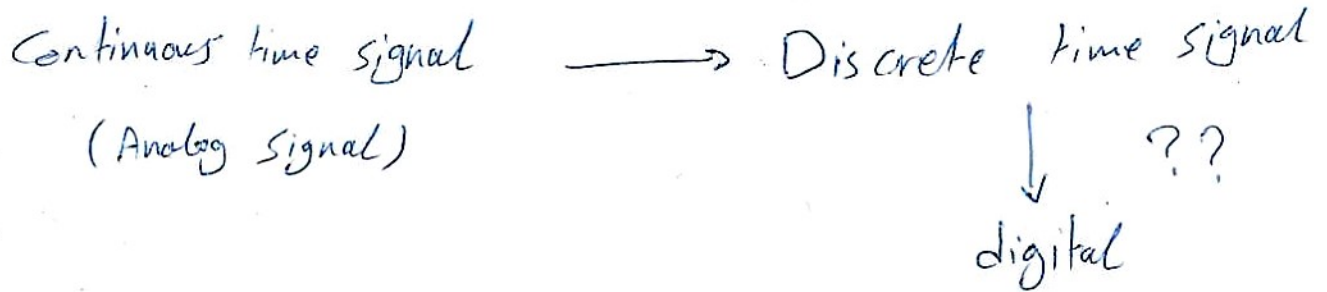


Digital Signal processing

Our concern is discrete time signals only

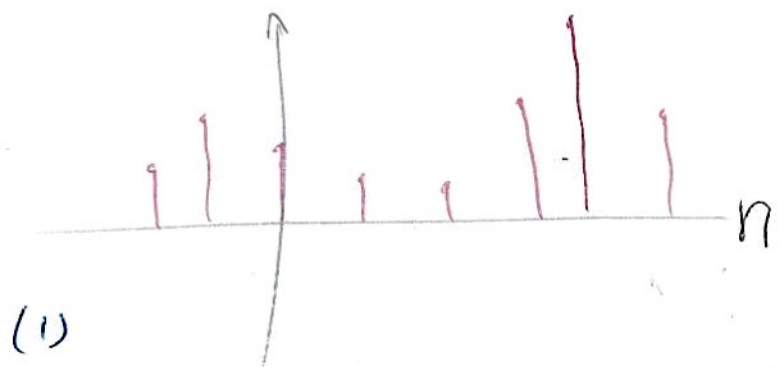
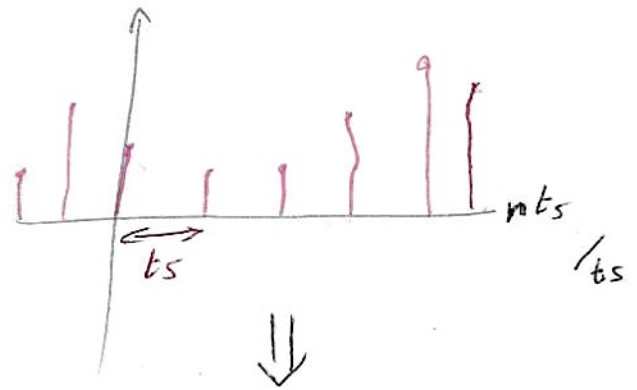
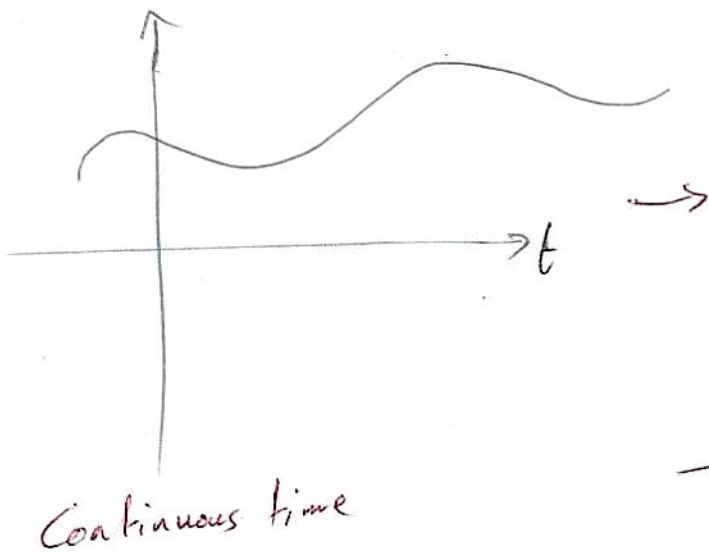


Signal is function of one or more independent variables

$X(t)$ \longleftarrow independent variable
↑
function (dependent)

$Z(x, y)$
image

$P(x, y, t)$
video



2.1 Basic sequence

① The unit sample sequence

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

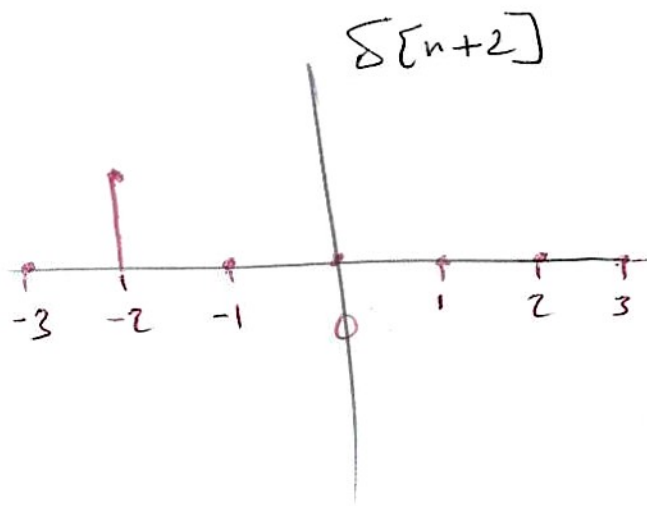
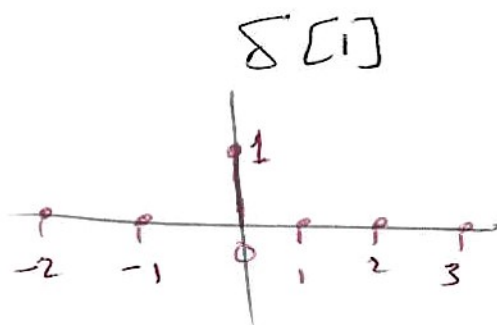
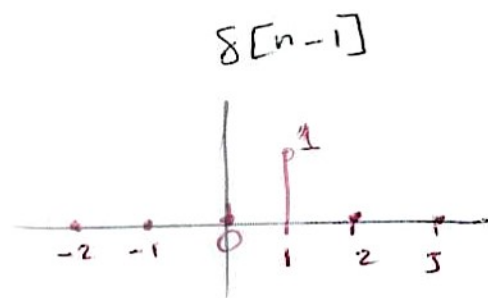
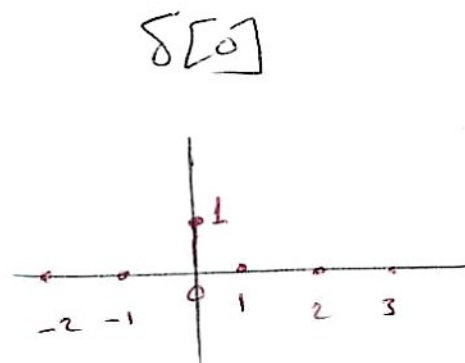
$\delta[n]$ is an even function

$$\delta[kn] = \delta[n]$$

$$X[n] \delta[n-n_0] = X[n_0] \delta[n-n_0]$$

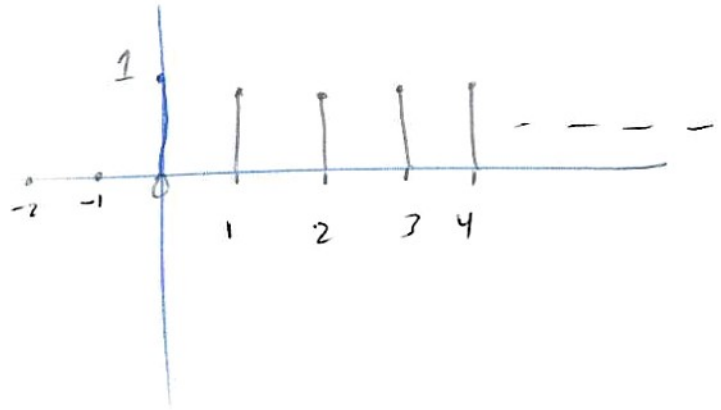
$$X[n] * \delta[n-n_0] = X[n-n_0]$$

$$\sum_{k=-\infty}^n \delta[k] = u[n]$$



2) The unit step sequence

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

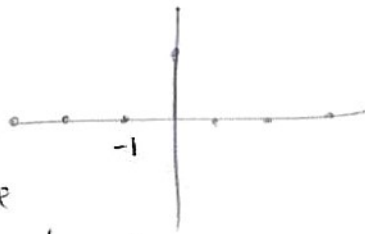


$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

using change of variables if $r = n - k$

$$u[n] = \sum_{r=-\infty}^{-\infty} \delta[r] = \sum_{r=-\infty}^n \delta(r) \quad \leftarrow \text{the definition}$$

$$\delta[-1] = 0$$



sum of all the previous value upto -1

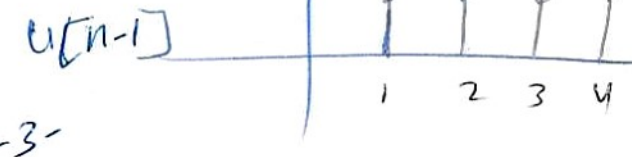
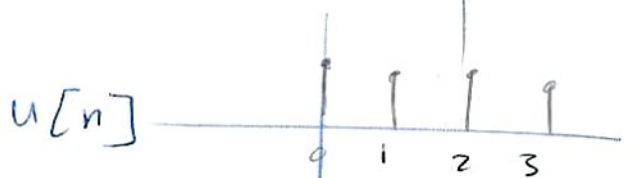
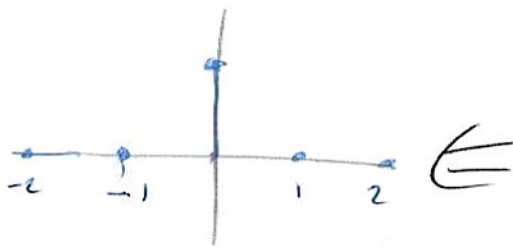
$$\delta[0] = 1$$



