Properties of LTI Discrete system

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

Causality of LTI System

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k]\, h[n_0-k]$$

the system is causal if $y[n]$ is independent of the futur values of the input

for $k < 0$ then $y[n] \longrightarrow x(n--k)$

$$x[n+k]$$

for this case now the output depends on the futur input, therfore to be causal $y[n]$ should be zero. So $h[n]$ should be zero

$h[n]$ should be zero for $k < 0$

Stability of LTI system

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x[k]|\, |h[n-k]|$$

It is given that $|x[n]|$ is bounded

So to have bounded $y[n]$, $\sum_{k=-\infty}^{\infty} |h[n-k]| < \infty$

bounded → finite

Finite impulse response (FIR)

infinite impulse response (IIR)

FIR

$$h[n] = \{0, 0, 0, \dots, 0, \underset{\uparrow}{a_1}, a_2, a_3, 0, 0, 0, 0\}$$

$h[n] = 0$ for all $n > N$

IIR

$$h[n] = \begin{cases} a^n & \text{for } n > 0 \\ 0 & \text{o.w} \end{cases} \qquad h = \{0, \dots, 0, \underset{\uparrow}{1}, a, a^2, a^3\}$$

Linear Difference Equations:-

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{r=0}^{m} b_r X[n-r]$$

Order ≡ No of delays in output sequence

E.g zero order

$$a_0 y[n] + 0 = \sum_{r=0}^{m} b_r X[n-r]$$

$$a_0 = 1 \quad h[n] = \begin{cases} b_n & , n = 0, 1, 2, 3 \cdots m \\ 0 & , o.w \end{cases}$$

1st order

1st

$$a_0 y[n] + a y[n-1] = \sum_{r=0}^{m} b_r X[n-r]$$

In general

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r X[n-r]$$

$$a_0 y[n] + \sum_{k=1}^{\infty} a_k y[n-k] = \sum_{r=0}^{M} b_r [n-r]$$

If we assume that $a_0 = 1$ and $a_k = 0$, $k = 1, 2, 3, 4 \cdots N$

$$y[n] = \sum^{M} b_m X[n-m] \quad , \quad h[n] = \begin{cases} b_n & , n = 0, 1, 2, 3 \cdots M \\ 0 & , o.w \end{cases}$$

Example:- Consider the following system described by the following difference equation:-

$$y[n] = x[n] + a\,y[n-1]$$

Evaluate the response of the system if

$$x[n] = K\,\delta[n] \quad \text{and} \quad y[-1] = C$$

Sol:- ~~Once~~ $x[n]$ is impulse then $y[n] = h[n]$

= impulse response

* for $n \geq 0$

$y[0] = K\,\delta[0] + a\,y[-1]$

$= K(1) + a\,C$

$y[1] = K\,\delta[1] + a\,y[0]$

$= 0 \quad + a\,[K + aC]$

$= aK + a^2 C$

$y[2] = K\,\delta[2] + a\,y[1]$

$= 0 \quad + a\,[aK + a^2 C]$

$= a^2 K + a^3 C$

$\therefore\ y[n] = a^n K + a^{n+1} C$

for $n \leq -1$

$$y[n-1] = a^{-1}\left[y[n] - x[n]\right]$$

$n = -1$

$$y[-2] = a^{-1}\left[y[-1] - x[-1]\right]$$

$$= a^{-1}\left[c - k\delta[-1]\right]$$

$$= a^{-1}[c - 0]$$

$$= a^{-1}c$$

$$y[-3] = a^{-1}\left[y[-2] - k\delta[-2]\right]$$

$$= a^{-1}\left[a^{-1}c - 0\right]$$

$$= a^{-2}c$$

$$y[n] = a^{n+1}c$$

$$y[n] = a^{n+1}c + a^{n}k\,u[n]$$

Example : Determine the impulse of the first order system given as follows :

$$y[n] = x[n] + a\,y[n-1], \text{ Assum } y[n] = 0,\ n < 0$$

for impulse response $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$$h[n] = \delta[n] + a\,h[n-1]$$

