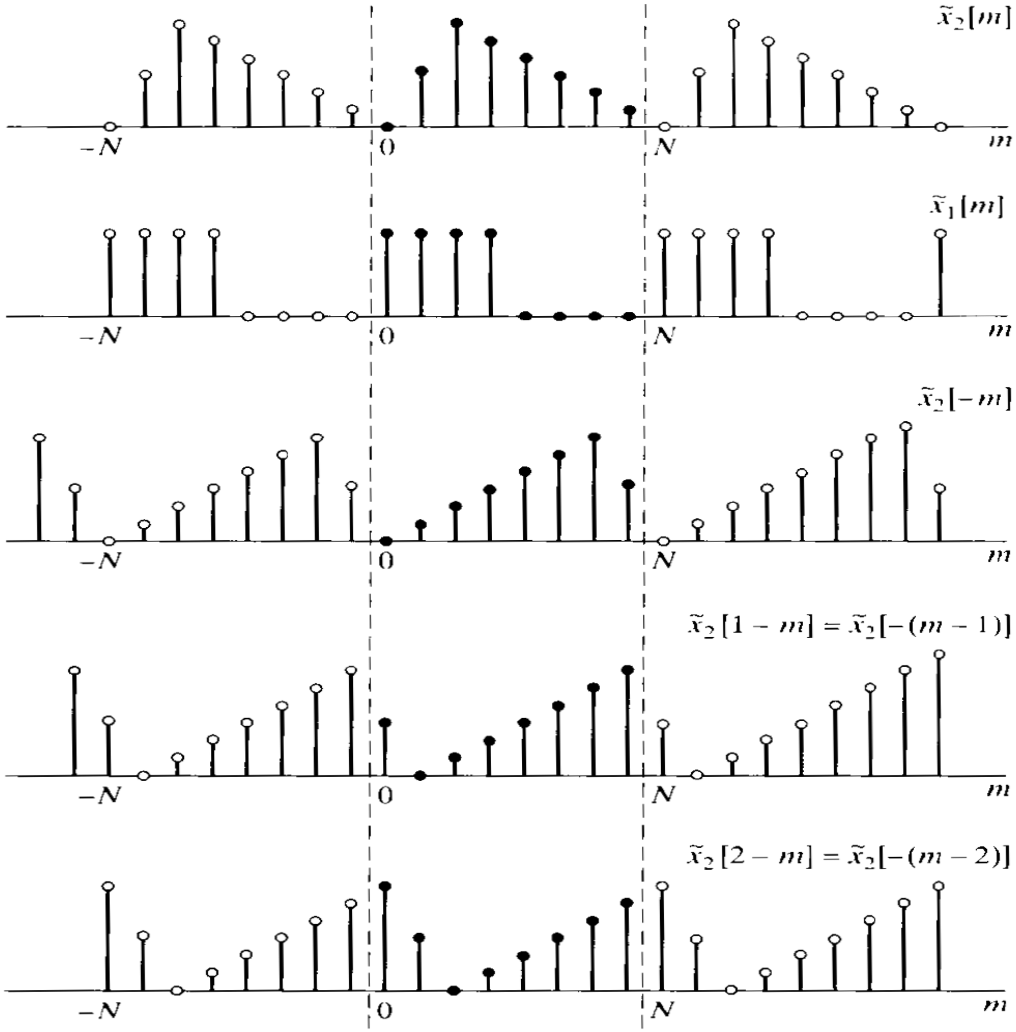


Figure 8.3 Procedure for forming the periodic convolution of two periodic sequences.

DFS of a periodic impulse train

- Periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

- The discrete Fourier series coefficients

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = 1$$

- By using synthesis equation, an alternative representation of $\tilde{x}[n]$ is

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}$$

Part II: The Fourier transform of periodic signals

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

The Fourier transform of periodic signals

- Fourier transform of complex exponentials

$$x[n] = \sum_k a_k e^{j\omega_k n}, \qquad -\infty < n < \infty$$

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_k 2\pi a_k \delta(\omega - \omega_k + 2\pi r)$$

- Fourier transform of $\tilde{x}[n]$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k] \delta(\omega - \frac{2\pi k}{N})$$

$\tilde{X}(e^{j\omega})$ has the required periodicity with period 2π

Fourier transform of a periodic impulse train

- Periodic impulse train $\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$
  - The discrete Fourier series coefficients
    $$\tilde{P}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = 1$$
  - Fourier transform
    $$\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta(\omega - \frac{2\pi k}{N})$$
- Finite duration signal $x[n]$ ($x[n] = 0$ outside of $[0, N-1]$)
  - Construct $\tilde{x}[n]$
    $$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta(n - rN) = \sum_{r=-\infty}^{\infty} x(n - rN)$$
  - Its Fourier transform
    $$\tilde{X}(e^{j\omega}) = X(e^{j\omega}) \tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j(2\pi/N)k}) \delta(\omega - \frac{2\pi k}{N})$$

The Fourier transform of periodic signals

- Compare

$$\tilde{X}(e^{j\omega}) = X(e^{j\omega})\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j(2\pi/N)k})\delta(\omega - \frac{2\pi k}{N})$$

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k]\delta(\omega - \frac{2\pi k}{N})$$

→ First represent it as Fourier series and then calculate Fourier transform

- Conclude that

$$\tilde{X}[k] = X(e^{j(2\pi/N)k}) = X(e^{j\omega})\big|_{\omega=(2\pi/N)k}$$

i.e. the DFS coefficients of $\tilde{x}[n]$ are samples of the Fourier transform of the one period of $\tilde{x}[n]$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Part III: Sampling the Fourier transform

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

Sampling the Fourier transform

- An aperiodic sequence and its Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \leftrightarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Sampling the Fourier transform

$$\tilde{X}[k] = X(e^{j\omega})\,|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k})$$

  - generates a periodic sequence in k with period N since the Fourier transform is periodic in ω with period 2π

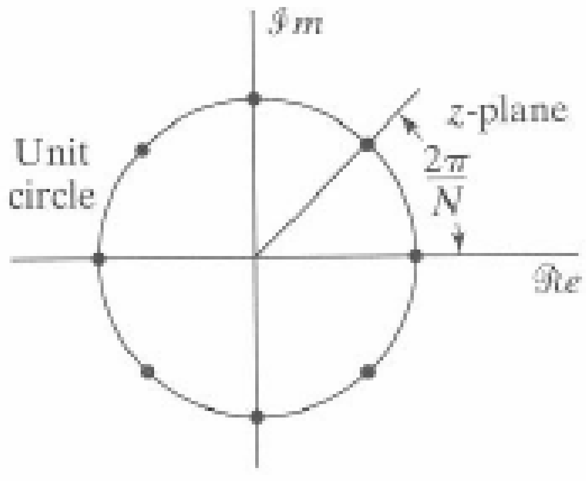


Figure 8.7 Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ ($N = 8$).

Sampling the Fourier transform

- Now we want to see if the sampling sequence $\tilde{X}[k]$ is the sequence of DFS coefficients of a sequence $\tilde{x}[n]$ this can be done by using the synthesis equation

$$\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$$

$$=\frac{1}{N}\sum_{k=0}^{N-1}\Big[\sum_{m=-\infty}^{\infty}x[m]e^{-j(2\pi/N)km}\Big]W_N^{-kn}$$

$$=\sum_{m=-\infty}^{\infty}x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1}W_N^{-k(n-m)}\right]=\sum_{m=-\infty}^{\infty}x[m]\tilde{p}[n-m]$$

$$=\sum_{r=-\infty}^{\infty}x[n-rN]$$

$=\tilde{x}[n]$ **A periodic sequence resulting from aperiodic convolution**

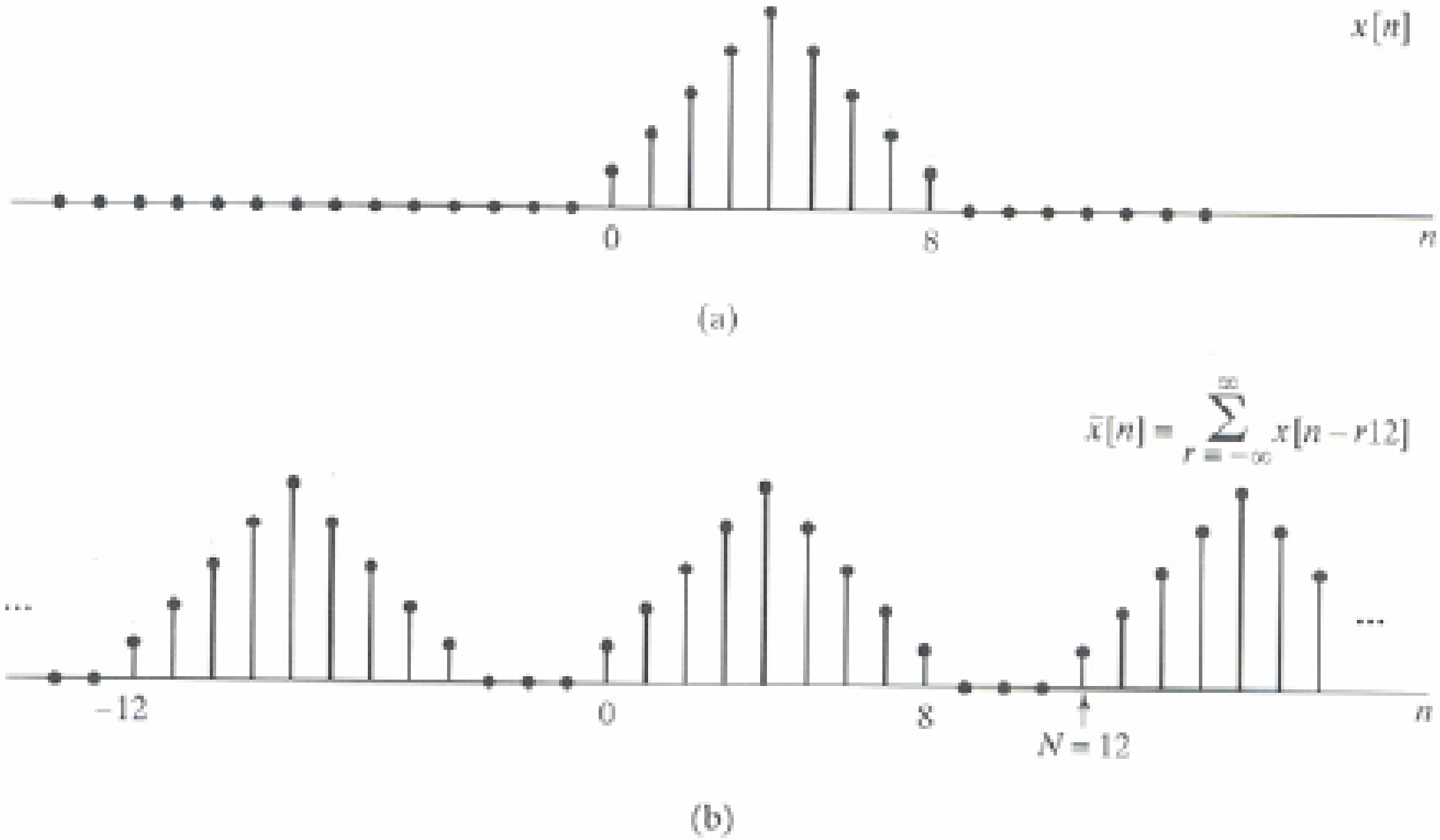


Figure 8.8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.

- In this case, the Fourier series coefficients for a periodic sequence are samples of the Fourier transform of one period

Examples

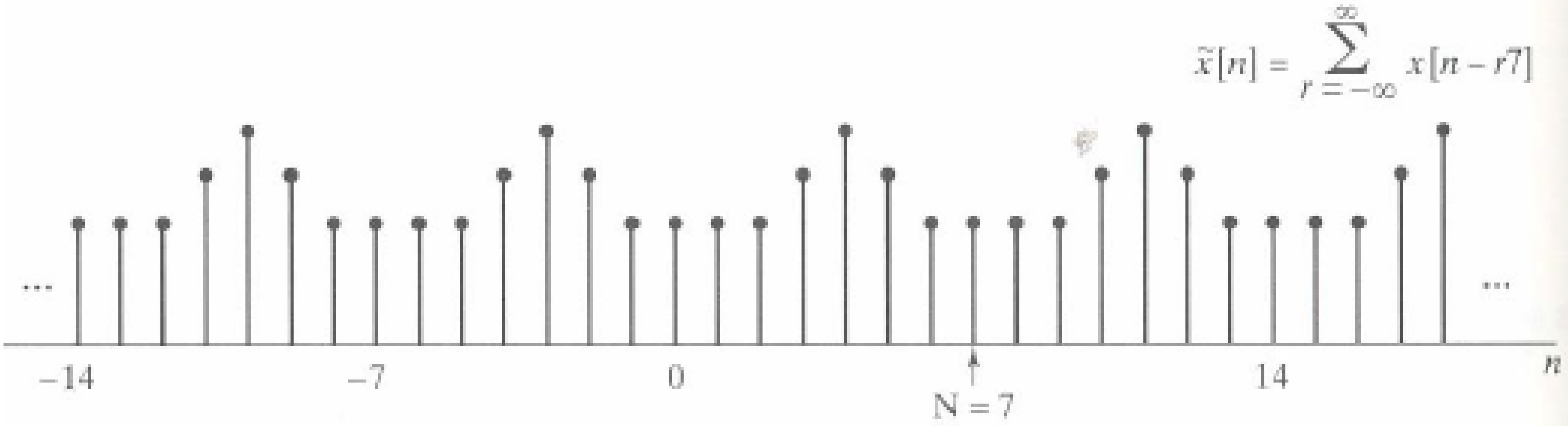


Figure 8.9 Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ in Figure 8.8(a) with $N = 7$.