$$\delta[1] = 1$$



$$\delta[n] = \frac{u[n] - u[n-1]}{\Delta n}$$



④ Sinusoidal sequence

$$X[n] = A \cos(\omega_0 n + \phi)$$

amplitude
phase shift

Digital Frequency (Ratio) of the original and sampling frequency

$$X(t) = A \cos(\Omega t + \phi) \quad \Omega = \frac{2\pi K}{T}$$

integer
real

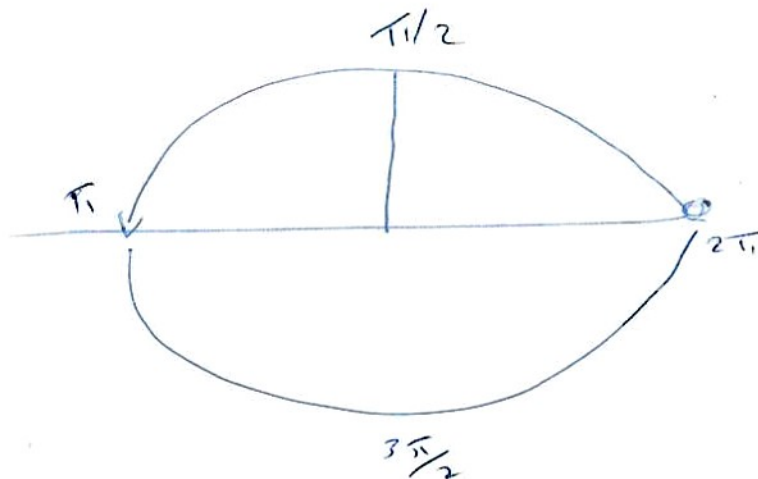
$$t = n T_s$$

$$X(t) = A \cos(\Omega n T_s + \phi)$$

$\omega_0 = \Omega T_s$
 rad/s (s) = rad only so has no physical meaning

$$\omega_0 = \frac{2\pi K}{N}$$

integer
integer



So $\cos(\omega_0) = \cos(\omega_0 + 2\pi)$

$$X[n] = A \cos((\omega_0 + 2\pi r)n + \phi)$$

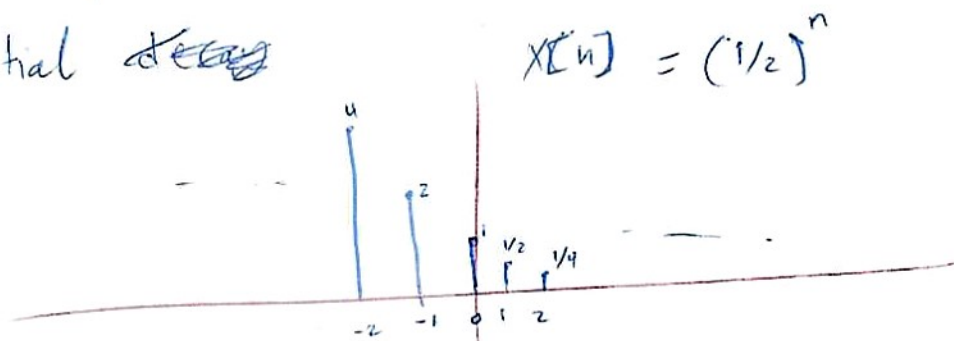
$$= A \cos(\omega_0 n + \phi)$$

③ Exponential sequence

$$x[n] = A \alpha^n$$

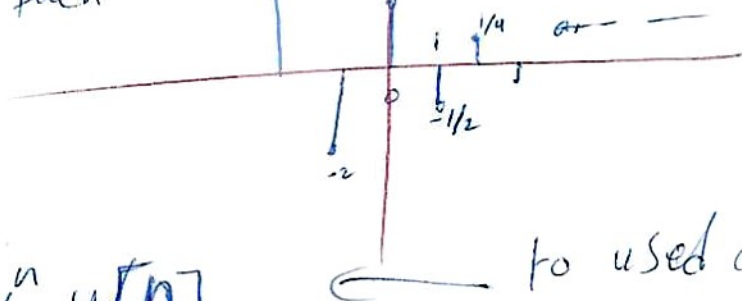
① If A & α are real

If $0 < \alpha < 1$ and A is positive, then the sequence is decaying exponential ~~decay~~



If $\alpha > 1$ and A is positive, then the sequence is growing exponential

If $-1 < \alpha < 0$ then the sequence is alternative



$$x[n] = A \alpha^n u[n]$$

$$= \begin{cases} A \alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

← to used as Causal Signal

If A & α are complex

$$\alpha = a + jb$$

↓

$$\alpha = |\alpha| e^{j\omega_0}$$

$$|\alpha| = \sqrt{a^2 + b^2} \quad \omega_0 = \tan^{-1} \frac{b}{a}$$

$$A = |A| e^{j\phi}$$

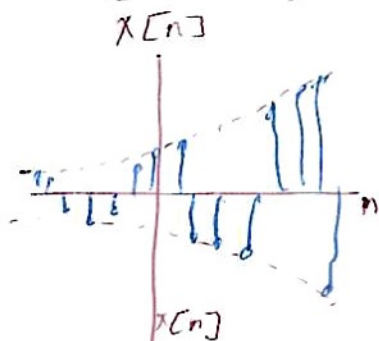
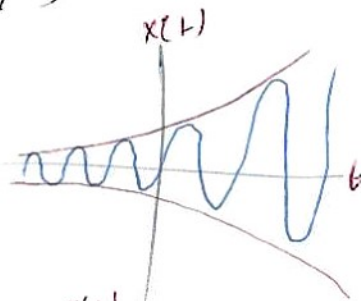
$$X[n] = A \alpha^n = |A| e^{j\phi} |\alpha|^n e^{j\omega_0 n}$$

$$= |A| |\alpha|^n e^{j(\phi + \omega_0 n)}$$

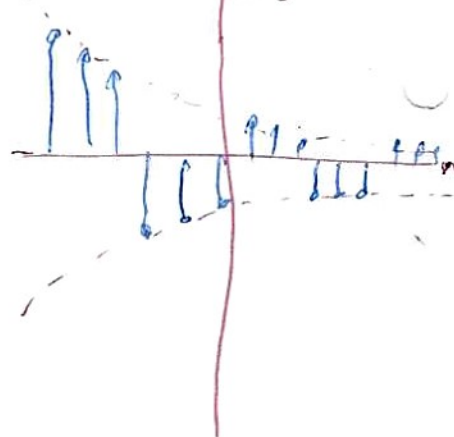
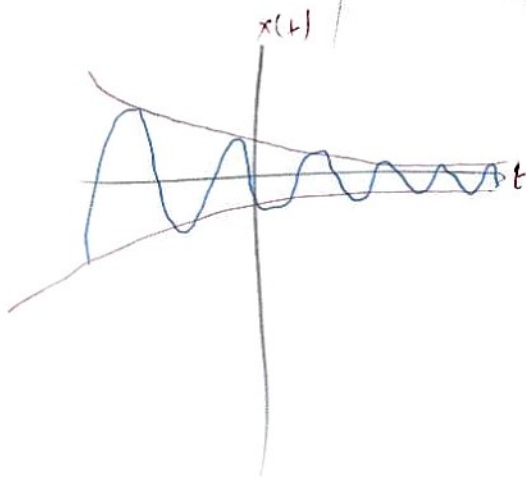
$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$= |A| |\alpha|^n \cos(\omega_0 n + \phi) + j |A| |\alpha|^n \sin(\omega_0 n + \phi)$$

If $|\alpha| > 1 \rightarrow$



If $|\alpha| < 1$



periodicity

$$X[n] = A e^{j(\omega_0 + 2\pi)n}$$

$$= A e^{j\omega_0 n} e^{j2\pi n} = A e^{j\omega_0 n}$$

$$e^{j2\pi n} = \cos(2\pi n) + j \sin(2\pi n)$$

periodicity

$$X[n] = \{1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4\}$$

$$X[n] = A \cos(\omega_0 n + \phi)$$

is periodic if and only if

$$A \cos(\omega_0(n+N) + \phi) = A \cos(\omega_0 n + \phi)$$

$$\omega_0 N = 2\pi K, \quad K \text{ is integer}$$

same for complex exponential

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} = e^{j\omega_0 n} e^{j\omega_0 N}$$

$$e^{j\omega_0 N} = 1, \quad \omega_0 N = 2\pi K$$

$$\omega_0 = \frac{2\pi K}{N}, \quad N = \frac{2\pi K}{\omega_0}$$

↑ integer ↑ integer

Example 2 - let $X_1[n] = \cos(\pi n/4)$, is this signal periodic?
show the fundamental period?

$$\omega_0 = \frac{\pi}{4} = \frac{2\pi K}{N}$$

$$N = \frac{2\pi K}{\frac{\pi}{4}} = 8K$$

If K integer then N integer so the signal is periodic

No = 8 samples

↳ to convert this to seconds, I need the sampling period.

Let us consider another signal

$$X_2[n] = \cos\left(\frac{3\pi n}{8}\right)$$

$$\omega_0 = \frac{3\pi}{8}$$

$$N = \frac{2\pi K}{\frac{3\pi}{8}} = \frac{16}{3} K \quad \text{at } K=3 \quad \text{periodic signal}$$

No = 16 samples

please note:

$$f = \frac{1}{t} \quad \underline{\text{Continuous}}$$

if f increased \Rightarrow t decreased

but for discrete signals
the relation is not correct

$$\omega_{01} = \frac{\pi}{4}$$
$$= \frac{2\pi}{8}$$

$$\omega_{02} = 3 \frac{\pi}{8}$$

$$\omega_{02} > \omega_{01}$$
$$\frac{3\pi}{8} > \frac{\pi}{4} = \frac{2\pi}{8}$$

$$N_{02} > N_{01}$$
$$16 > 8$$

this is clear from the previous example, the physical period here depends on the sampling rate.

$$\text{let } x_3[n] = \cos(n)$$
$$\omega_0 = 1$$

$$N = \frac{2\pi K}{\omega_0} = 2\pi K$$

there is N_0 integer to make N integer

so $x_3[n]$ is not periodic (aperiodic)