$y[n] = 0$ for $n < 0$ Causal system

$y[-1] = 0 \Rightarrow h[-1] = 0$

$y[-2] = 0 \quad h[-2] = 0$

$$h[0] = \delta[0] + a\,\underbrace{h[-1]}_{0}$$

$$= 1$$

$$h[1] = y[1] = a$$

$$h[2] = a^2$$

$$h[n] = a^n u[n]$$

$$y[n] = x[n] + a\,y[n-1]$$

$$y[n-1] = \frac{y[n] - x[n]}{a}$$

$$= a^{-1}\left[y[n] - x[n]\right]$$

Let us assume tha $y[n] = 0$ for $n > 0$

non Causal system

and $x[n] = \delta[n]$

$$y[n-1] = a^{-1}\left[y[n] - \delta[n]\right]$$

$$y[1] = 0$$

$$y[0] = 0$$

$$y[-1] = a^{-1}\left[\overset{0}{y[0]} - \overset{1}{\delta[0]}\right]$$

$$= -a^{-1}$$

$$y[-2] = a^{-1}\left[y[-1] - \delta[-1]\right]$$

$$= a^{-1}\left[-a^{-1} -\right.$$

$$= -a^{-2}$$

$$y[n] = -a^{n}\,u[-n-1]$$

* frequency Response of the system

$$x[n] \longrightarrow \boxed{\begin{matrix} LTI \\ h[n] \end{matrix}} \longrightarrow y[n]$$

$$x[n] = e^{jw_0 n}$$

for LTI

$$y[n] = \sum_{k=-\infty}^{\infty} h(k)\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{jw_0(n-k)}$$

$$= \underbrace{e^{jw_0 n}}_{\text{same as the input}} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-jw_0 k}}_{H(e^{jw_0}) \text{ function of } w_0}$$

Eigen Function { Same as function

$$y[n] = H(e^{jw_0})\, e^{jw_0 n}$$

where $H(e^{jw_0}) = |H e^{(jw_0)}| \angle \theta_{H(e^{jw_0})}$

→ when input to the system, produces the same input signal.

$|H(e^{jwn})| = |H(e^{-jwn})|$ even function

$\theta(H(e^{jwn})) = -\theta(H(e^{-jwn}))$ odd function

Example: Consider the ideal delay system defined by

$$y[n] = x[n-n_d]$$

find the frequency response of the system?

$$x[n] = e^{j\omega_0 n}$$

$$y[n] = x[n-n_d] = e^{j\omega_0 (n-n_d)} = e^{j\omega_0 n} \underbrace{e^{-j\omega_0 n_d}}$$

$$H(e^{j\omega_0}) = e^{\underbrace{-j\omega_0 n_d}_{\text{phase}}}$$

magnitud

Example: Consider $h[n] = \delta(n-n_d)$ evaluate the frequency response

$$H(e^{j\omega_0}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega_0 k}$$

$$= \sum_{k=-\infty}^{\infty} \delta(n-n_d)\, e^{-j\omega_0 k}$$

$$= e^{-j\omega_0 \underbrace{n_d}_{\text{phase}}}$$

magnitude

$$H(e^{j\omega_0}) = \cos(\omega_0 n_d) - j\sin(\omega_0 n_d)$$

$$H_R(e^{j\omega}) + j H_I(e^{j\omega_0})$$

where $H_R(e^{j\omega}) = \cos(\omega_0 n_d)$ and $H_I(e^{j\omega_0}) = -\sin(\omega_0 n_d)$