- In this case, still the Fourier series coefficients for $\tilde{x}[n]$ are samples of the Fourier transform of $x[n]$. But, one period of $\tilde{x}[n]$ is no longer identical to $x[n]$
- This is just sampling in the frequency domain as compared in the time domain discussed before.

Sampling in the frequency domain

- The relationship between $x[n]$ and one period of $\tilde{x}[n]$ in the undersampled case is considered a form of time domain aliasing.
- Time domain aliasing can be avoided only if $x[n]$ has finite length, just as frequency domain aliasing can be avoided only for signals being bandlimited.
- If $x[n]$ has finite length and we take a sufficient number of equally spaced samples of its Fourier transform (specifically, a number greater than or equal to the length of $x[n]$), then the Fourier transform is recoverable from these samples, equivalently $x[n]$ is recoverable from $\tilde{x}[n]$.

Sampling in the frequency domain

- Recovering $x[n]$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

  i.e. recovering $x[n]$ does not require to know its Fourier transform at all frequencies

- Application: represent finite length sequence by using Fourier series (coefficients) → DFT

$$x[n] \rightarrow \tilde{x}[n] \rightarrow \underline{DFS, \tilde{X}[k]} \rightarrow \tilde{x}[n] \rightarrow x[n]$$

Sampling the Fourier transform

- Fourier transform $X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

- Discrete-time Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Discrete Fourier transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

Part IV: The DFT

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

The discrete Fourier transform

- Consider a finite length sequence $x[n]$ of length N samples (if smaller than N, appending zeros)
  - Construct a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

Assuming no overlap btw $x[n - rN]$

$$\tilde{x}[n] = x[(n \text{ modulo } N)] = x[((n))_N]$$

  - Recover the finite length sequence

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

- To maintain a duality btw the time and frequency domains, choose one period of $\tilde{X}[k]$ as the DFT

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

The DFT

- Periodic sequence and DFS coefficients

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \underline{\tilde{x}[n]} W_N^{kn}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \underline{\tilde{X}[k]} W_N^{-kn}$$

- Since summations are calculated btw 0 and (N-1)

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} \underline{x[n]} W_N^{kn}, & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \underline{X[k]} W_N^{-kn}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Generally

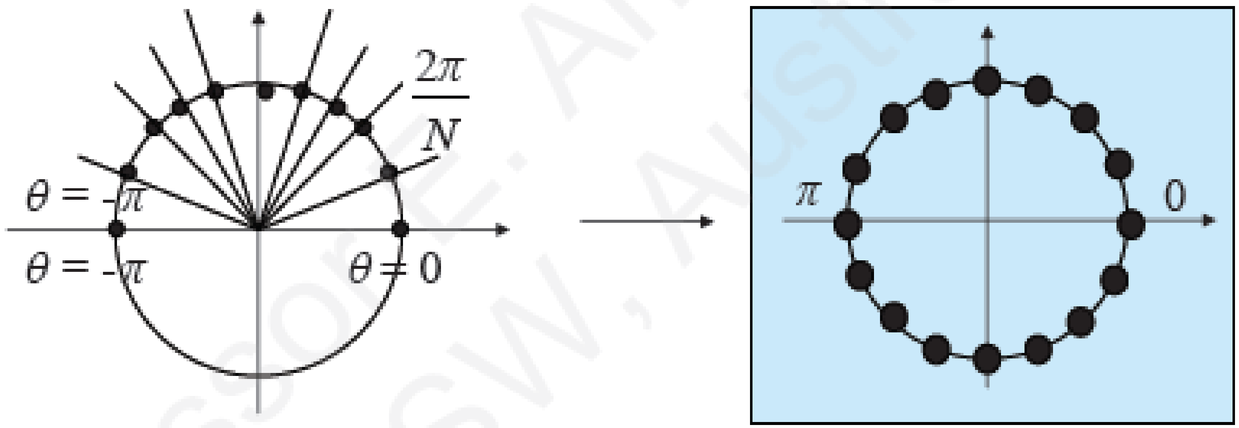
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

The DFT

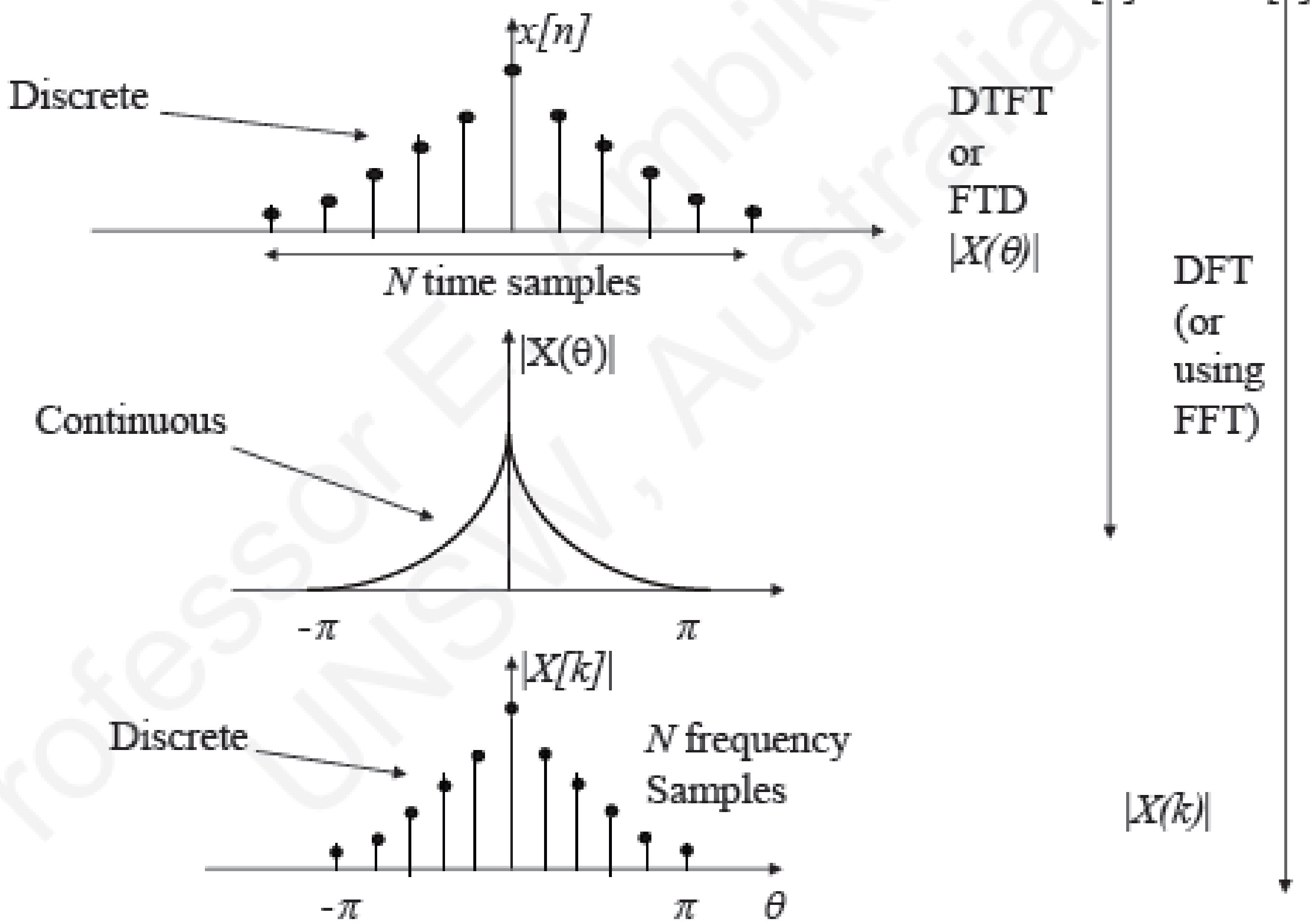
- A finite or periodic sequence has only N unique values, $x[n]$ for $0<=n<N$
- Spectrum is completely defined by N distinct frequency samples
- DFT: uniform sampling of DTFT spectrum

- The DFT corresponds to sampling the z-transform of $X(z)$ at N-points equally spaced in angle around the unit circle.



N points equally spaced on the unit circle

Note:



x[n]
Discrete
N time samples
DTFT
or
FTD
|X(θ)|
|X(θ)|
Continuous
-π
π
|X[k]|
Discrete
N frequency
Samples
-π
π
θ
x[n]
x[n]
DFT
(or
using
FFT)
|X(k)|

Example:

Let $f_s = 8000$ Hz, Number of sample(N) = 1000

Frequency resolution = $\dfrac{f_s}{N} = \dfrac{8000}{1000} = 8Hz$

$f_0 = 0$, $f_1 = 8\text{Hz}$, $f_2 = 16$ Hz , …. $f_{999} = 8000\text{Hz}$

$$X[k] = \sum_{0}^{999} x[n] e^{-jn\left(\frac{2\pi k}{1000}\right)}$$

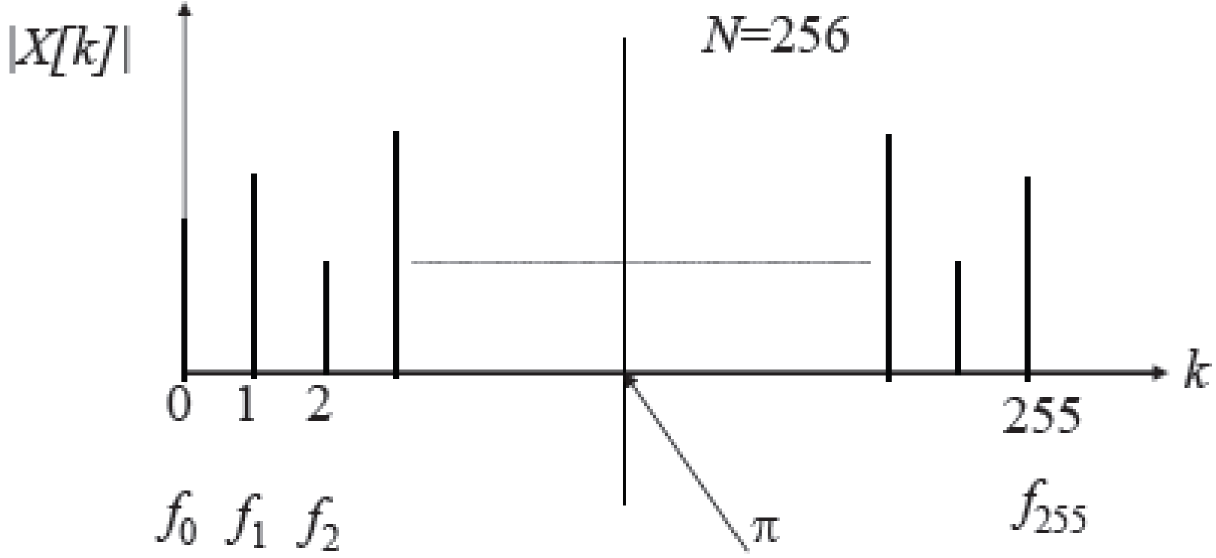
$k = 0,1,2,3, \ldots$

Example :

- A speech signal is sampled at a rate of 20000 samples/sec. A sequence of length (N) 1024 samples is selected and the 1024-point DFT is computed.

  (1) What is the time duration of segment of speech?
  Duration = no of samples $\times$ sampling period.
  = 1024 (1/20000) = 51.2 ms
  (2) What is the frequency resolution (spacing in Hz) between the DFT values.

$$\text{Resolution} = \frac{f_s}{N} = \frac{20000}{1024} = 19.531\,Hz$$



The frequency resolution (Δf) can be made as small as desired by increasing the value of N (window size being analysed)

Padding with Zeros and frequency Resolution

$$DFT: \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-jn\frac{2\pi k}{N}} \qquad k = 0,1,2,3,\ldots.N$$

- To obtain more points in the DFT sequence, we can always increase the duration of $x[n]$ by adding additional zero-valued elements. This procedure is called padding with zeros.