Operations on Signals

- amplitude scaling

$2X[n] \rightarrow$ amplification

$\frac{1}{2}X[n] \rightarrow$ attenuation

$-2X[n] \leftarrow$ Reflection

- time scaling

$X[2n] \leftarrow$ compression
double the speed

$X[\frac{1}{2}n] \leftarrow$ expansion

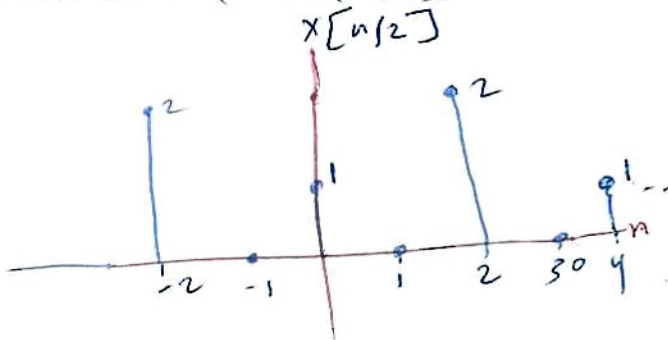
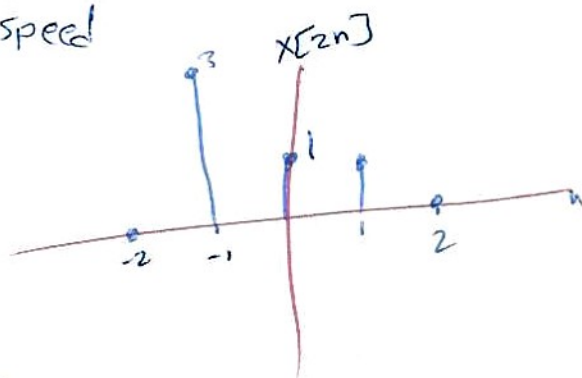
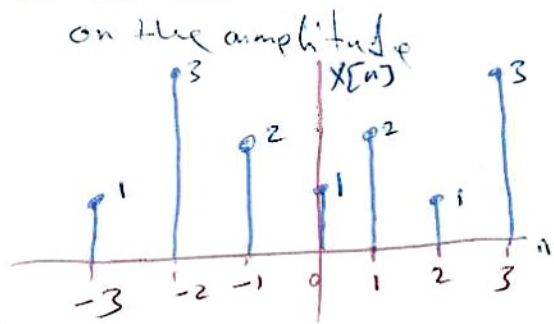
$X[-n] \leftarrow$ Reflection on the time axis

- Amplitude shifting

$X[n] + 2$ shift up

$X[n] - 2$ shift down

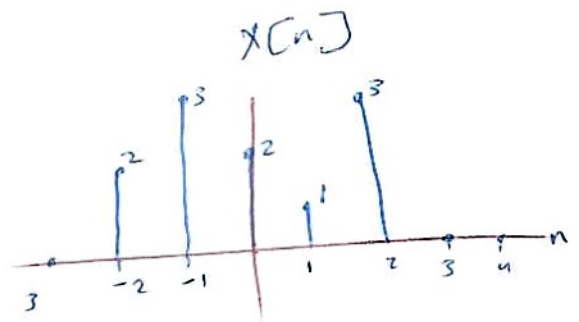
continuous in n and t



Time shifting

$X[n-1]$ ← delayed

$X[n+1]$ ← advanced

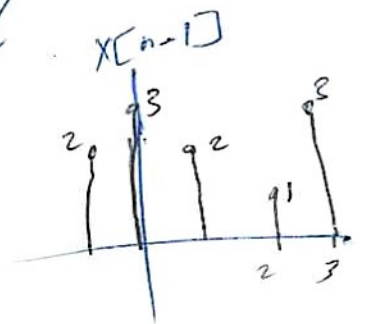


$$X[n] = \{2, 3, 2, 1, 3\}$$

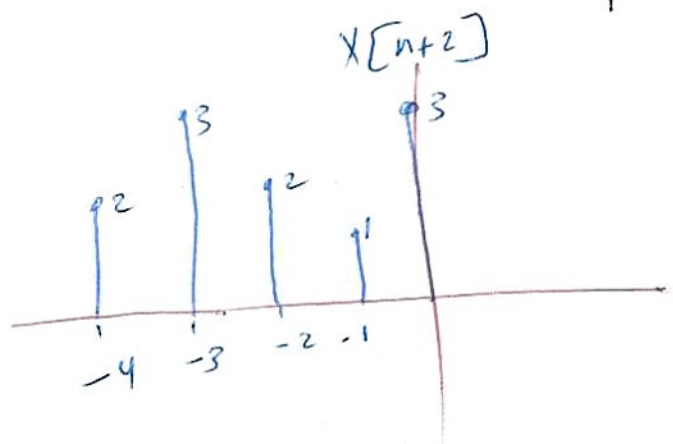
$X[2n-4]$ the best way to solve such a problem is to take $X[2(n-2)]$

time scaling to shifted signal

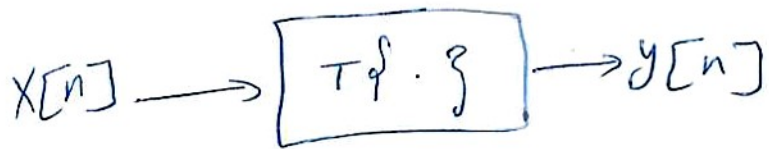
$$X[2-n] \equiv X[-(n-2)]$$



$$X[n-1] = \{2, 3, 2, 1, 3\}$$



Discrete time systems



$$Y[n] = T\{X[n]\}$$

$$X[n] \longrightarrow Y[n]$$

* Linear

$$\text{if } X_1[n] \longrightarrow Y_1[n]$$

$$\text{and } X_2[n] \longrightarrow Y_2[n]$$

$$Y_3 = \alpha Y_1 + \alpha Y_2$$

$$Y_4 = \alpha [X_1 + X_2]$$

$$Y_3 \stackrel{??}{=} Y_4$$

if yes then linear

$$\text{then } a X_1[n] + b X_2[n] \longrightarrow a Y_1[n] + b Y_2[n]$$

Example $Y[n] = 3X[n] + 5$

* Time invariant (shift invariant)

$$X[n] \longrightarrow Y[n]$$

$$X[n - n_0] \longrightarrow Y[n - n_0]$$

$$Y_1[n] = F(X_1[n]) = X[n - n_0]$$

$$Y[n - n_0] \stackrel{?}{=} Y_1[n]$$

if yes \Rightarrow invariant

$$Y[n] = 3X[n] + 5$$

$$Y[n] = 1X[n]$$

So for any LTI system



if $x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \quad \text{by property of LTI systems}$$

\downarrow same

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k] h[k]$$

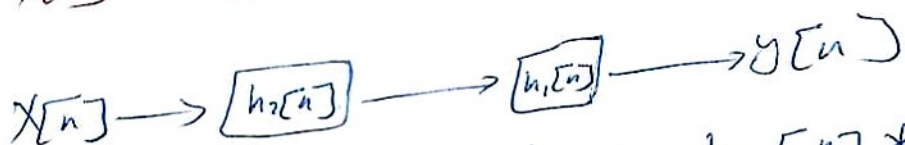
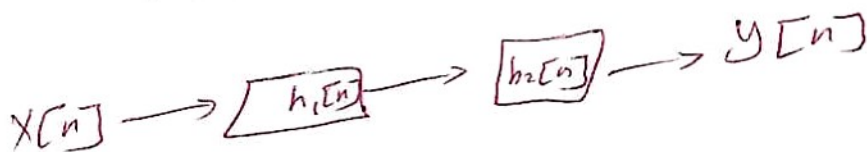
\uparrow
Convolution Sum

$$y[n] = x[n] * h[n]$$

$$= h[n] * x[n]$$

it is commutative

if we have series of system (cascade), the order is not important



$$h_1 * h_2[n] = h_2[n] * h_1[n]$$

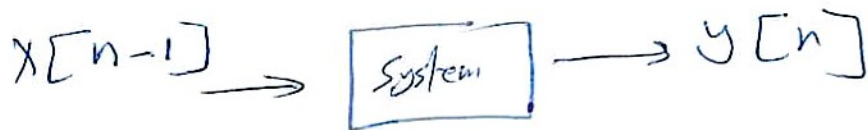
Example- Ideal delay system

$$y[n] = x[n - n_d] \quad -\infty < n < \infty$$

n_d : constant positive integer, called the delay of the system

the system could be dependent on the

present input or on past input (difference equation)



with out feedback or with feedback



delay system is applicable but the advanced system not

Example- moving averager

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

$$= \frac{1}{M_1 + M_2 + 1} \{ x[n + M_1] + x[n + M_1 - 1] + \dots + x[n] \}$$

$$+ x[n-1] + \dots + x[n - M_2] \}$$

[14]

For $M_1 = 0$, $M_2 = 5$

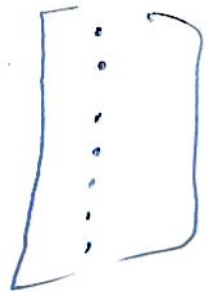
$$y[7] = \frac{1}{6} [x[7] + x[6] + x[5] + x[4] + x[3] + x[2]]$$

$$y[8] = \frac{1}{6} [x[8] + x[7] + x[6] + x[5] + x[4] + x[3]]$$

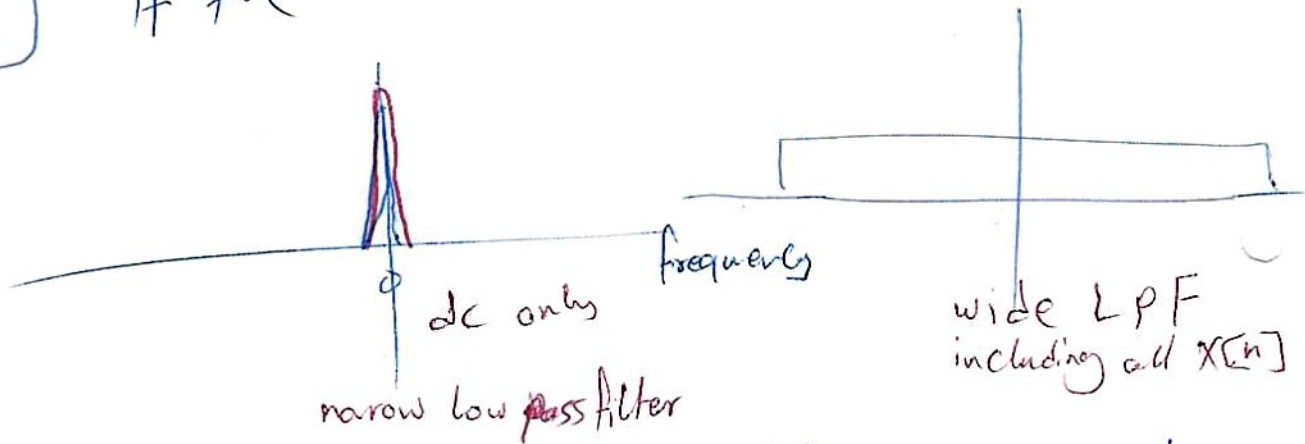
and so on

relation of the window size and filter bandwidth

when the window size = all the data
then $y[n]$ will be one value only (dc)



If the window size = 1 \rightarrow then $y[n] = x[n]$



the design of filter and choosing the correct window will be discussed later in chapters 7 and 9

Example on matlab

$n = 0:1000$

$$x = \cos(2\pi \times 500 \times n / 8000) + \cos(2\pi \times 1000 \times n / 8000) + \cos(2\pi \times 4000 \times n / 8000)$$

plot(n,x)
stem(n,x)

increase the window size to cancel frequencies

* Memoryless / Memory

A system is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n .