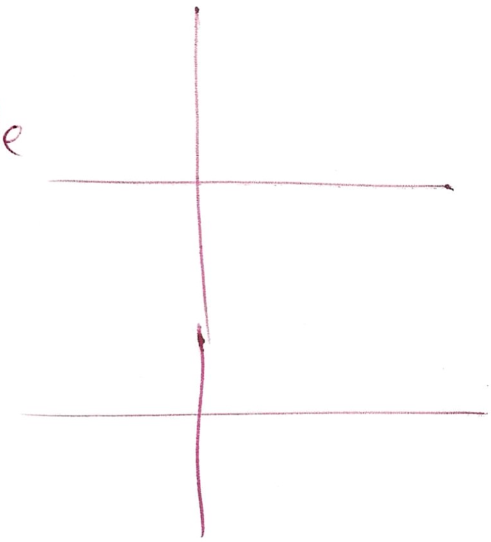
* Sinusoidal response

$$x[n] = A\cos(w_o n + \phi)$$

$$H(e^{jw_o}) = |H(e^{jw_o})|\, e^{j\theta(w_o)}$$

$$y[n] = A\,|H(e^{jw_o})| \cdot \cos(w_o n + \phi + \theta)$$

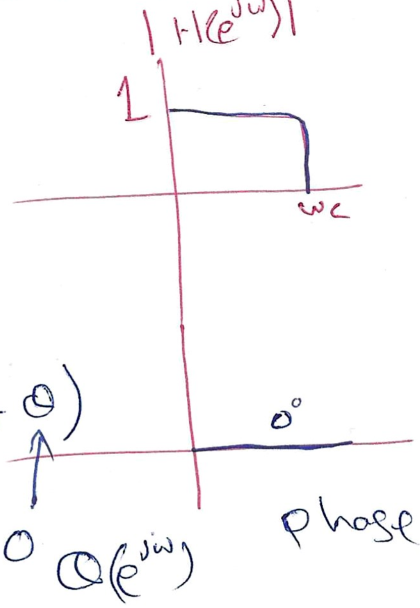
$H(e^{jw_o})$ → magnitude
→ phase



Example :- consider the system as low pass filter

$$x[n] = A\cos(w_o n + \phi)$$

based on the magnitud and the phase of the frequency response

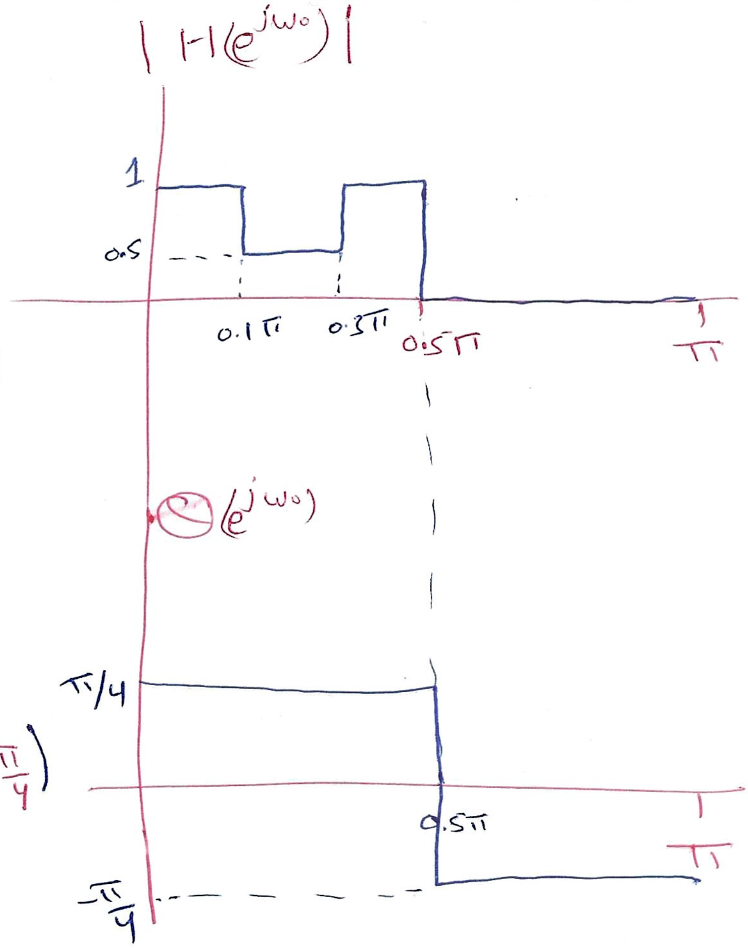


$$y[n] = A\,|H(e^{jw_o})| \cos(w_o n + \phi + \theta)$$

θ: 0

$|H(e^{jw_o})|$: 1 if $w_o < w_c$, 0 if $w_o > w_c$

Example:



$$x_1[n] = 2\cos(0.2\pi n)$$

$$y_1[n] = (0.5)2\cos(0.2\pi n + \tfrac{\pi}{4})$$

$$x_2[n] = 4\cos(0.4\pi n)$$

$$y_2[n] = (1)4\cos(0.4\pi n + \pi/4)$$

$$x_3[n] = 3\cos(0.8\pi n)$$

$$y_3[n] = (0)(3)\cos(0.8\pi n - \pi/4)$$

$$= 0$$

Examples
consider the following System described by the following Difference equation:

$$y[n] - y[n-1] = x[n]$$

assume the system is Causal and stable.