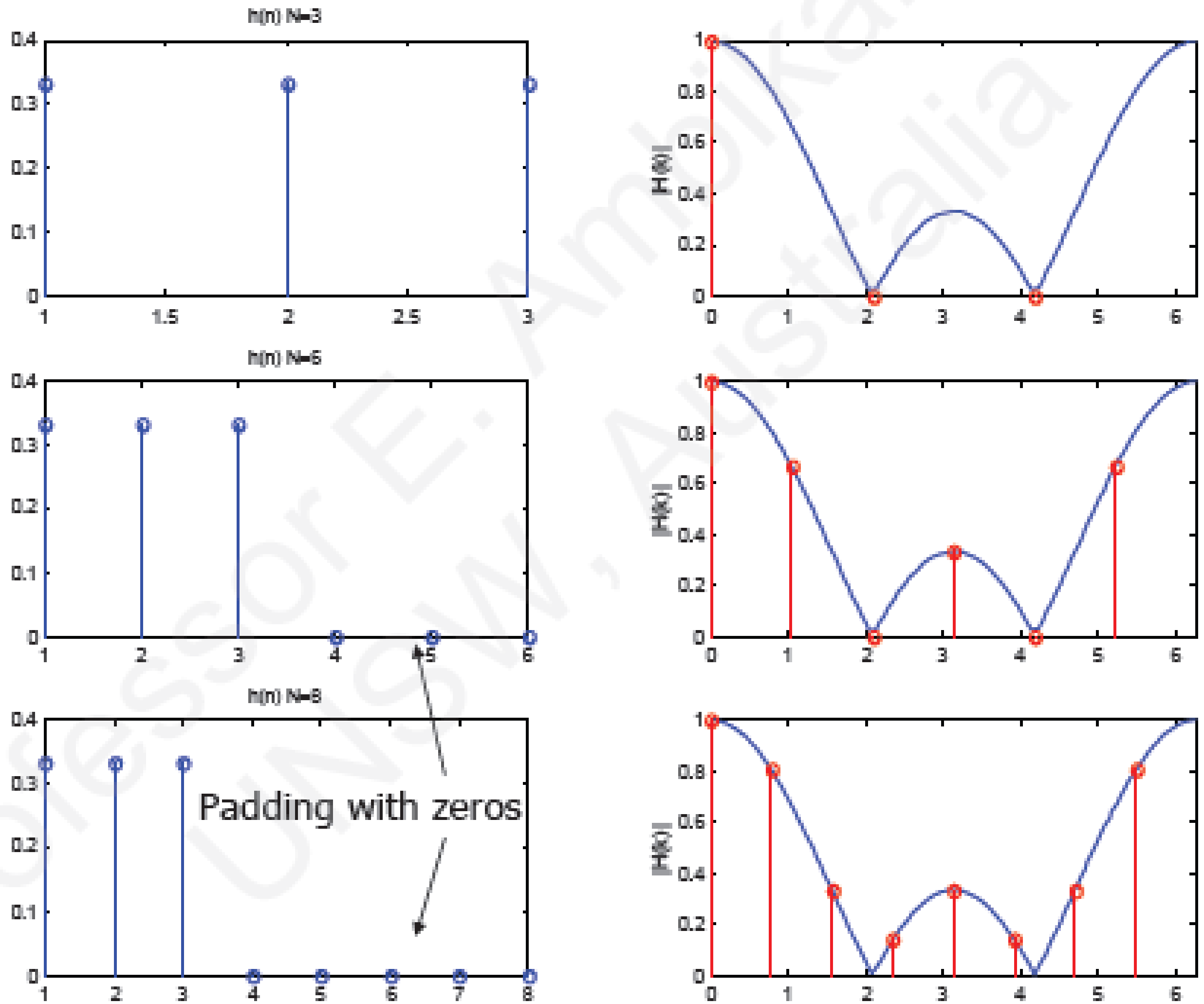
- These zero-valued elements contribute nothing to the sum in the above equation, but act to decrease the frequency spacing $(2\pi/N)$.

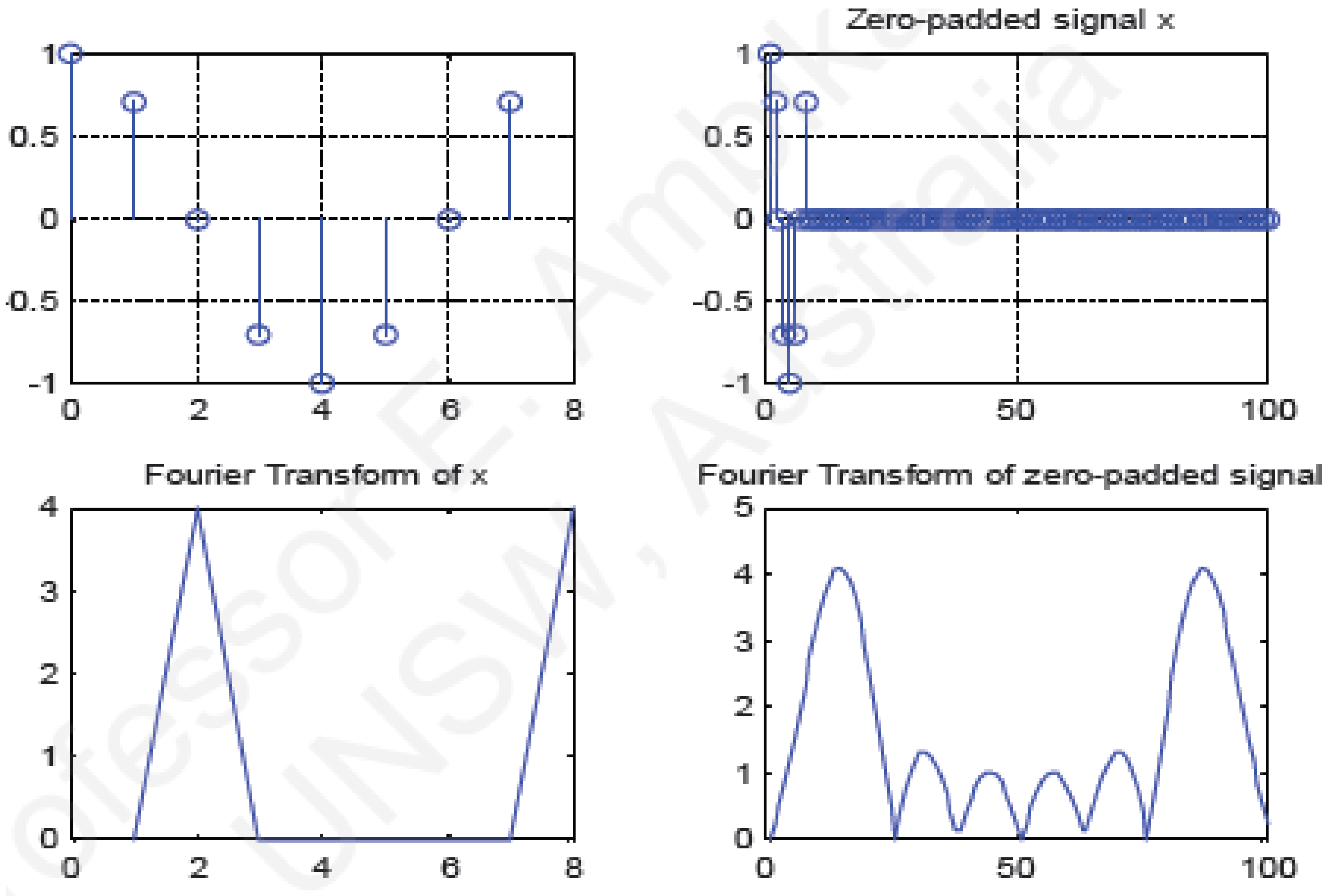
- The zero padding gives us a high-density spectrum and provided a better displayed version for plotting.

- But it does not give us a high resolution spectrum because no new information is added to the signal.

- Only additional zeros are added in the data.

Three Sample Averager





Zero-padded signal x
Fourier Transform of x
Fourier Transform of zero-padded signal
1
0.5
0
-0.5
-1
0
2
4
6
8
50
100
3
5

Part V: Properties of the DFT

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
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- Linear convolution using the DFT

Properties of the DFT – linearity

Linearity

$$a x_1[n] + b x_2[n] \stackrel{DFT}{\leftrightarrow} a X_1[k] + b X_2[k]$$

The lengths of sequences and their DFTs are all equal to the maximum of the lengths of $x_1[n]$ and $x_2[n]$

Circular shift of a sequence

- Given

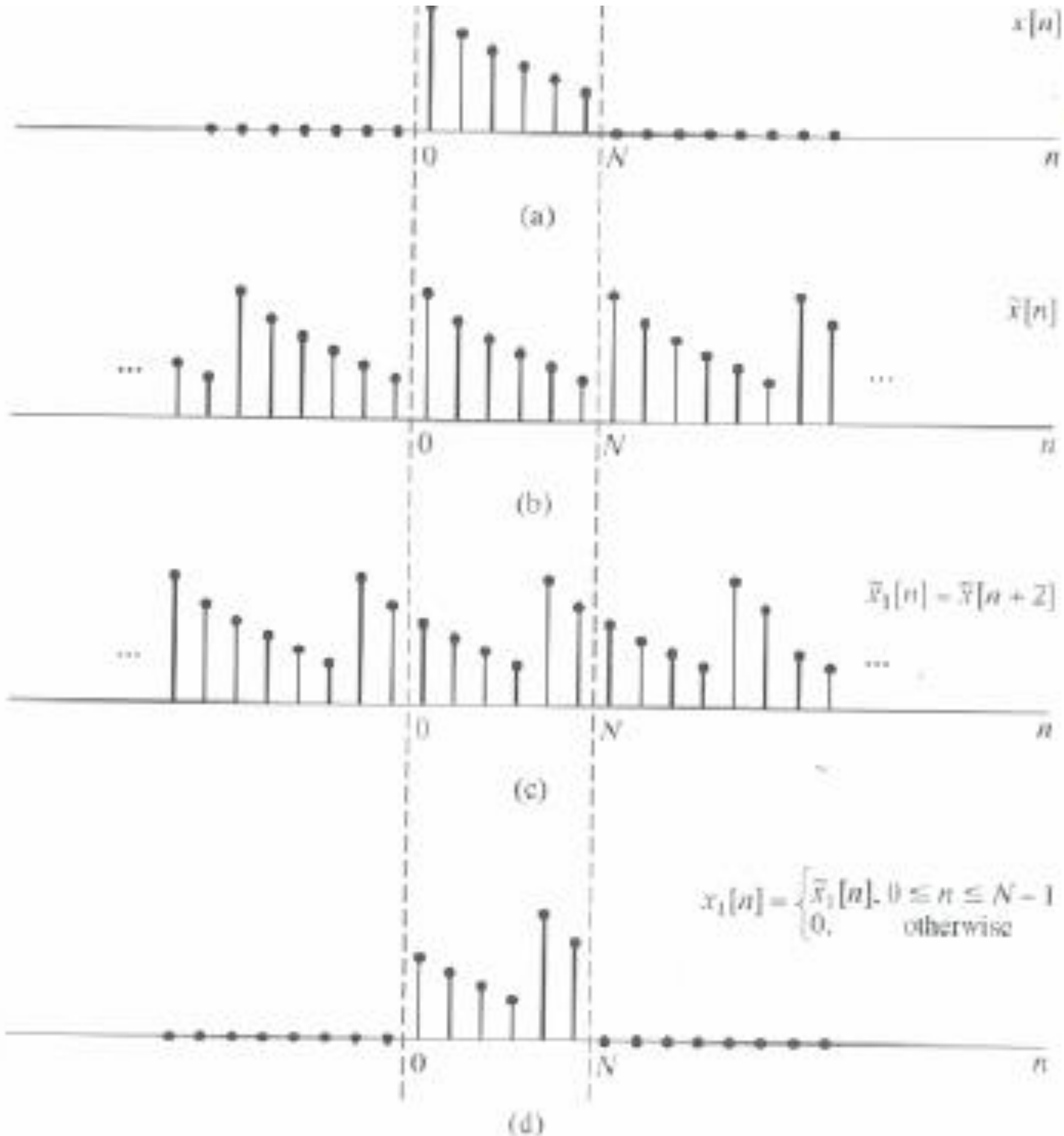
$$x[n] \overset{DFT}{\leftrightarrow} X[k]$$

$$x_1[n] \overset{DFT}{\leftrightarrow} X_1[k] = e^{-j(2\pi k/N)m} X[k]$$

- Then

$$x_1[n] = \begin{cases} \tilde{x}_1[n] = \tilde{x}[n-m] = x[((n-m))_N], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

an example



$x[n]$
0
N
n
(a)
$\tilde{x}[n]$
...
(b)
$\tilde{x}_1[n] = \tilde{x}[n+2]$
(c)
$x_1[n] = \begin{cases} \tilde{x}_1[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$
(d)