Example $y[n] = (x[n])^2$ for each value of n
memoryless system

$$y[n] = x[n^2]$$

memory

Ideal delay system / Moving average filter
memory

Example $y[n] = \sum_{k=-\infty}^n x[k]$ is it linear?

Linear

Example \circ - $w[n] = \log(|x[n]|)$

non linear

Example \circ - $y[n] = \sum_{k=-\infty}^n x[k]$, is it time invariant?

time invariant \rightarrow change of variables

Example :- "Compressor System" Time Variant System

$$y[n] = X[Mn] \quad -\infty < n < \infty$$

* Causality :- The output at any time depends on the instant value of the input or at past value

$$y[n] = X[n] - X[n-1]$$

$$y[n] = \sum_{n=-\infty}^{\infty} X[n]$$

$$y[n] = \sum_{n=-\infty}^{n+1} X[n]$$

$$y[n] = X[n^2]$$

* stability :- [bounded-Input Bounded-output]

if $X[n]$ bounded i.e. $|X[n]| < B_x < \infty$
then $y[n]$ is bounded i.e. $|y[n]| < B_y < \infty$, all n

LTI $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ (absolutely summable)

e.g. $h[n] = 2^n u[n] \rightarrow$ unstable
 $h[n] = (\frac{1}{2})^n u[n] \rightarrow$ stable

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

converge

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}, \quad |\alpha| < 1$$

$$x(t) = \sum_{k=n_0}^n x(k) \quad \text{check the stability}$$

stable?

$$\left| \sum_{k=n_0}^n x[k] \right| = \sum_{k=n_0}^n |x| \leq M < \infty$$

$$= \underbrace{(n-n_0)}_{\text{finite}} M < \infty \rightarrow \text{stable}$$

Causality = ?? depends on n_0

$$\text{Example 2- } x[n] = 4 \cos(0.2\pi n) - 3 \sin(0.3\pi n) + 5 \cos(0.4\pi n)$$

$$\omega_1 N_1 = 2\pi K_1$$

$$0.2\pi N_1 = 2\pi K_1$$

$$\frac{N_1}{K_1} = \frac{2}{0.2} = \frac{10}{1} \Rightarrow N_1 = 10 \text{ Samples}$$

$$\omega_2 N_2 = 2\pi K_2$$

$$0.3\pi N_2 = 2\pi K_2$$

$$\frac{N_2}{K_2} = \frac{2\pi}{0.3\pi} = \frac{2}{0.3} = \frac{20}{3} \Rightarrow N_2 = 20 \text{ Samples}$$

$$\omega_3 N_3 = 2\pi K_3$$

$$0.4\pi N_3 = 2\pi K_3$$

$$\frac{N_3}{K_3} = \frac{2\pi}{0.4\pi} = \frac{20}{4} = 5 \Rightarrow N_3 = 5 \text{ Samples}$$

$$\therefore N_0 = \text{LCM}(N_1, N_2, N_3) = \text{LCM}(10, 20, 5) \\ = 20 \text{ Samples}$$

* Note :- The summation of two periodic discrete signal is always periodic

Example :- Is the signal $4 \cos(0.2\pi n) u[n]$ a periodic signal

It is not because it is not periodic for $n < 0$, periodic signals should be periodic for the infinite interval from $-\infty$ to ∞

Example :- Is the following signal periodic

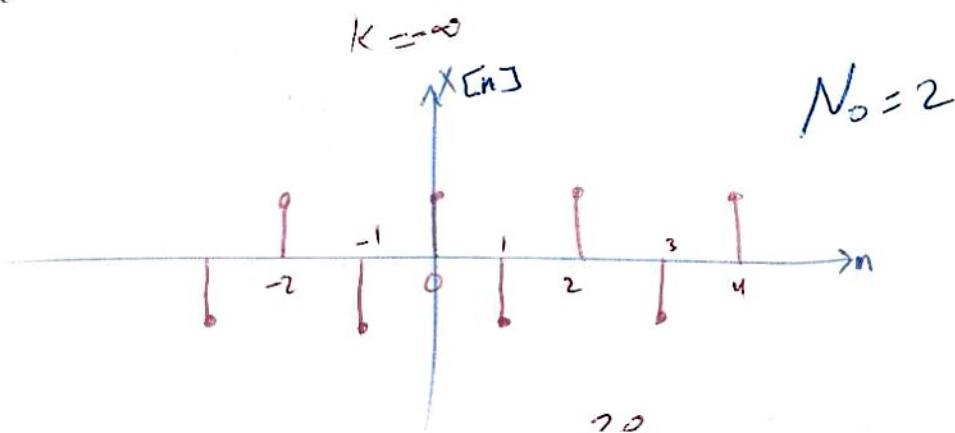
$$x[n] = e^{j\omega n} \quad A_0 e^{j\omega_0 n}$$

$$\omega_0 N = 2\pi k$$

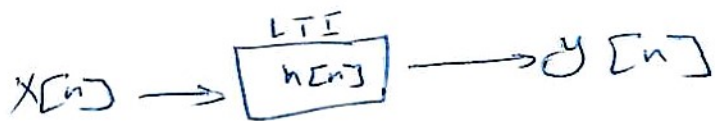
$$2N = 2\pi k$$

$$N = \frac{2\pi k}{2} = \pi k \quad \text{not periodic}$$

Example :- $x[n] = \sum_{k=-\infty}^{\infty} (-1)^k \delta(n-k)$



* linear time invariant system



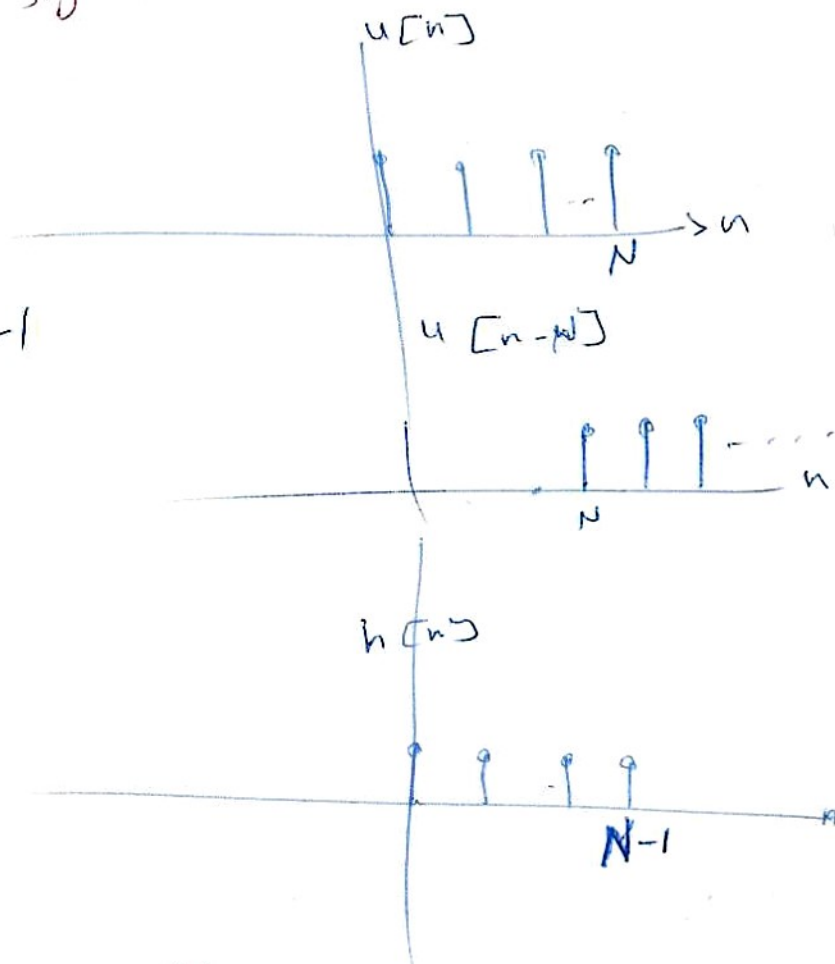
$$y[n] = x[n] * h[n]$$

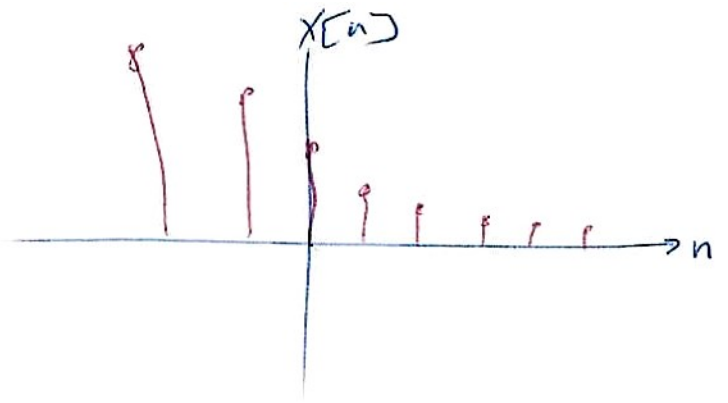
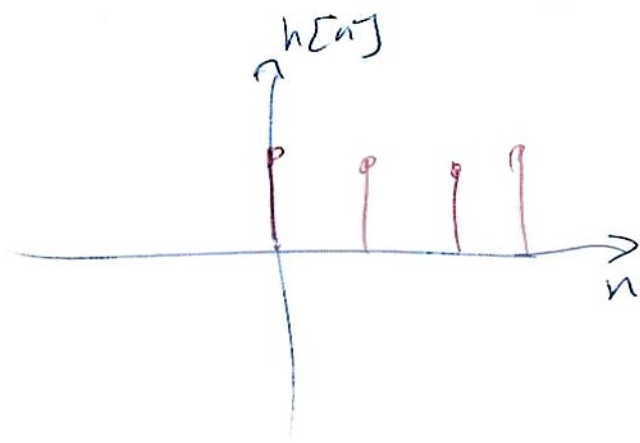
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