from previous example $h[n] = a^n u[n]$ → Causality

$0 < a < 1$ → stability

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (\underset{\text{magnitude}}{a} \underset{\text{phase}}{e^{-jw}})^n$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

$$0 < |\alpha| < 1$$

sum of geomatric series

$$= \frac{1}{1 - a e^{-jw}}$$

$$|H(e^{jw})|^2 = \frac{1}{1 - a e^{jw}} \cdot \frac{1}{1 - a e^{-jw}}$$

$$= \frac{1}{1 + a^2 - 2a \cos w}$$

$$H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$= \frac{1}{1 - a[\cos(w) - j\sin(w)]}$$

$$= \frac{1\angle 0}{1 - a\cos(w) + ja\sin(w)}$$

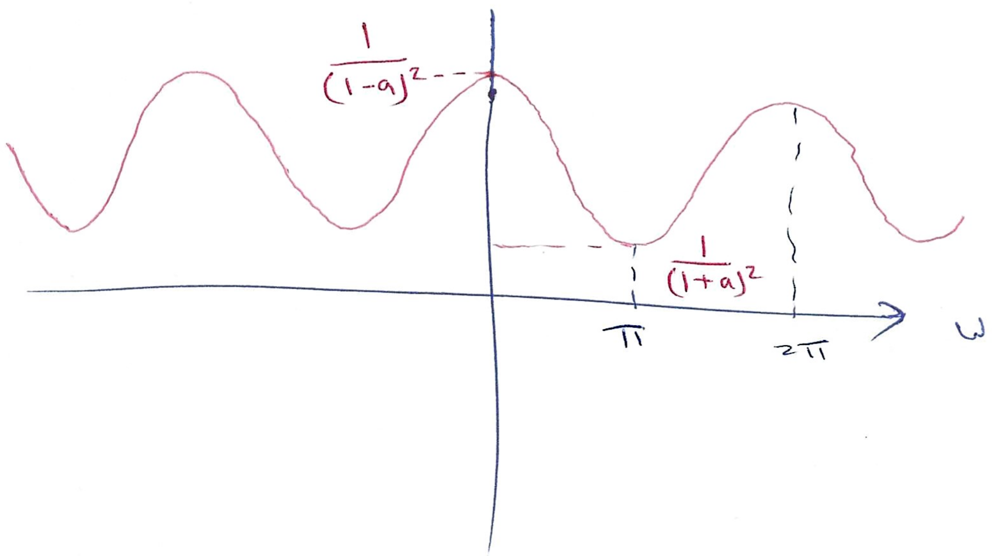
$$= \frac{1\angle 0}{\sqrt{(1 - a\cos(w))^2 + (a\sin(w))^2}\angle \tan^{-1}\left(\frac{a\sin(w)}{1 - a\cos(w)}\right)}$$

$$= \frac{1\angle 0}{\sqrt{1 - 2a\cos(w) + a^2\cos^2(w) + a^2\sin(w)}\angle \tan^{-1}\left(\frac{a\sin(w)}{1 - a\cos(w)}\right)}$$

$$|H(e^{jw})| = \frac{1}{\sqrt{1 - 2a\cos(w) + a^2}}, \quad \Theta_{H(e^{jw})} = -\tan^{-1}\left(\frac{a\sin(w)}{1 - a\cos(w)}\right)$$

$$|H(e^{jw})|^2 = \frac{1}{1 - 2a\cos(w) + a^2}$$ easier to sketch

$$|H(e^{jw})|^2$$



the response is low pass filter

the frequency response is continuous

suppose $x[n] = 2\cos(0.2\pi n)$, find $y[n]$

$y[n] = (\ )2\cos(0.2\pi n + \ )$

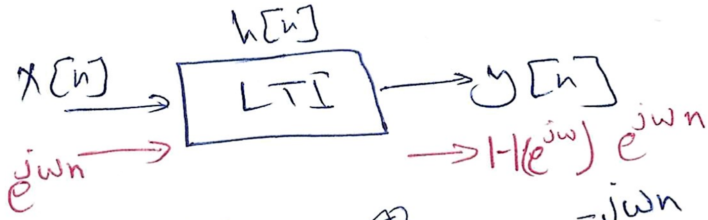
from frequency response

$$\angle He^{jw} = -\tan^{-1}\left(\frac{a\sin w}{1 - a\cos w}\right)$$

* properties of frequency Response

- continuous
- Periodic

Fourier Transform for DT Systems



$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn}$$ (The frequency response of the system)