Duality

$$x[n] \overset{DFT}{\leftrightarrow} X[k]$$

$$X[n] \overset{DFT}{\leftrightarrow} Nx[((-k))_N], \quad 0 \le k \le N-1$$

Circular convolution

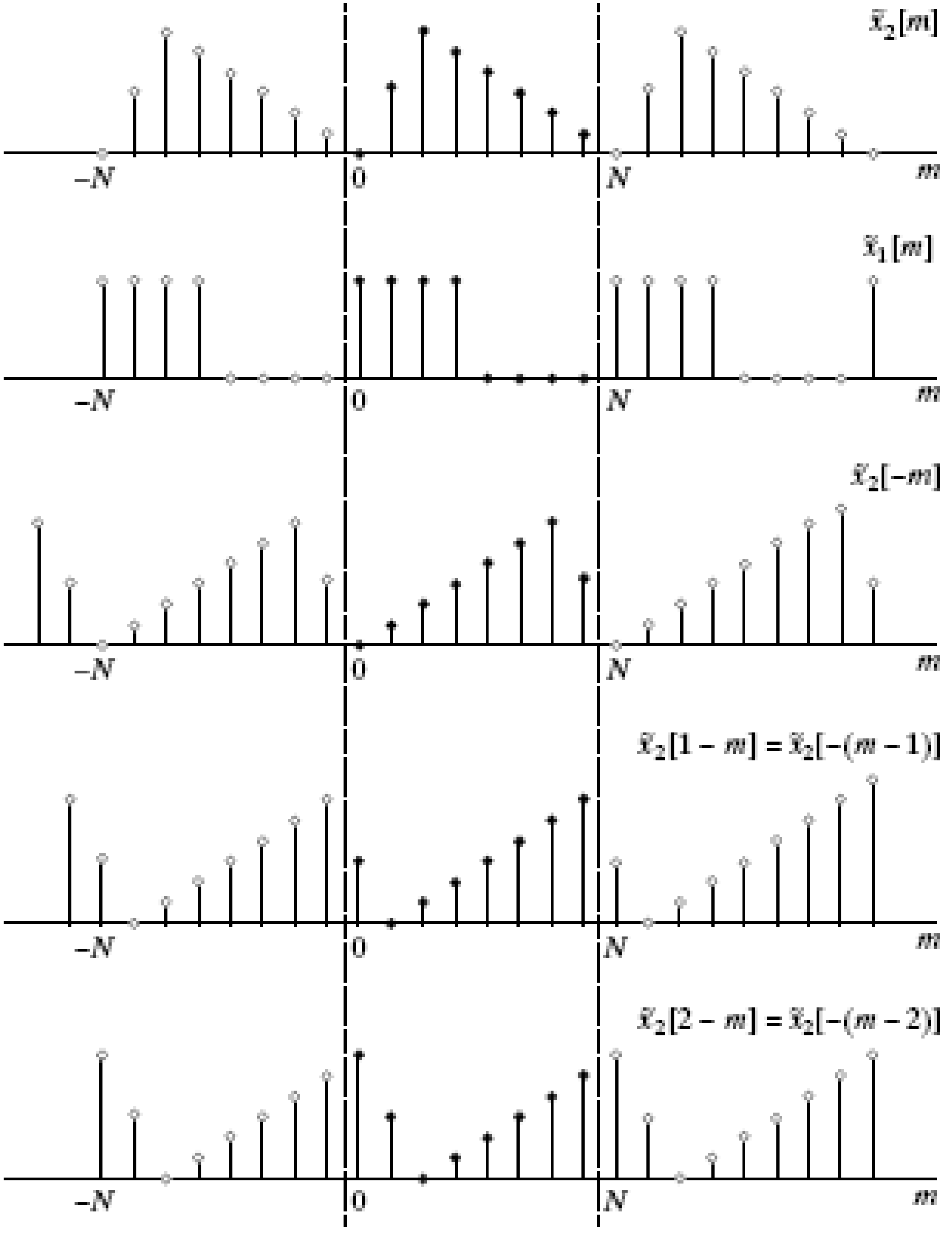
$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], \quad 0 \le n \le N-1$$

$$= \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N], \quad 0 \le n \le N-1$$

$$= \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N], \quad 0 \le n \le N-1$$

- In linear convolution, one sequence is multiplied by a time –reversed and linearly shifted version of the other. For convolution here, the second sequence is circularly time reversed and circularly shifted. So it is called an N-point circular convolution

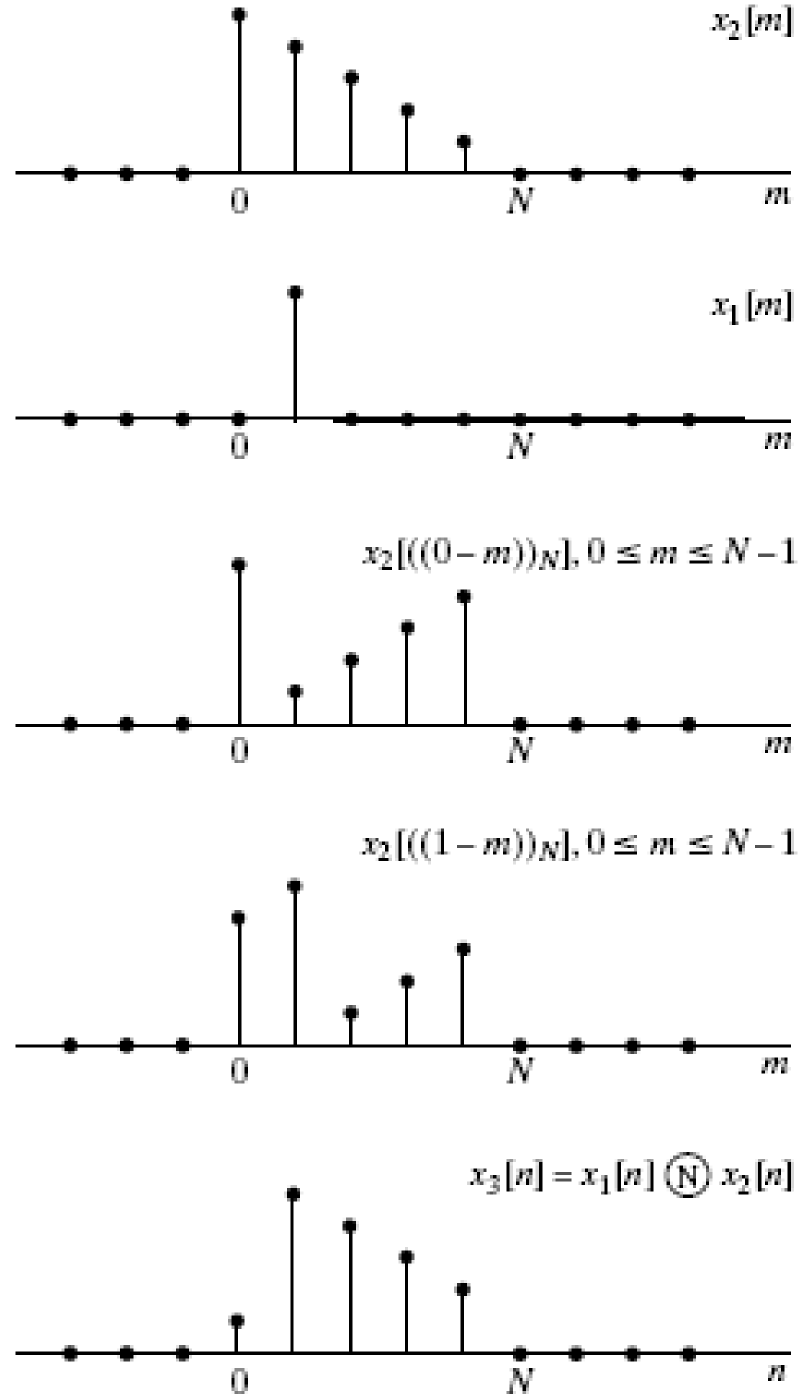
$$x_3[n] = x_1[n] \, \text{Ⓝ} \, x_2[n]$$



Convolution of two periodic sequences

Example: circular convolution with a delayed impulse sequence

$$x_1[n] = \delta[n - n_0], \quad 0 \le n_0 \le N$$



Circular convolution

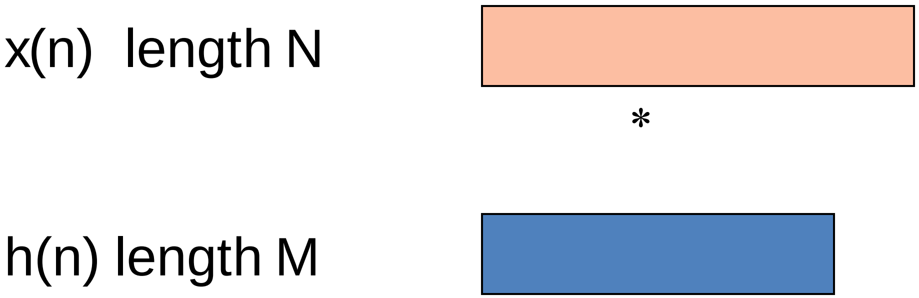
• Circular convolution of x(n) and h(n) is defined as the convolution of h(n) with a periodic signal $x_p(n)$:

$$y_p(n) = x_p(n) * h(n)$$

where

$$x_p(n) = x(n \bmod N), \qquad -\infty < n < \infty$$

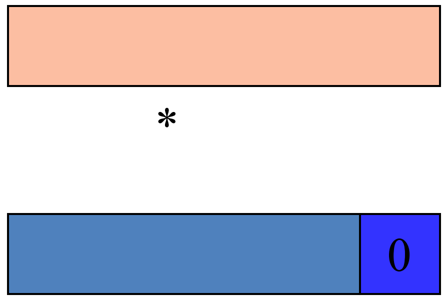
Circular Convolution



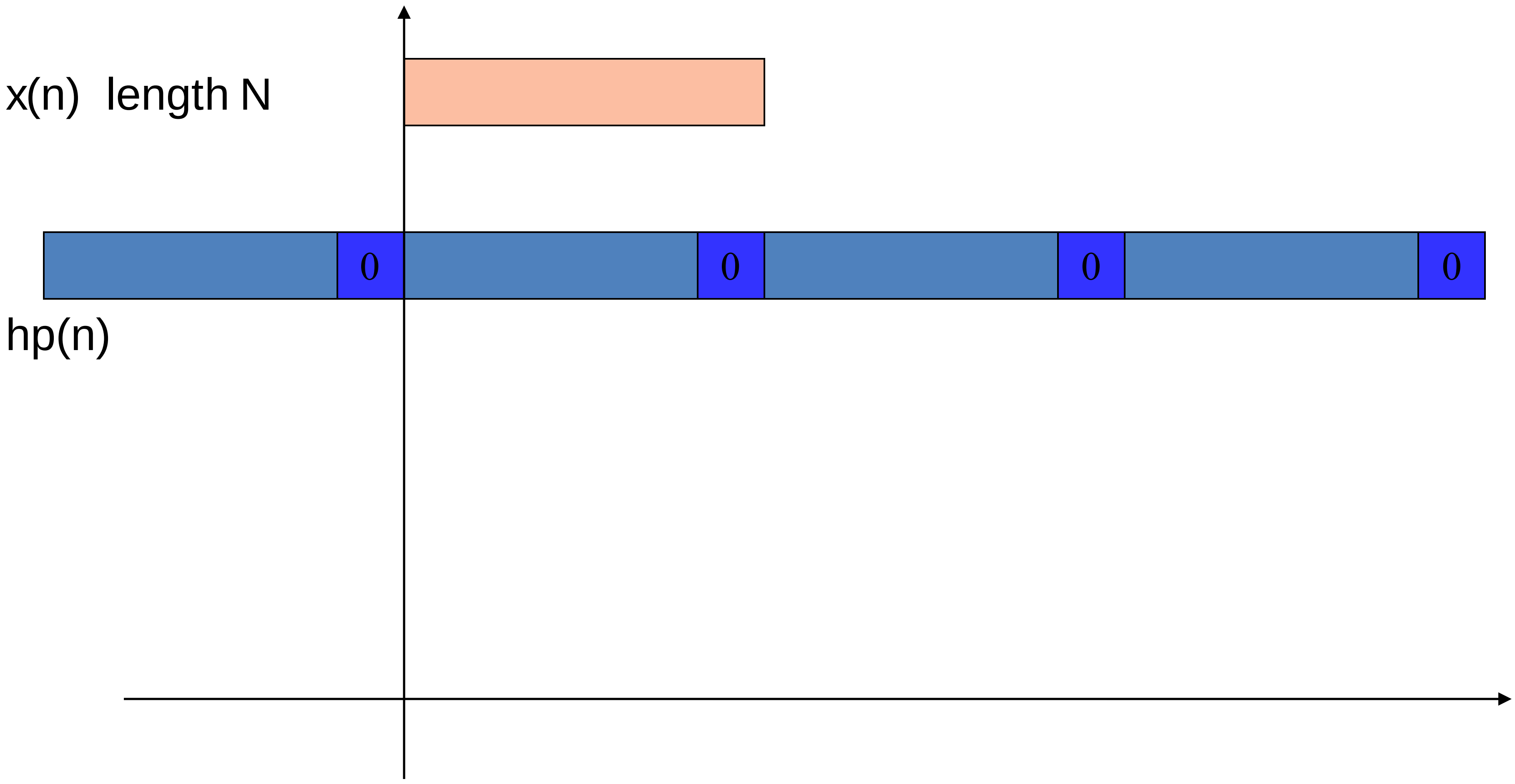
Circular Convolution

x(n) length N

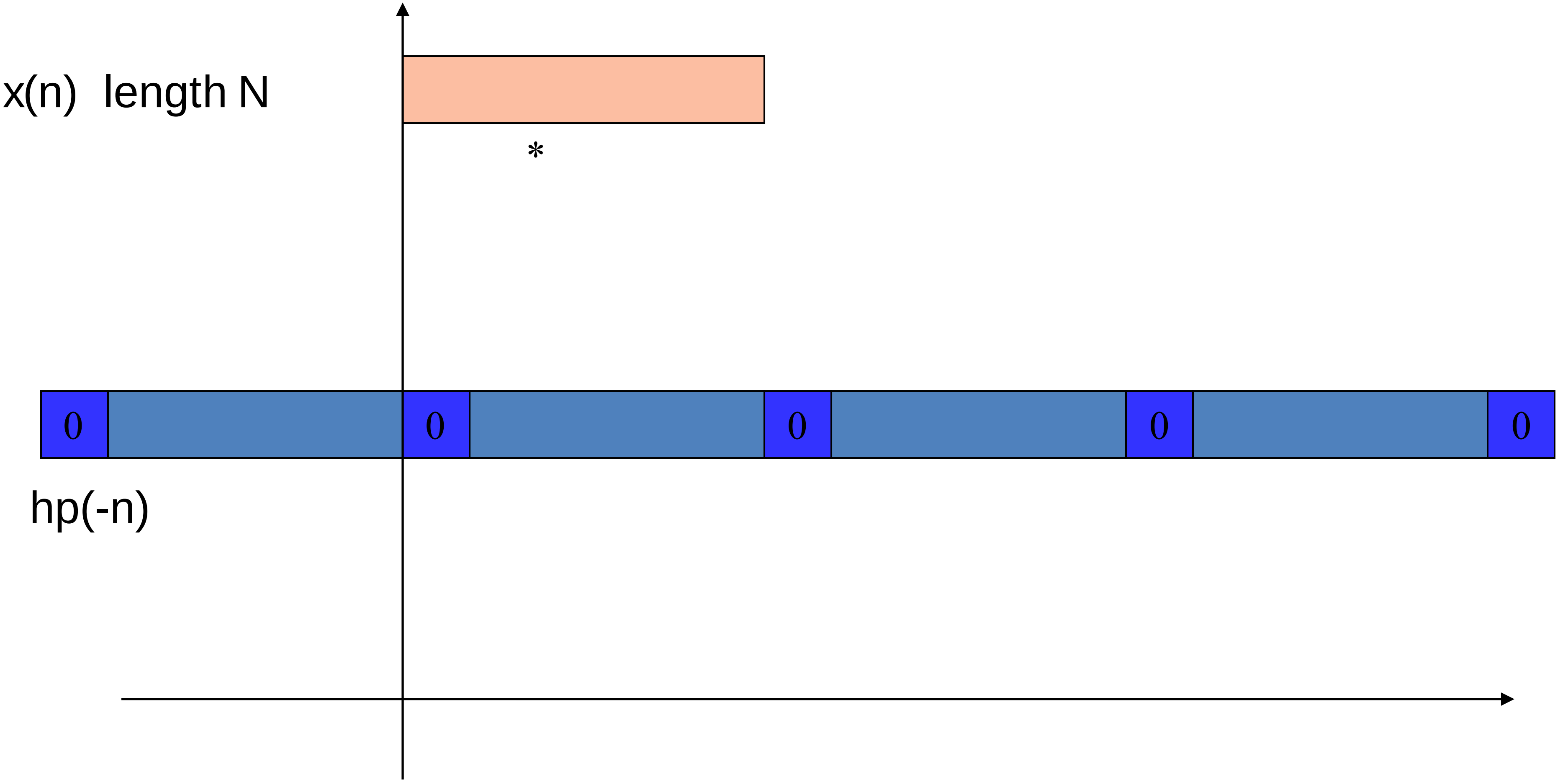
h(n) length M



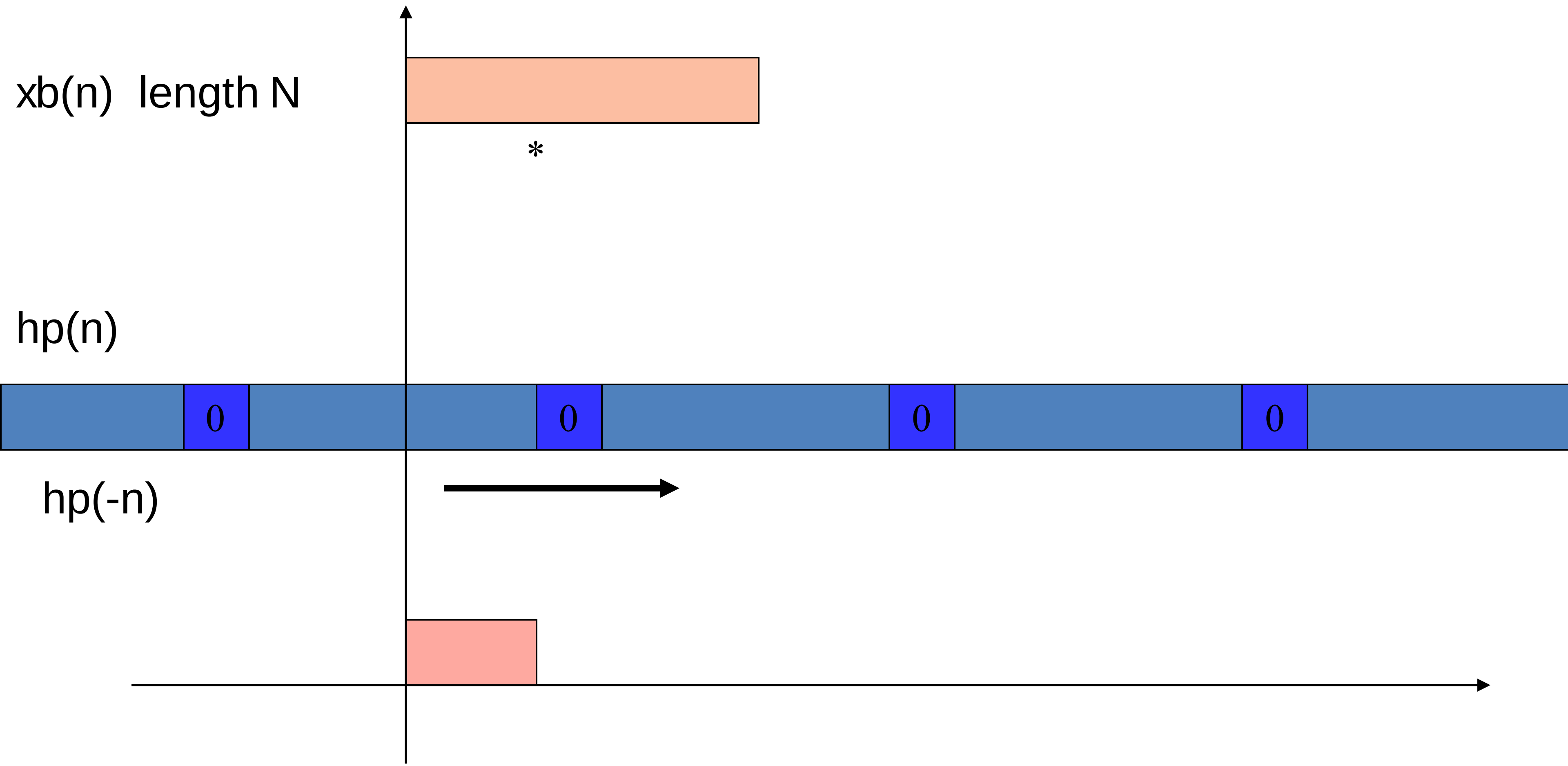
Circular Convolution



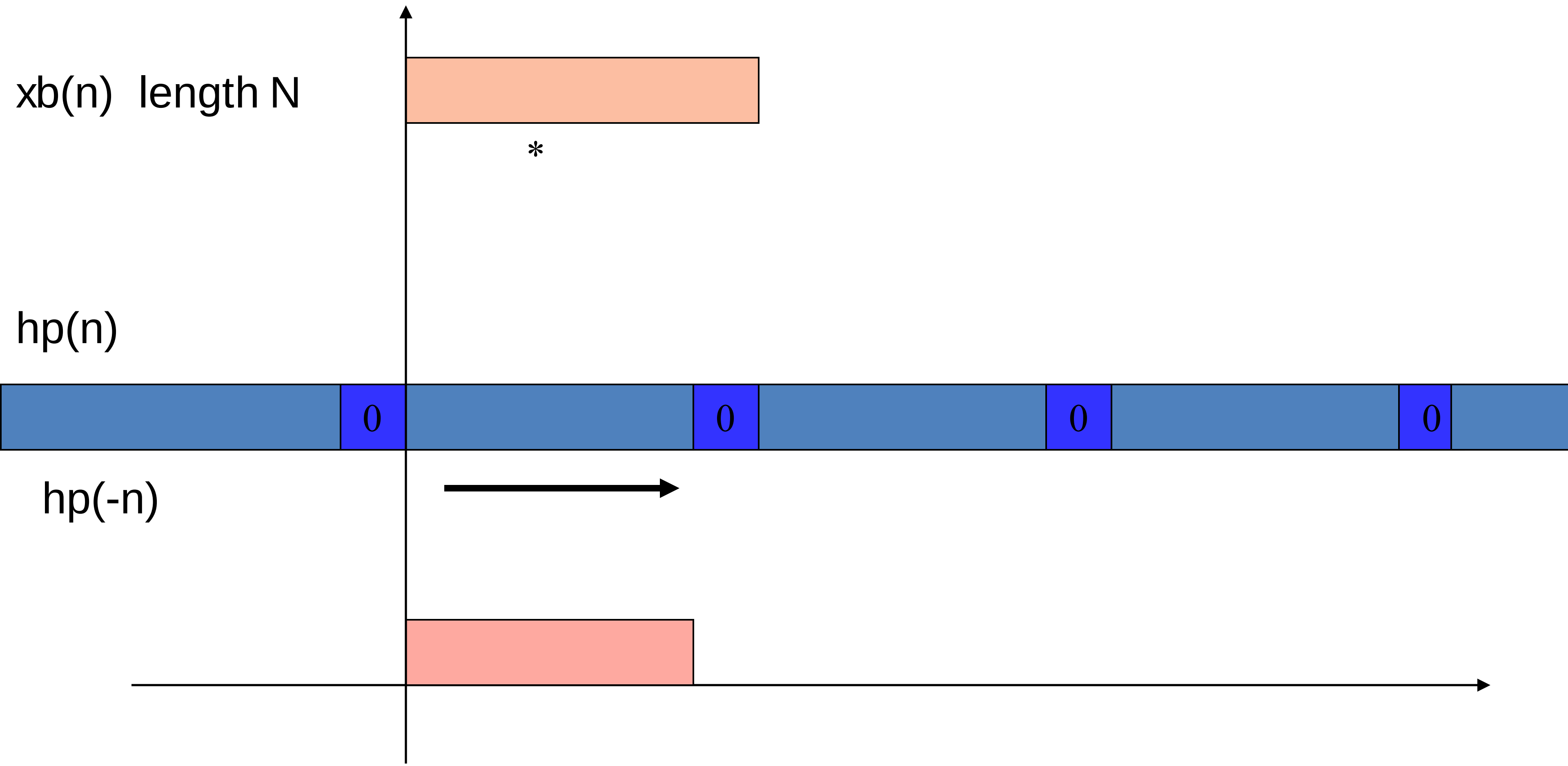
Circular Convolution



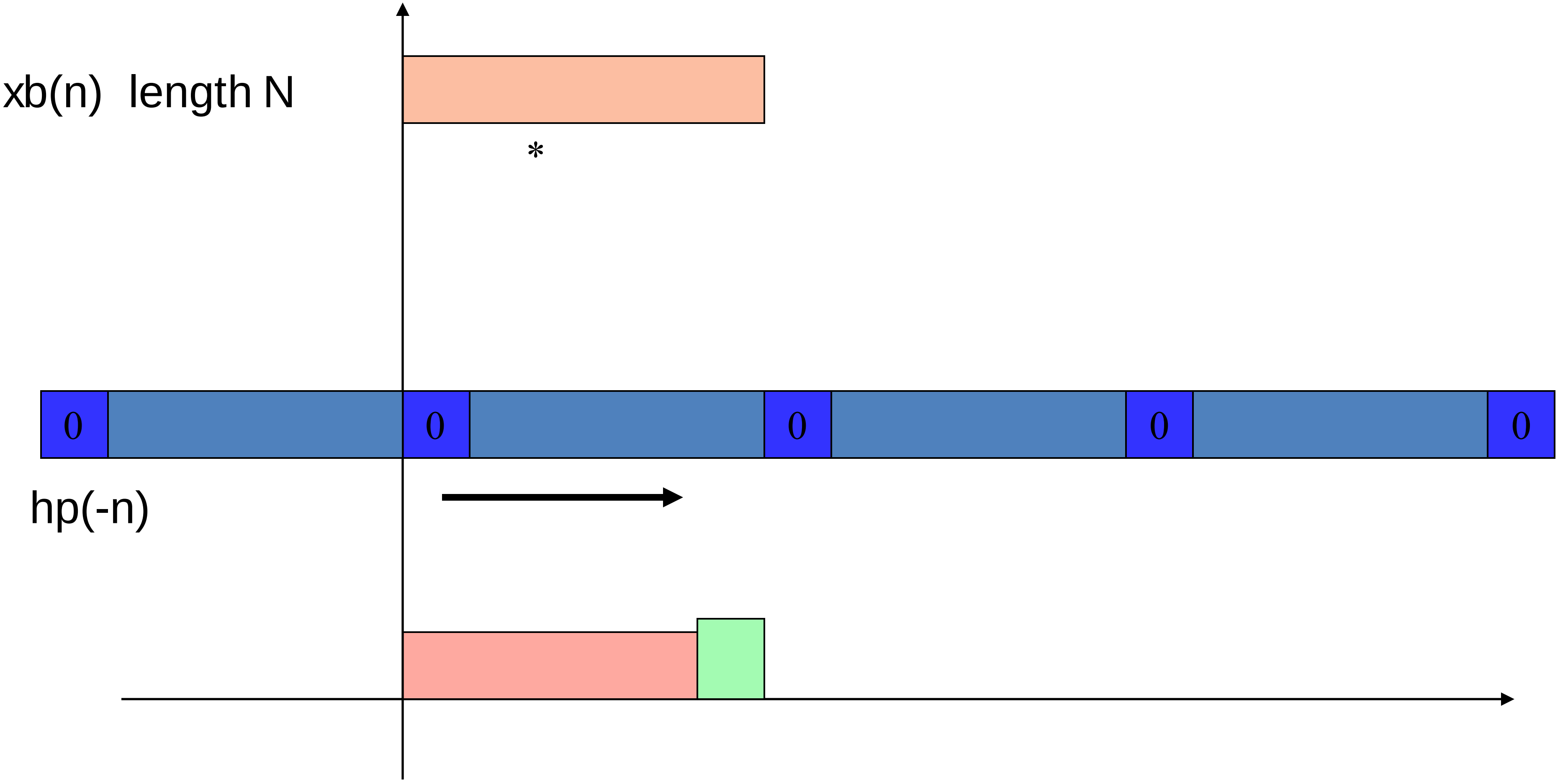
Circular Convolution



Circular Convolution



Circular Convolution



N-point circular convolution can be computed using Matrix form:

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h(0) & h(N-1) & h(N-2) & \dots & h(1) \\ h(1) & h(0) & h(N-1) & \dots & h(2) \\ h(2) & h(1) & h(0) & \dots & h(3) \\ \vdots & & & & \\ h(N-1) & h(N-2) & h(N-3) & \dots & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

- Elements in each row are obtained by circularly rotating the elements of the previous row to the right by 1.