Example 0 - consider LTI system with $h[n] = u[n] - u[n-N]$
and the input signal $x[n] = a^n u[n]$ $0 < a < 1$

Find $y[n]$

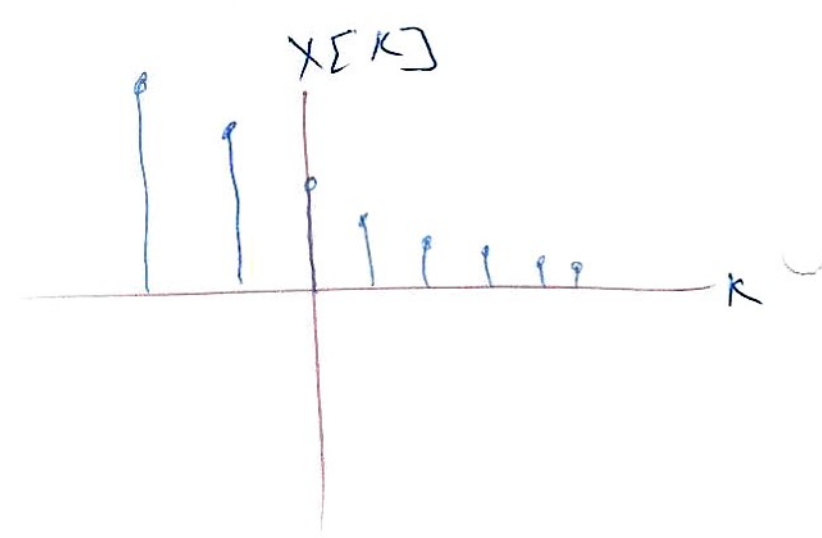
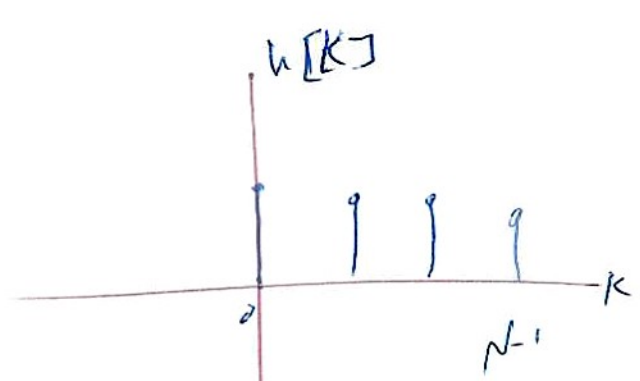
$$h[n] = \begin{cases} 1 & 0 < n < N-1 \\ 0 & \text{o.w} \end{cases}$$



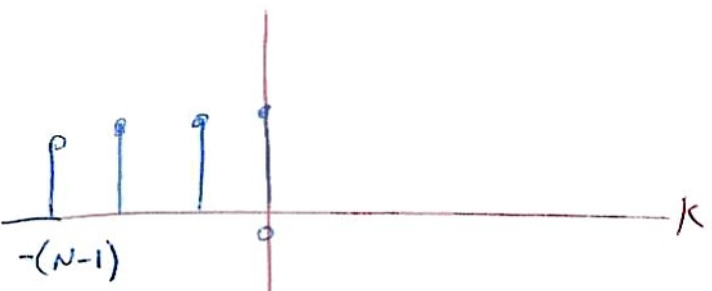


$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

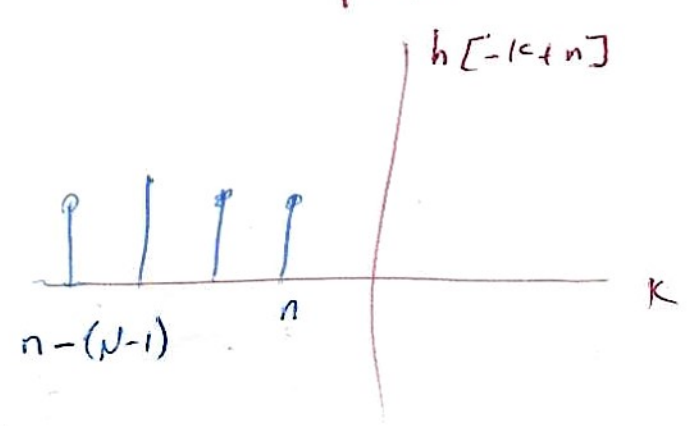
$$= \sum_{k=-\infty}^{\infty} x[n-k] h[k]$$



⇓
h[-k]



shift + n
+ n

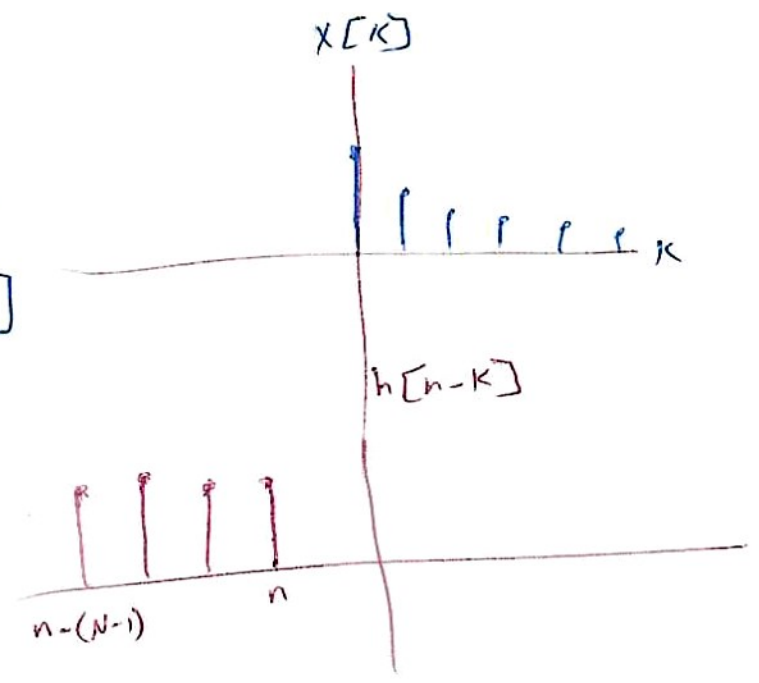


⇒ do the summation

1) for $n < 0$

$$y[n] = 0$$

$[0, \infty]$
~~+~~
 $[0, N-1]$
 $[0, N-1, \infty]$



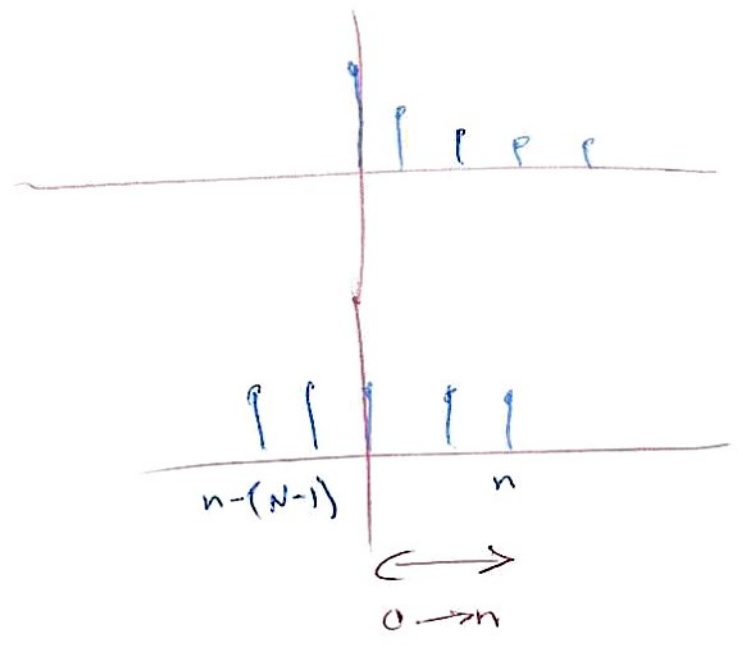
2) for $0 \leq n < N-1$

$$y[n] = \sum_{k=0}^n a^k$$

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$N_2 > N_1$

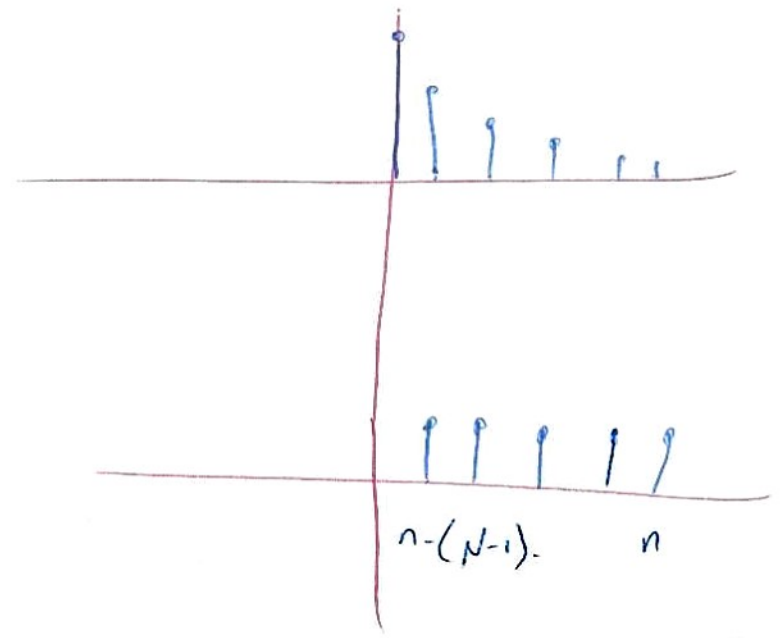
$$\sum_{k=0}^n a^k = \frac{a^0 - a^{n+1}}{1-a}$$



for $n \geq N-1$

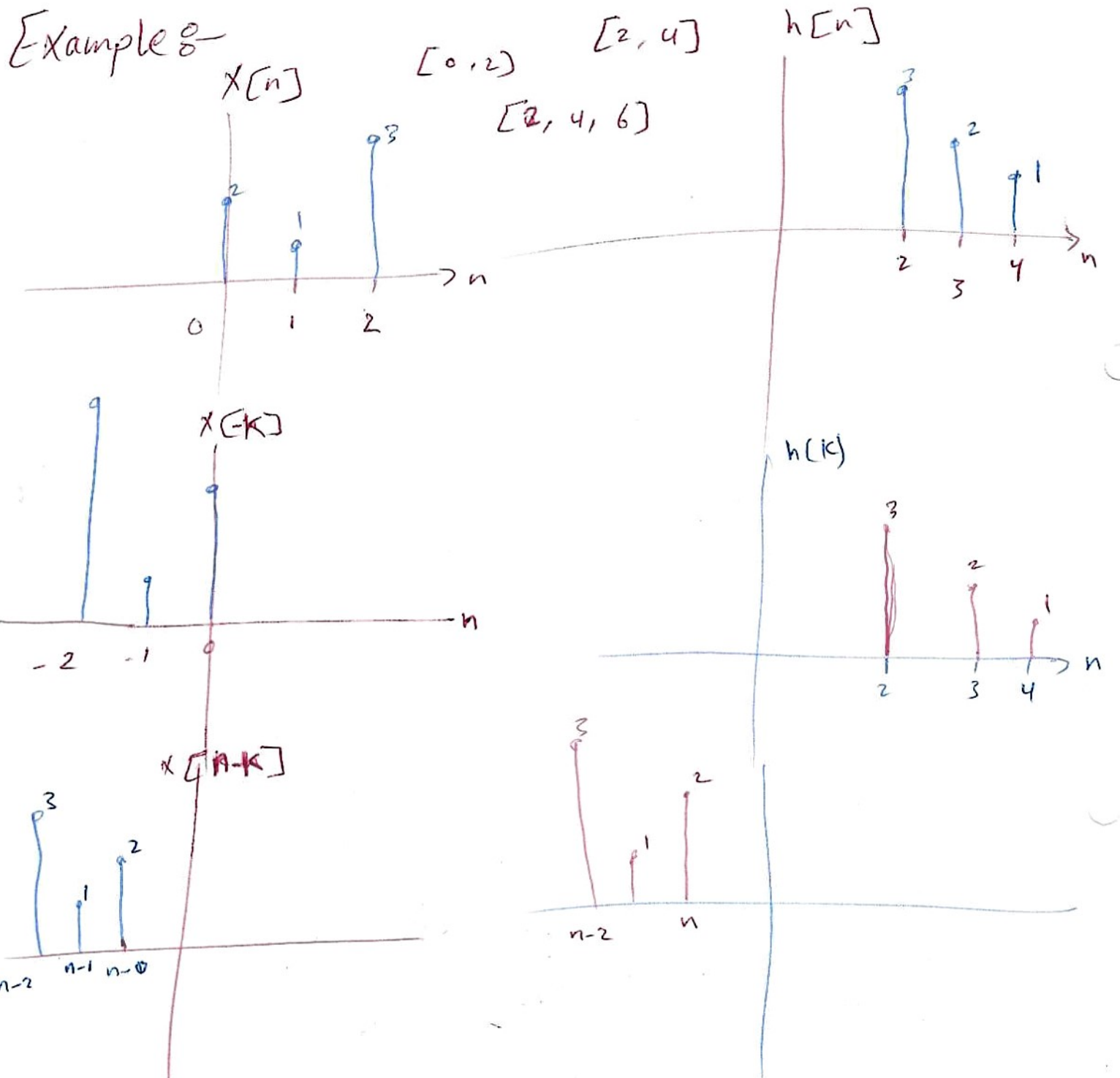
$$y[n] = \sum_{k=n-(N-1)}^n a^k$$

$$= \frac{a^{n-(N-1)} - a^{n+1}}{1-a}$$



$$y[n] = \begin{cases} 0 & n < 0 \\ \frac{1-a^{n+1}}{1-a} & 0 \leq n < M \\ \frac{a^{n-(M-1)} - a^n}{1-a} & \end{cases}$$

Example 8-



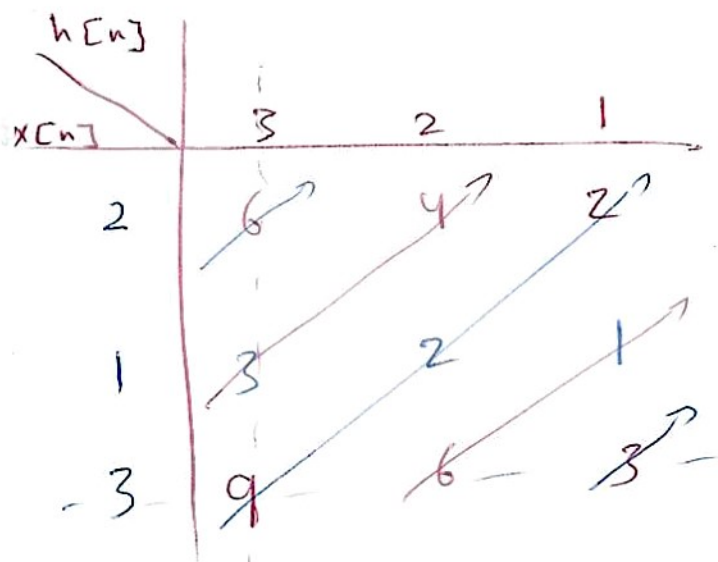
for $n < 2$ $y[n] = 0$
 for $n = 2$ $y[2] = 6$
 for $n = 3$ $y[3] = 1 \times 3 + 2 \times 2 = 7$

for ~~n=4~~ $n=4$, $y[4] = 9 + 2 + 2 = 13$

for $n=5$ $y[5] = 6 + 1 = 7$

for $n=6$ $y[6] = 3$

for $n > 6$ $y[n] = 0$



$$y[n] = \{6, 7, 13, 7, 3\}$$

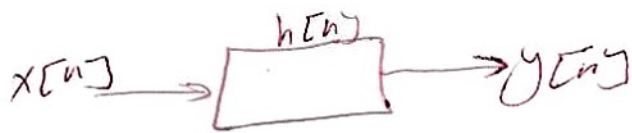
Example 2
consider the following system with impulse response $h[n]$

=

n	0	1	2	3	4	5	6	7
$X[n]$	$X[0]$	$X[1]$	$X[2]$	0	0	0	0	0
$h[n]$	0	0	$h[2]$	$h[3]$	$h[4]$	0	0	0
	0	0	$h[2]X[0]$	$h[3]X[0]$	$h[4]X[0]$			
	0	0	0	$h[3]X[1]$	$h[4]X[1]$	$h[5]X[2]$		
	0	0	0	0	$h[4]X[2]$	$h[5]X[1]$	$h[6]X[2]$	0
	0	0	0	0	0	$h[5]X[0]$	$h[6]X[1]$	$h[7]X[2]$
Σ			6	7	13	7	7	0

Example 2 Consider the system with $h[n] = \{ -1, 2, 0, 1 \}$

and $x[n] = \{ 3, 1, 0, -1 \}$, Find $y[n]$



$$y[n] = x[n] * h[n]$$

$x[n]$ \ $h[n]$	-1	2	0	1
3	-3	6	0	3
1	-1	2	0	1
0	0	0	0	0
-1	1	-2	0	-1

$$y[n] = \{ -3, 5, 2, 4, -1, 0, -1 \}$$

exercise 2 $h[n] = \{ 1, -1, 1 \}$, $x[n] = \{ 2, -1, 1 \}$
 find $y[n] = ??$

Example- Determine the impulse response of an LTI

$$\text{System } y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$$

To find the impulse response $x[n] = \delta[n]$

So in the equation above replace each $x[n]$ by $\delta[n]$

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n]$$

$$\delta[n] \rightarrow \boxed{\text{LTI}} \rightarrow h[n]$$

$$\begin{aligned} \text{So } h[n] &= a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] \\ &\quad + a_4 \delta[n-3] \\ &= \{a_1, a_2, a_3, a_4\} \end{aligned}$$

Example- Consider the LTI discrete system with

$$h[n] = \{1, 2, 0, -3\}, \text{ and plot } y[n] \text{ for}$$

$$\textcircled{1} x_1[n] = \delta[n]$$

$$\textcircled{2} x_2[n] = \delta[n+1] + \delta[n-2]$$

$$\textcircled{3} x_3[n] = \{1, 1, 1\}$$

$$\textcircled{4} x_4[n] = \{2, 1, -1, -2, -3\}$$

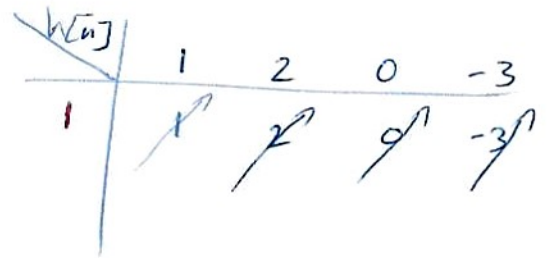
Solution :-



$$y[n] = x[n] * h[n]$$

① $x_1[n] = \delta[n]$

$$\begin{aligned} y[n] &= x_1[n] * h[n] \\ &= \delta[n] * h[n] \\ &= h[n] \end{aligned}$$



$$= \{1, 2, 0, -3\}$$

② $x_2[n] = \delta[n+1] + \delta[n-2]$

$$\begin{aligned} y[n] &= x_2[n] * h[n] \\ &= [\delta[n+1] + \delta[n-2]] * h[n] \\ &= \delta[n+1] * h[n] + \delta[n-2] * h[n] \\ &= h[n+1] + h[n-2] \end{aligned}$$

$$= \{1, 2, 0, -3\} + \{0, 1, 2, 0, -3\}$$

$$y[n] = \{1, 2, 0, -2, 2, 0, -3\}$$

$$\textcircled{3} X_3[n] = \{1, 1, 1\}$$

$$y_3[n] = X_3[n] * h[n]$$

$$= X_3[n] * h[n]$$

$$y[n] = \{1, 3, 3, -1, -3, -3\}$$

$h[n]$	1	2	0	-3
$X_3[n]$				
→ 1	1	2	0	-3
1	1	2	0	-3
1	1	2	0	-3

$$\text{or } X_3[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

$$y[n] = [\delta[n] + \delta[n-1] + \delta[n-2]] * h[n]$$

$$= h[n] + h[n-1] + h[n-2]$$

$$= \{1, 2, 0, -3\} + \{1, 2, 0, -3\} + \{0, 1, 2, 0, -3\}$$

$$= \{1, 3, 3, -1, -3, -3\}$$

$$\textcircled{4} X_4[n] = \{2, 1, -1, -2, 3\}$$

$$y[n] = X_4[n] * h[n]$$

$$= X_4[n] * h[n]$$

$$y[n] = \{2, 5, 1, -10, -10, 9, 6, 9\}$$

$h[n]$	1	2	0	-3
$X_4[n]$				
2	2	4	0	-6
1	1	2	0	-3
→ -1	-1	-2	0	3
-2	-2	-4	0	6
-3	-3	-6	0	9

or

$$x_4[n] = 2\delta[n+2] + \delta[n+1] - \delta[n] - 2\delta[n-1] - 3\delta[n-2]$$