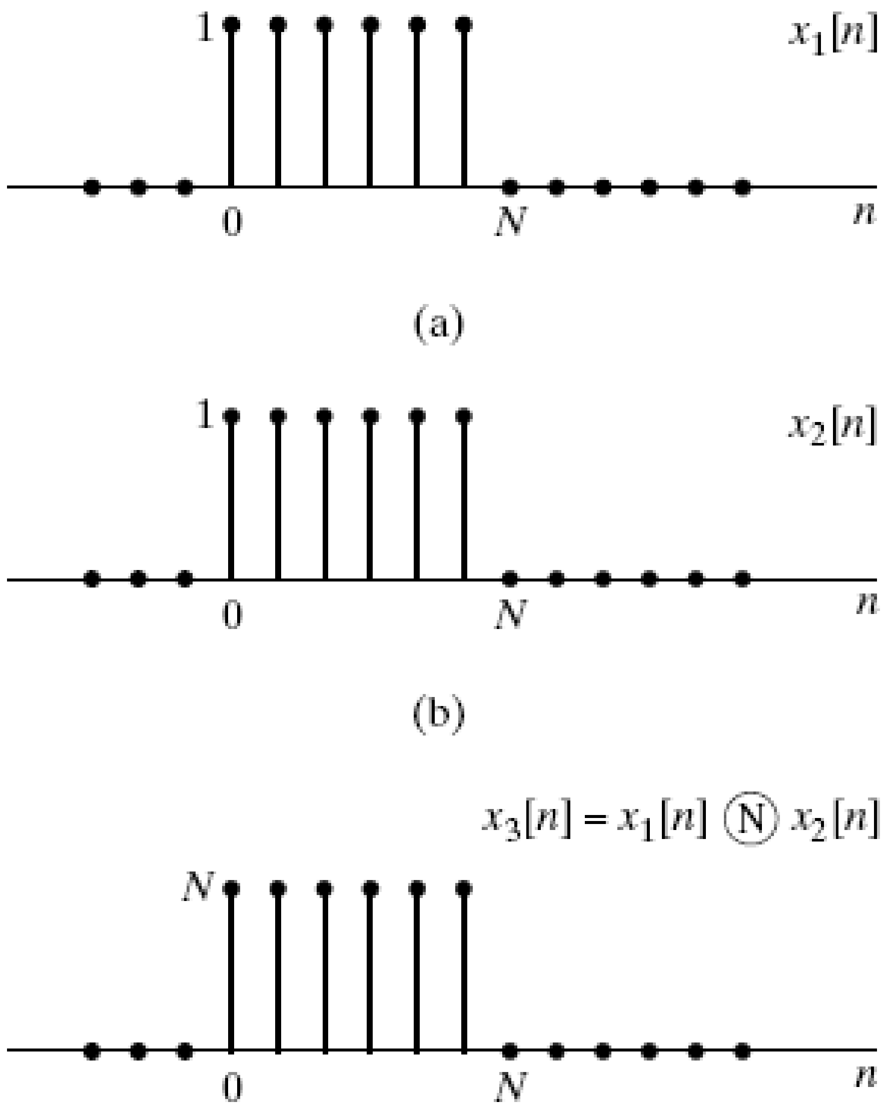
Example: Determine 4-point circular convolution of the two length-4 sequences **x(n) = {1 2 0 1}** and **h(n)={2 2 1 1}**, $0 \leq n \leq 3$.

Method1: use DFT equation

Method2: Graphical (cylinders)

Method3: use Matrix computation method

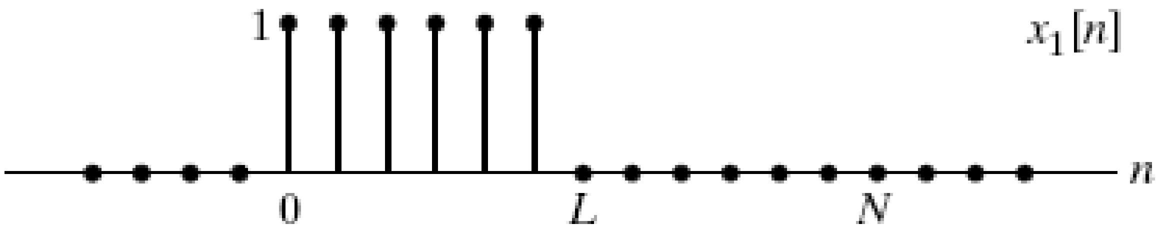
Example: circular convolution of two rectangular pulses



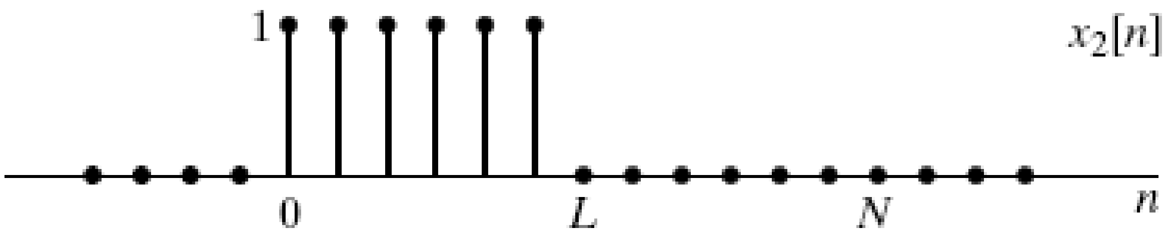
N-point circular convolution of two sequences of length N.

Example: circular convolution of two rectangular pulses (continue)

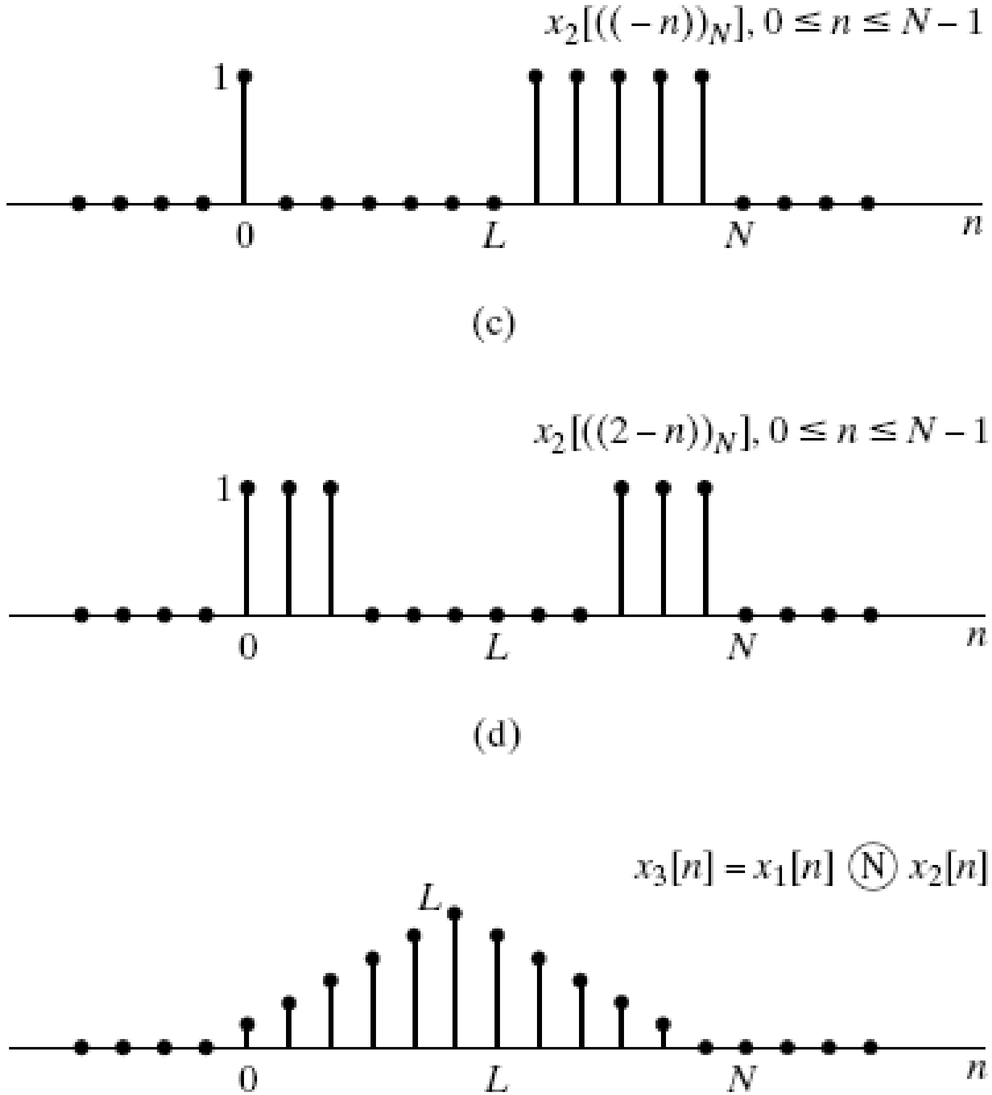
Given two sequences of length L, assume that we add L zeros on its end, making an N=2L point sequence – referred to as zero padding



(a)



N-point circular convolution of two sequences of length L, where N=2L.



N-point circular convolution of two sequences of length L, where N=2L (continue).

Note that by zero padding, we can use circular convolution to compute convolution of two finite length sequences.

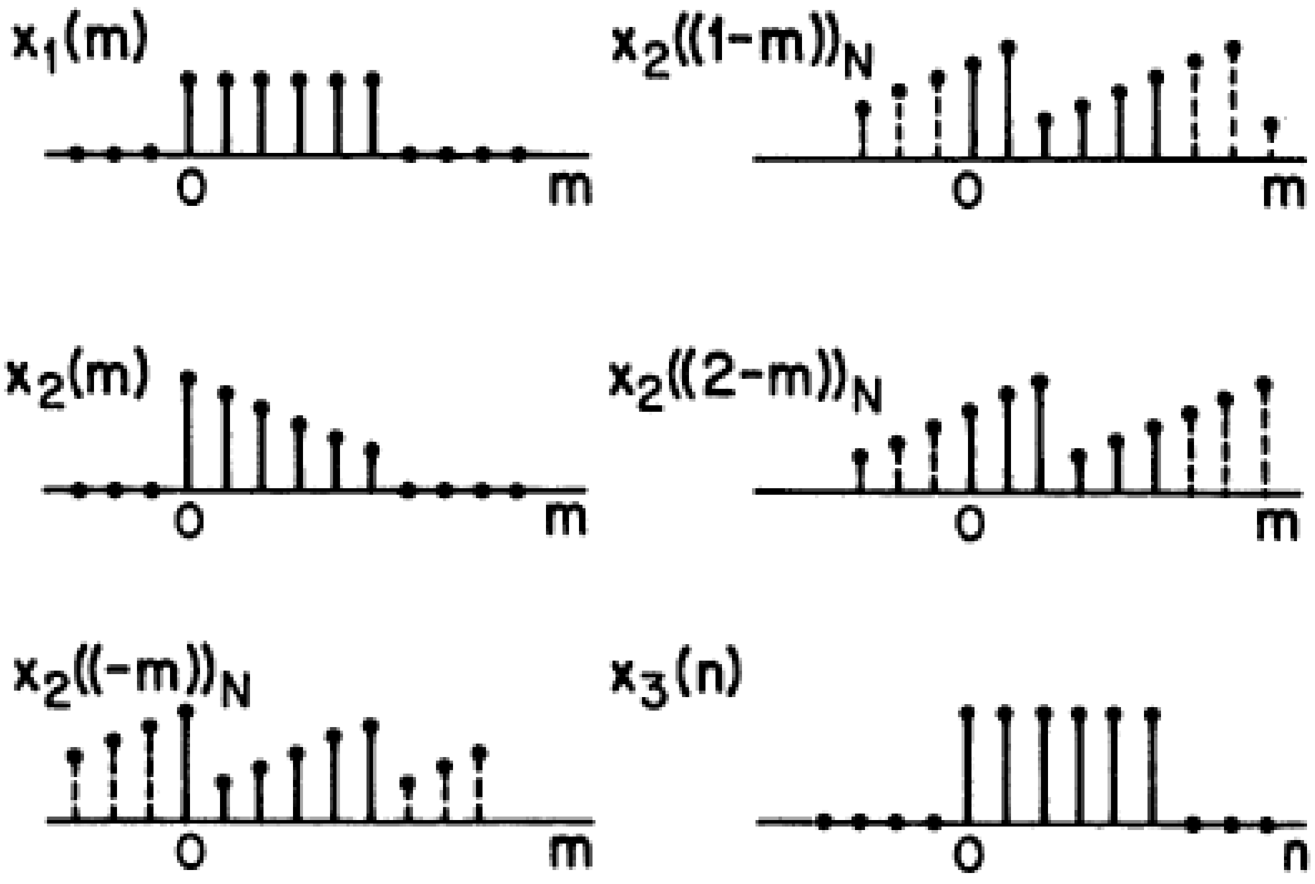
Part VI: Linear convolution of the DFT

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

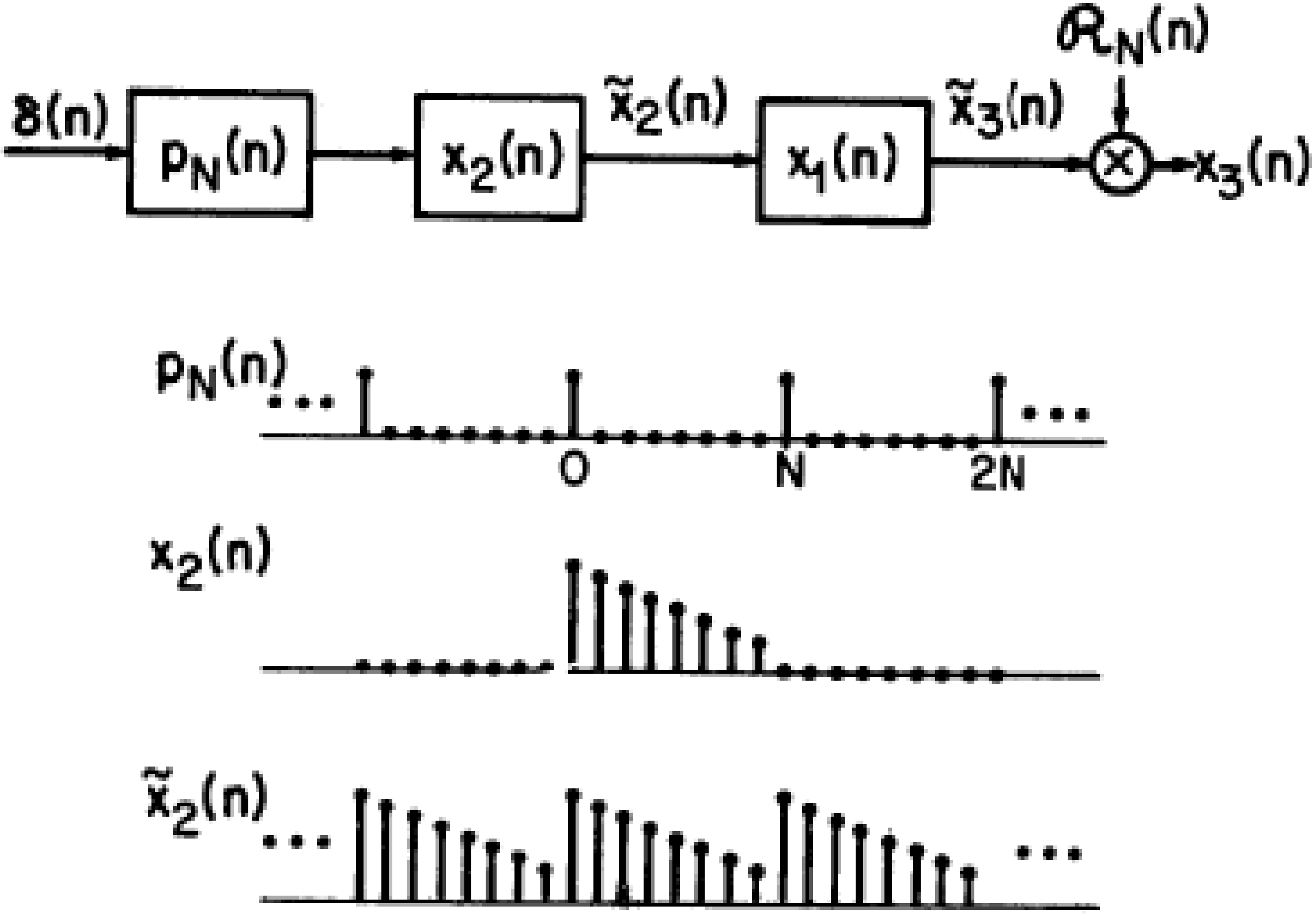
Linear convolution using the DFT

- Procedure
  - Compute the N-point DFTs $X_1[k]$ and $X_2[k]$ of two sequences $x_1[n]$ and $x_2[n]$, respectively
  - Compute the product of $X_3[k] = X_1[k]X_2[k]$ for $0 \le k \le N-1$
  - Compute the sequence $x_3[n] = x_1[n] \circledN x_2[n]$ as the inverse DFT of $X_3[k]$
- As we know, the multiplication of DFTs corresponds to a circular convolution of the sequences. To obtain a linear convolution, we must ensure that circular convolution has the effect of linear convolution.

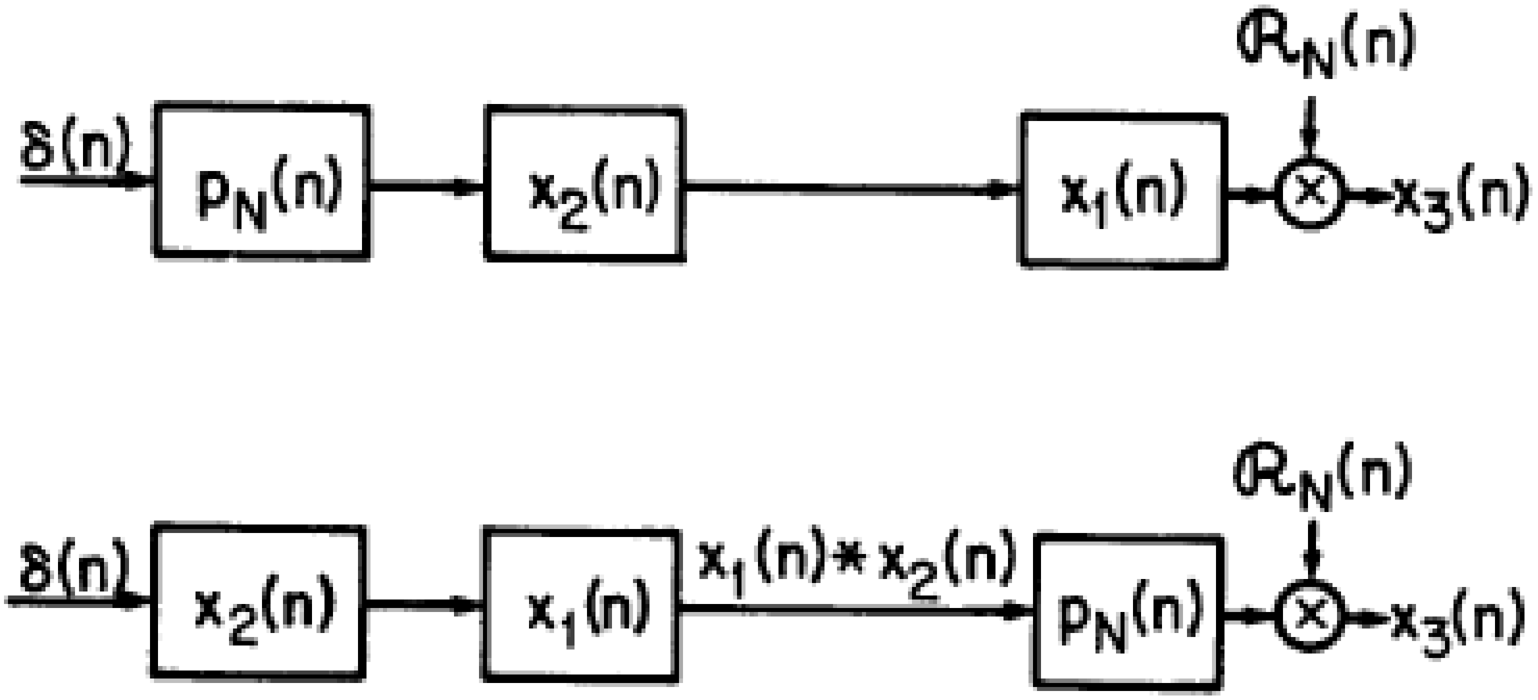
Example of circular convolution of two sequences



An Interpolation of circular convolution



Re-Arrangement of the operations of forming circular convolution



"Circular Convolution =

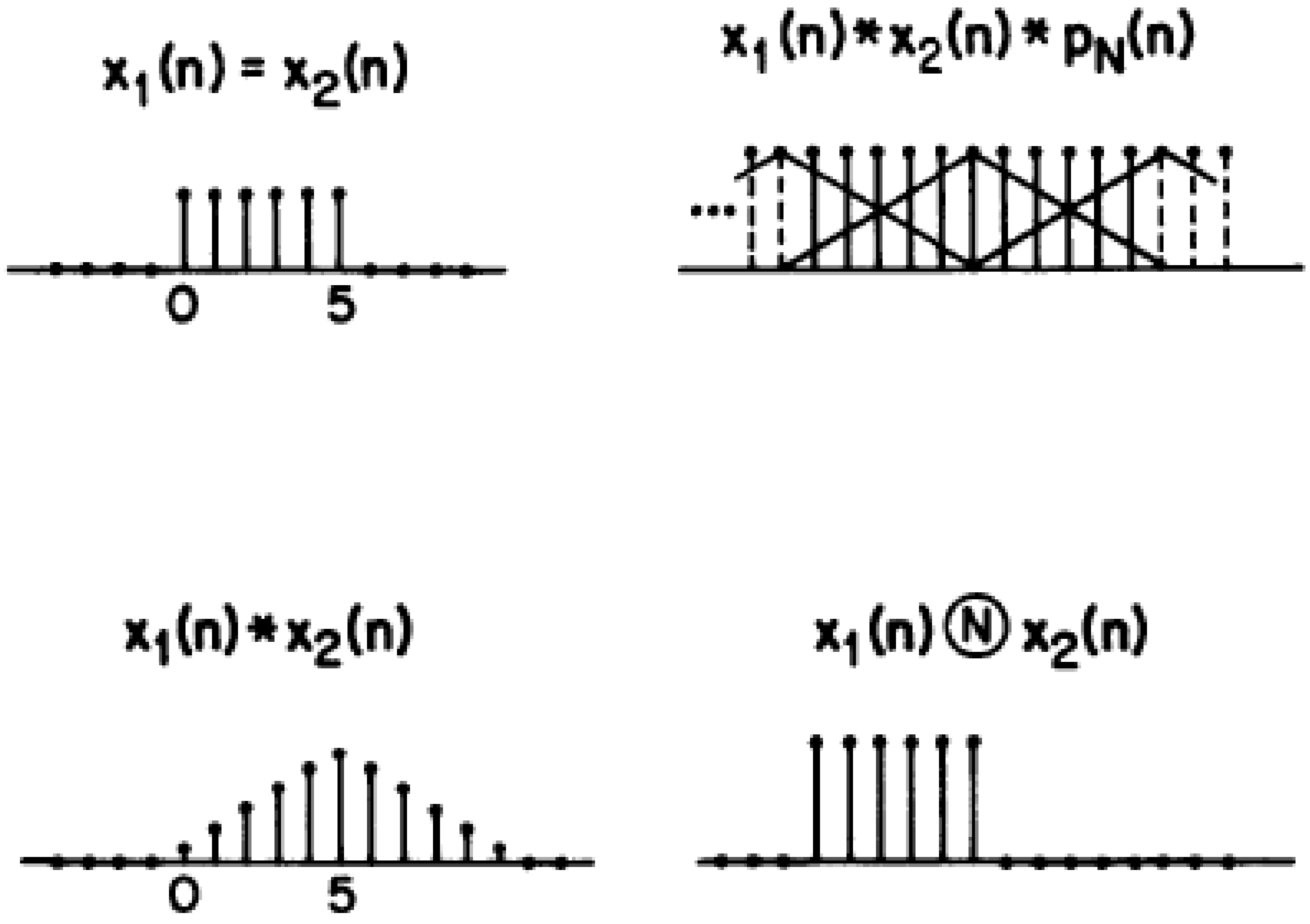
Linear Convolution + Aliasing"