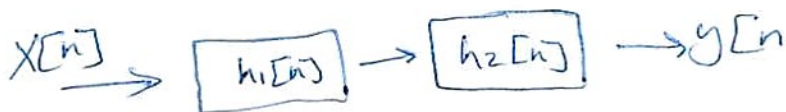
$$y[n] = [2\delta[n+2] + \delta[n+1] - \delta[n] - 2\delta[n-1] - 3\delta[n-2]] * h[n]$$

$$= 2h[n+2] + h[n+1] - h[n] - 2h[n-1] - 3h[n-2]$$

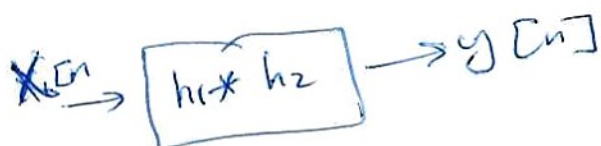
$$= 2 \int \begin{matrix} 1, 2, 0, -3 \\ \uparrow \end{matrix} + \int \begin{matrix} 1, 2, 0, -3 \\ \uparrow \end{matrix} - \int \begin{matrix} 1, 2, 0, -3 \\ \uparrow \end{matrix} - 2 \int \begin{matrix} 1, 2, 0, -3 \\ \uparrow \end{matrix} - 3 \int \begin{matrix} 0, 1, 2, 0, 3 \\ \uparrow \end{matrix}$$

	-3	-2	-1	0	1	2	3	4	5
$-h[n]$			-1	-2	0	3	0	0	
$h[n+1]$		1	2	0	-3	0	0	0	
$2h[n+2]$	2	4	0	-6	0	0	0	0	
$-2h[n-1]$	0	0	0	-2	-4	0	6	0	
$-3h[n-2]$	0	0	0	0	-3	-6	9	9	
	2	5	1	-10	-10	-3	6	9	

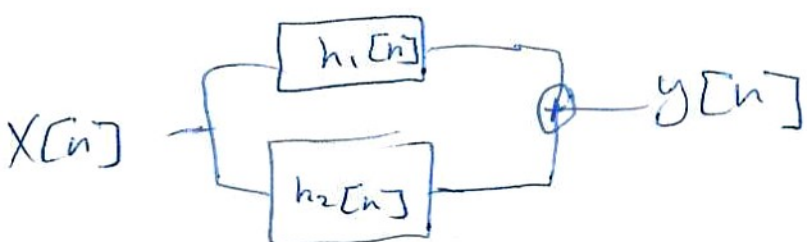
* Connections of LTI system
Series (Cascade) connection



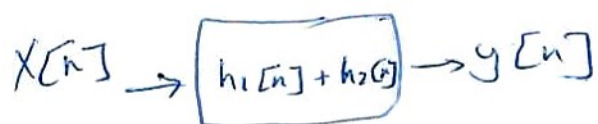
$$x[n] * h_1[n] * h_2[n] = y[n]$$



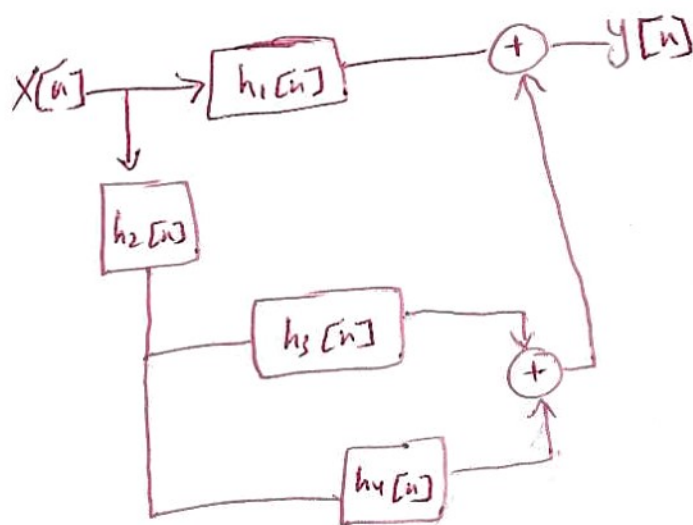
Parallel connection



$$y[n] = X[n] * [h_1[n] + h_2[n]]$$
$$= X[n] * h_1[n] + X[n] * h_2[n]$$



Example Consider the following system



$$\text{If } h_1[n] = \delta[n] + \frac{1}{2} \delta[n-1]$$

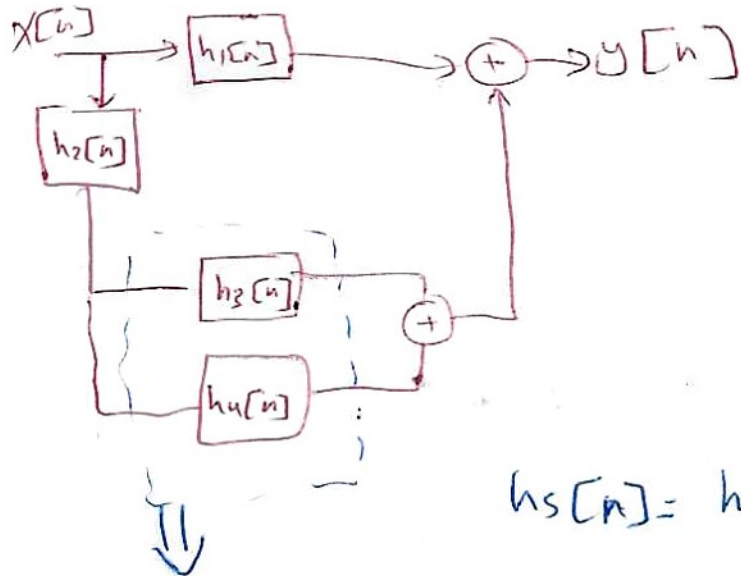
$$h_2[n] = \frac{1}{2} \delta[n] - \frac{1}{4} \delta[n-1]$$

$$h_3[n] = 2\delta[n]$$

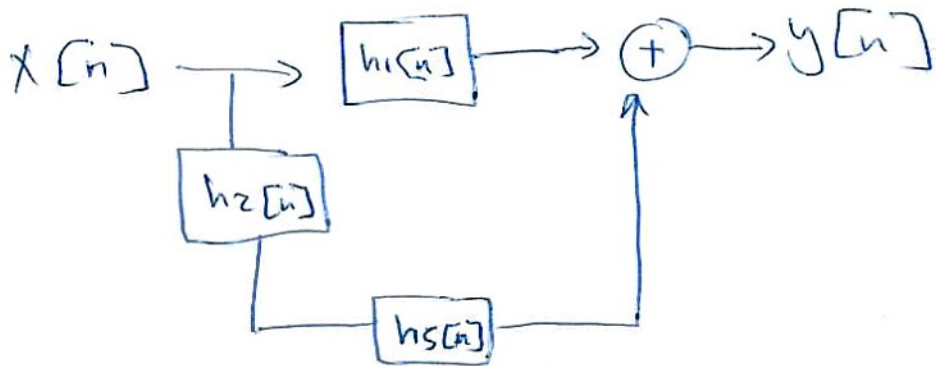
$$h_4[n] = \dots \left(\frac{1}{2}\right)^n u[n] \quad ?2$$

Evaluate the overall impulse response $h[n]$

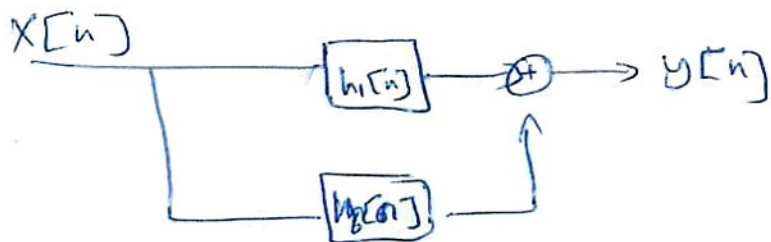
Solution



$$h_5[n] = h_3[n] + h_4[n]$$



$$h_6 = h_2[n] * h_5[n]$$



$$y[n] = [h_1[n] + h_6[n]] * x[n]$$



$$h_7[n] = h_1[n] + h_6[n]$$

$$y[n] = \underbrace{\left[h_1[n] + h_2[n] * (h_3[n] + h_4[n]) \right]}_{h[n]} * x[n]$$

$$h[n] = h_1[n] + h_2[n] * [h_3[n] + h_4[n]]$$

$$= h_1[n] + h_2[n] * h_3[n] + h_2[n] * h_4[n]$$

$$h_2[n] * h_4[n] = \left(\frac{1}{2} \delta[n] - \frac{1}{4} \delta[n-1] \right) * \left(-2 \left(\frac{1}{2} \right)^n u[n] \right)$$

$$= \left(\frac{1}{2} \right) (-2) \left(\frac{1}{2} \right)^n u[n] * \delta[n] - \left(\frac{1}{4} \right) (-2) \left(\frac{1}{2} \right)^n u[n] * \delta[n-1]$$

$$= -1 \left(\frac{1}{2} \right)^n u[n] + \left(\frac{1}{2} \right) \left(\frac{1}{2} \right)^{n-1} u[n-1]$$

$$= -1 \left(\frac{1}{2} \right)^n u[n] + \left(\frac{1}{2} \right)^n u[n-1]$$

$$= -1 \left(\frac{1}{2} \right)^n [u[n] - u[n-1]]$$

$$= -\left(\frac{1}{2} \right)^n \delta[n] = -1 \left(\frac{1}{2} \right)^0 \delta[n] = -\delta[n]$$

$$h_2[n] * h_3[n] = \left[\frac{1}{2} \delta[n] - \frac{1}{4} \delta[n-1] \right] * 2 \delta[n]$$

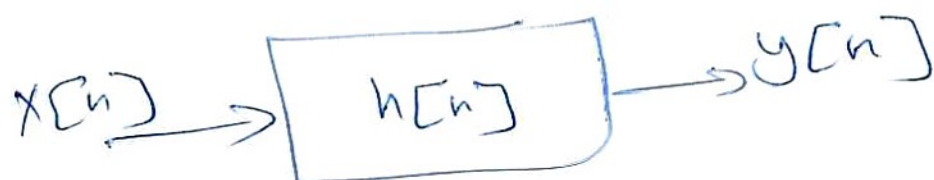
$$= \left(\frac{1}{2}\right)(2) \delta[n] * \delta[n] - \left(\frac{1}{4}\right)(2) \delta[n-1] * \delta[n]$$

$$= \delta[n] - \frac{1}{2} \delta[n-1]$$

$$h[n] = h_1[n] + [h_2 * h_4] + [h_2 * h_3]$$

$$= [\delta[n] + \frac{1}{2} \delta[n-1]] + [\delta[n]] + [\delta[n] - \frac{1}{2} \delta[n-1]]$$

$$= \delta[n]$$



$$y[n] = x[n] * h[n]$$

$$= x[n] * \delta[n]$$

$$= x[n]$$

$$y[n] = x[n] * \delta[n - n_0] \quad \text{convolution theorem}$$

$$= x[n - n_0]$$

$$y[n] = x[n] \delta[n - n_0]$$

$$= x[n_0] \delta[n - n_0]$$

* Correlation of Signals :-

measure the similarity of two signals

↳ Cross Correlation of two different signals ^{processing}

↳ Auto Correlation of the signal and shifted version of the same signal

$$r_{x,y}(L) = \sum_{n=-\infty}^{\infty} x[n] y[n-L]$$

- Convolution → reflect + shifted

- Correlation → only delay

$$L = 0, \pm 1, \pm 2, \dots$$

↳ shift value

$$r_{x,y} \neq r_{y,x}$$

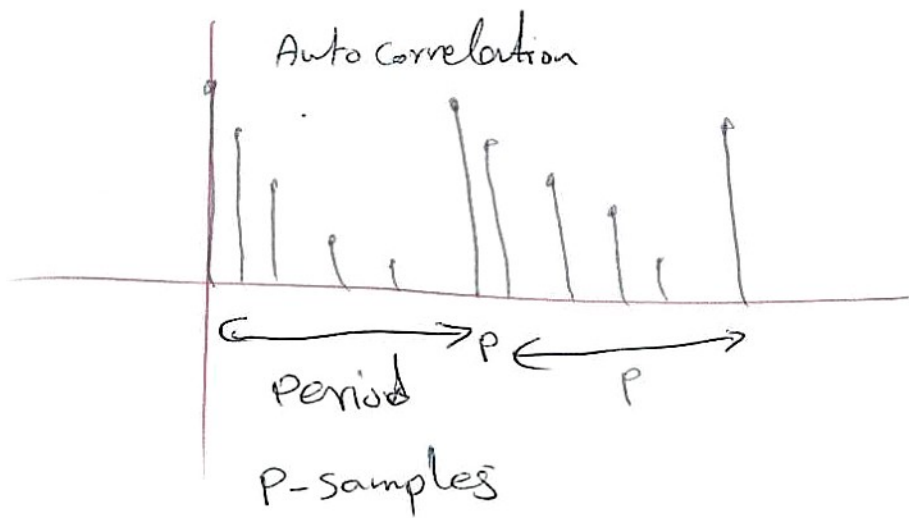
$$r_{y,x}(L) = \sum_{n=-\infty}^{\infty} y[n] x[n-L]$$

$$\text{if } n-L = m$$

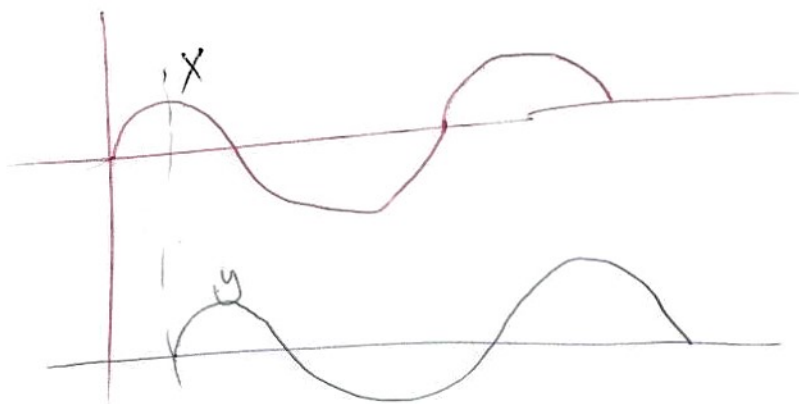
$$= \sum_{n=-\infty}^{\infty} y[m+L] x[m] = r_{x,y}[-L]$$

time reversing

Auto-correlation is a technique to find the periodicity of the signal



Cross-correlation is used to find the delay between two signals



$$r_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2[n] = \text{Energy of the signal}$$

$$r_{xx}[L] = r_{xx}[-L], \quad r_{xx} \text{ is an even function}$$

$$r_{xy}[L] = \sum_{n=-\infty}^{\infty} x[n]y[-L-n] = x[L] * y[-L]$$

Convolution

$$= \text{conv}(x, y)$$

$$r_{xy}(L) = \text{conv}(x, \text{flip}(y))$$

$$r_{xy}[L] = X_{\text{corr}}(x, y) \quad \text{Cross-correlation}$$

$$r_{xx}[L] = X_{\text{corr}}(x) \quad \text{Auto-correlation}$$

Correlation ~~Coefficient~~ Normalization

$$C \quad \rho_{xx}(L) = \frac{r_{xx}[L]}{r_{xx}[0]} \quad |\rho_{xx}[L]| \leq 1$$

$$\rho_{xy}(L) = \frac{r_{xy}[L]}{\sqrt{r_{xx}[0] r_{yy}[0]}} \quad |\rho_{xy}[L]| \leq 1$$

Auto correlation

$$X[n] = \{-1, 2, 1\}$$

		-1	2	1	
-1	2	1			
		-1			

-1

	-1	2	1	
-1	2	1		
	-2	2		

0

	-1	2	1	
	-1	2	1	
	1	4	1	

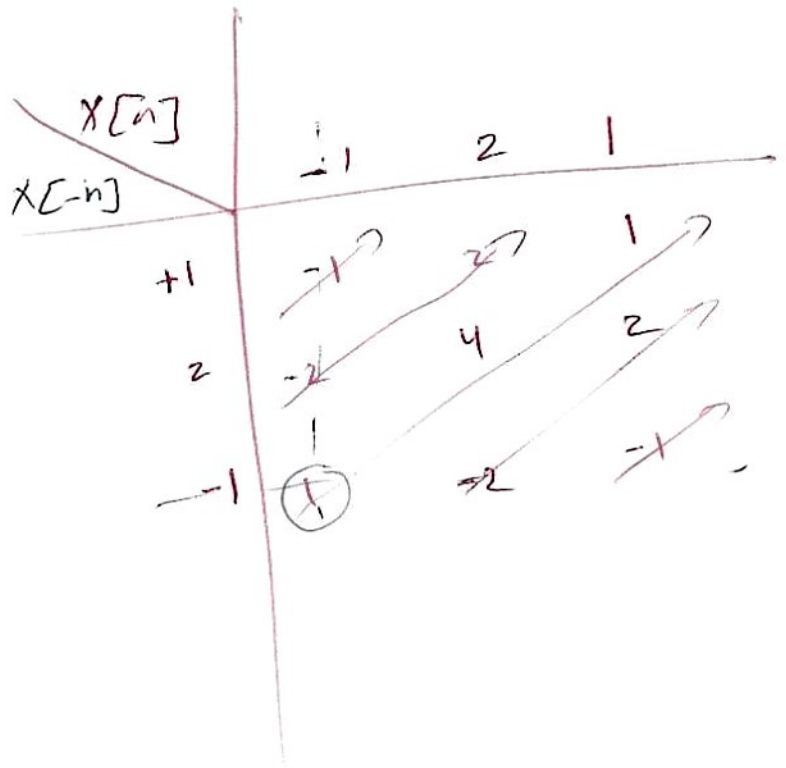
6

	-1	2	1	
		-1	2	1
		-2	2	

0

	-1	2	1		
			-1	2	1
			-1		

-1



$$-1, 0, 6, 0, -1$$

$$P_{xx}(k) = \{-1, 0, 6, 0, -1\}$$

↑
6

$$P_{xx}[0] = 1$$

$$P_{xx}[-1] = 0$$

$$P_{xx}[L] = \{-0.166, 0, 1, 0, -0.166\}$$

Cross correlation

Example 2

Given that

$$x[n] = \{1, 1, 2, 2\}, y[n] = \{1, 2, 3, 4\}$$

Find $r_{xy}[L]$, and $\rho_{xy}[L]$

$$r_{xy} = \{4, 7, 13, 17, 11, 6, 2\}$$

$$r_{xx}[0] = 10$$

$$r_{yy}[0] = 30$$

$$\rho_{xy}[L] = \frac{\{4, 7, 13, 17, 11, 6, 2\}}{\sqrt{10 \times 30}}$$

$$= \frac{\{4, 7, 13, 17, 11, 6, 2\}}{17.32}$$

$x[n]$				
$y[n]$	1	1	2	2
4	4	4	8	8
3	3	3	6	6
2	2	2	4	4
1	1	1	2	2

Example 3

$$x[n] = \{1, 3, -2, 4\}$$

$$y[n] = \{2, 3, -1, 3\}$$

$$z[n] = \{-2, 4, -1, 2\}$$

Find r_{xy}

r_{xz}

ρ_{xy}

ρ_{xz}

X _i \ Z _j	1	3	-2	4
2	2	6	-4	8
-1	-1	-3	2	-4
4	4	12	-8	16
-2	-2	-6	4	-8

$$r_{XZ} = \{ 2, 5, -3, 20, -18, 20, -8 \}$$

$$r_{XX}[0] = 30$$

$$r_{ZZ}[0] = 25$$

$$\begin{aligned}
 \rho_{XZ}[L] &= \frac{\{ 2, 5, -3, 20, -18, 20, -8 \}}{\sqrt{30 \times 25}} \\
 &= \frac{\{ 2, 5, -3, 20, -18, 20, -8 \}}{\sqrt{24.39}}
 \end{aligned}$$

X _i \ Y _j	1	3	-2	4
3	3	9	-6	12
-1	-1	-3	2	-4
3	3	9	-6	12
2	2	6	-4	8

$$r_{XY}[L] = \{ 3, 8, -6, 25, -4, 8, 8 \}$$

$$r_{XX}[0] = 30$$

$$r_{YY}[0] = 23$$

$$\begin{aligned}
 \rho_{XY}[L] &= \frac{\{ 3, 8, -6, 25, -4, 8, 8 \}}{\sqrt{30 \times 23}} \\
 &= \frac{\{ 3, 8, -6, 25, -4, 8, 8 \}}{\sqrt{26.27}}
 \end{aligned}$$

we can conclude that y is more similar to X than Z