$$\hat{x}_3(n) = x_1(n) * x_2(n)$$

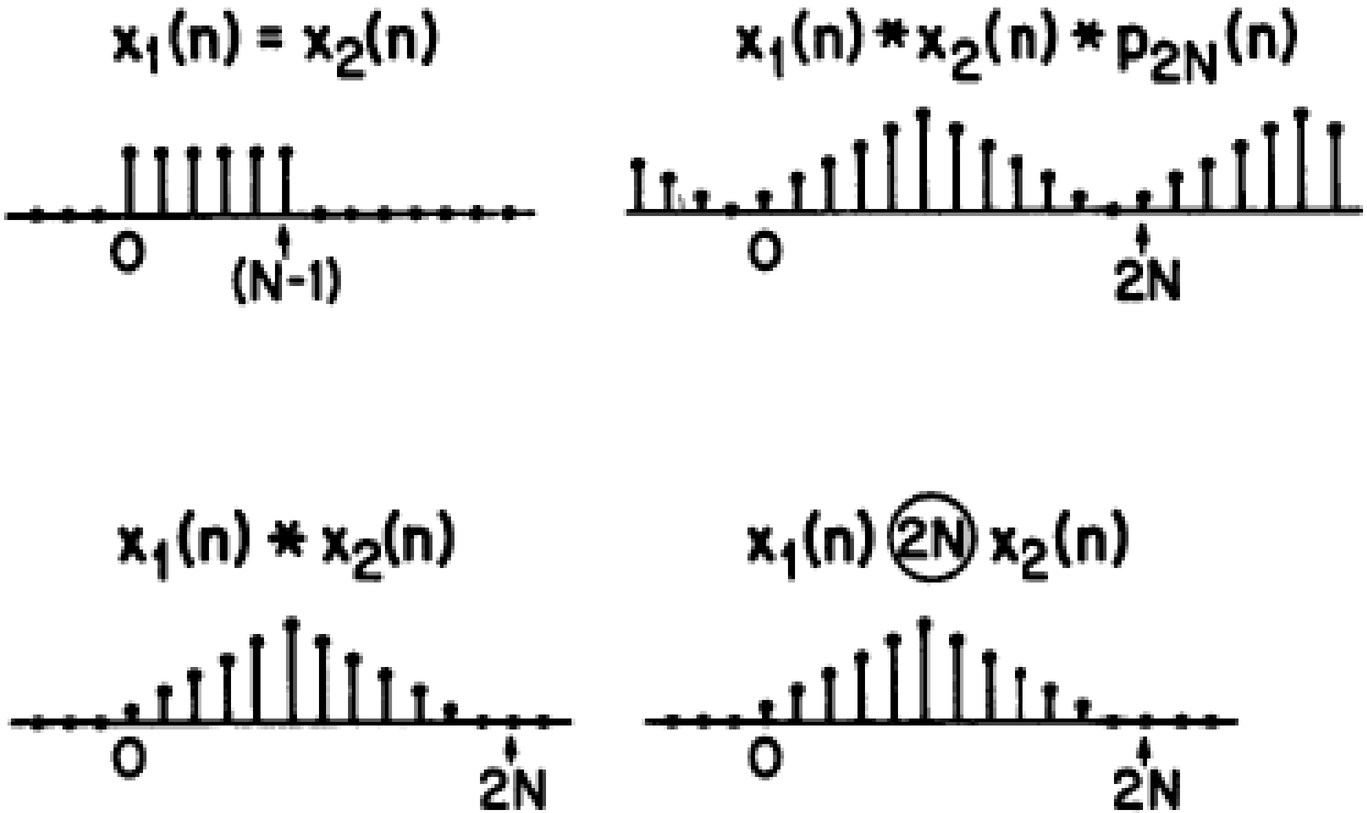
$$x_3(n) = x_1(n) \circledN x_2(n)$$

$$x_3(n) = \left[\sum_{r=-\infty}^{+\infty} \hat{x}_3(n+rN)\right] \mathcal{R}_N(n)$$

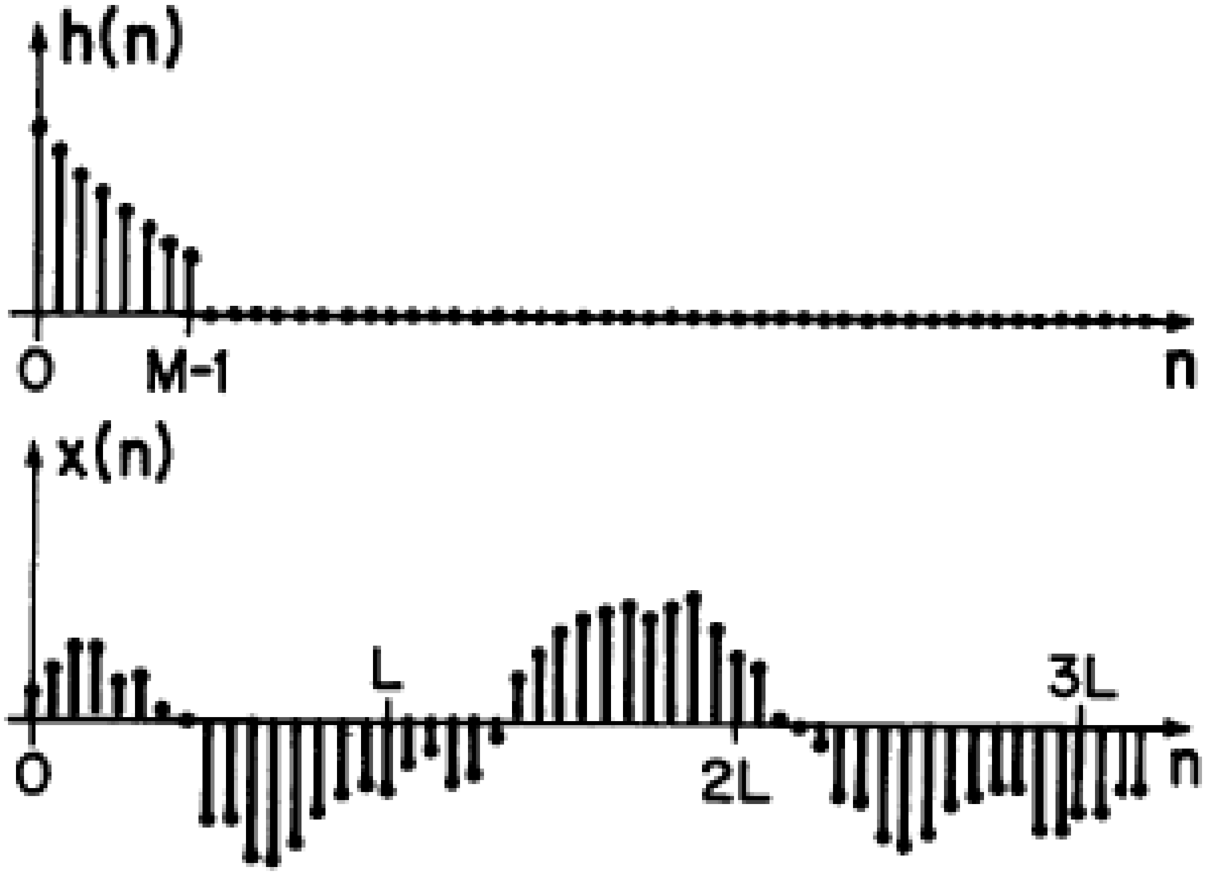
Example of forming circular convolution by linear convolution followed by aliasing:



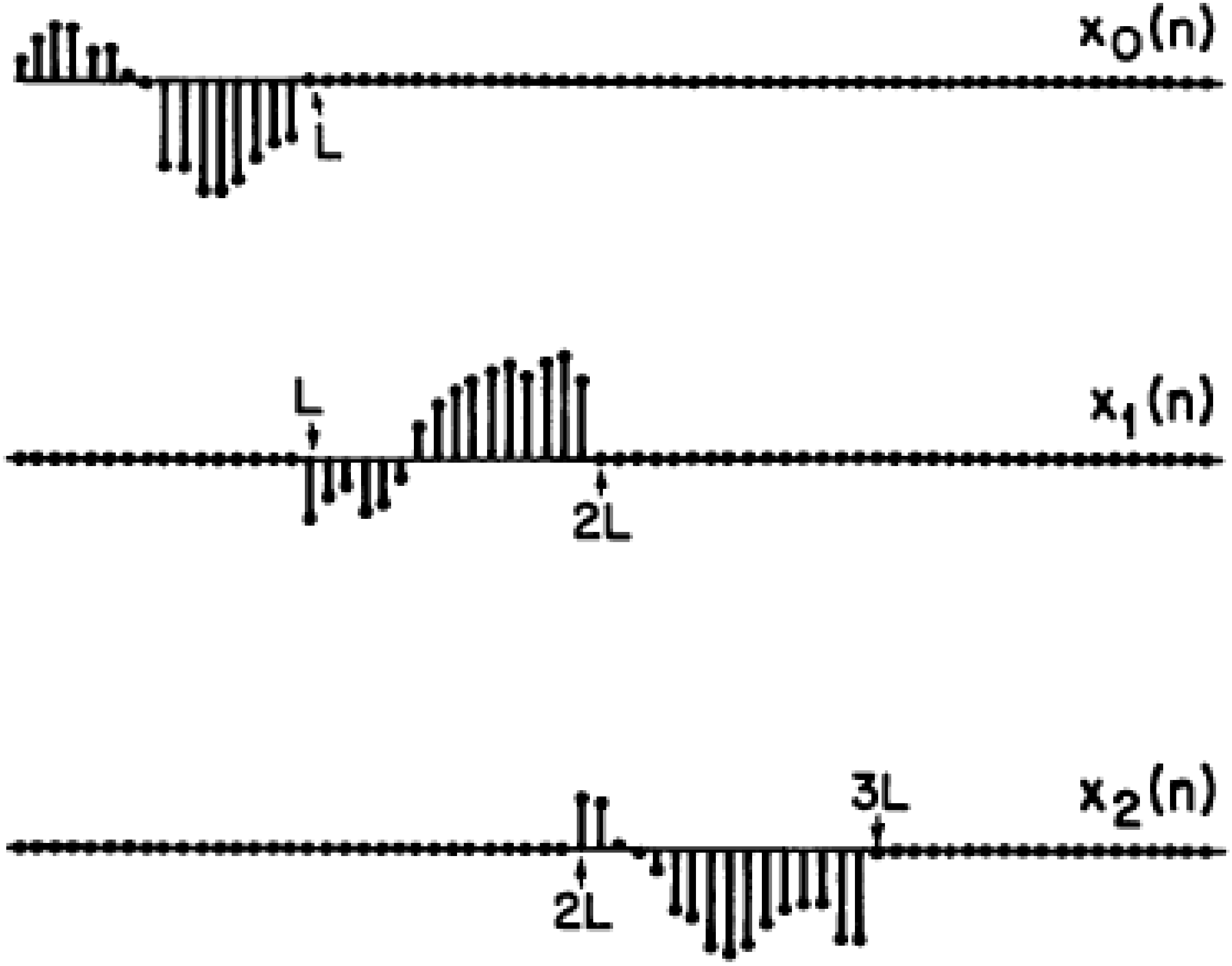
Obtaining linear convolution through the use of circular convolution



Overlap – Add Method



Overlap – Add Method



Overlap – Add Method