$\equiv$ Fourier transform for DT systems

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw}) e^{jwn} \, dw$$, w is a Continuous Variable

n is a discrete Variable

to prove the relation

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw}) e^{jwn} \, dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-jwk}\right) e^{jwn} \, dw$$

$$= \sum_{k=-\infty}^{\infty} h[k] \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw(n-k)} \, dw$$

$$= \sum_{k=-\infty}^{\infty} h[k] \delta[n-k]$$

$$= h[n]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw(n-k)} = \frac{\sin\pi(n-k)}{\pi(n-k)}$$

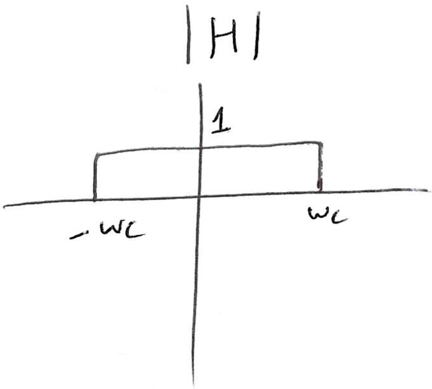
$$= \frac{\sin \pi x}{\pi x} = \mathrm{Sinc}\,\pi x$$

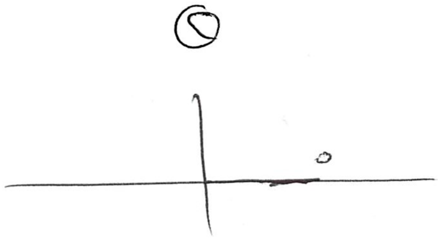
$$= \begin{cases} 1 , & n = k \\ 0 , & \text{o.w} \end{cases}$$

$$= \delta(n-k)$$

Examples: Ideal low pass filter

$$H_{LP}(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & w_c \le w < 2\pi \end{cases}$$





$$h[n] = \frac{1}{2\pi}\int_{-w_c}^{w_c} 1\, e^{jwn} = \frac{1}{2\pi jn}\left(e^{jw_cn} - e^{-jw_cn}\right)$$

$$= \frac{\sin w_c n}{\pi n}$$

* Theorems of fourier transform for DT

1- Linearity :

$$\mathcal{F}\left[a\,x_1[n] + b\,x_2[n]\right] = a\,X_1(e^{jw}) + b\,(X_2(e^{jw}))$$

2- Time shifting

$$\mathcal{F}(x[n-nd]) = X(e^{jw})\,e^{-jwnd}$$

3- Frequency shifting

$$\mathcal{F}[x[n]\,e^{jw_0 n}] = X(e^{j(w-w_0)})$$

4- Time reversal

$$x[n] \xrightarrow{\mathcal{F}} X(e^{jw})$$

$$x[-n] \xrightarrow{\mathcal{F}} X(e^{-jw})$$

$$X^*(e^{jw})$$

$|X(e^{jw})| = |X(e^{-jw})|$

even function

but the phase is odd

5- Differential in frequency

$$n\,X[n] \xleftrightarrow{\mathcal{F}} j\,\frac{d\,X(e^{jw})}{dw}$$

6- Parseval's theorem

$$E = \sum_{n=-\infty}^{\infty} |X[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{jw})|^2\,dw$$

$$|X(e^{jw})|^2 \text{ : energy spectral density)}$$

7- Convolution theorem

$$\mathcal{F}\left(X[n] * h[n]\right) = X(e^{jw})\,H(e^{jw})$$
$$= Y(e^{jw})$$

8- Modulation and windowing theorem

$$X[n]\,w[n] \xleftrightarrow{\mathcal{F}} X(e^{jw}) * W(e^{jw})$$

Example: Consider the following system

$$y[n] = x[n] \cos(w_o n)$$

Evaluate $Y(e^{jw})$

Solution: $Y(e^{jw}) = \mathcal{F}\left[x[n] \cos(w_o n)\right]$

$$= \mathcal{F}\left[\frac{1}{2} x[n] e^{jw_o n} + \frac{1}{2} x[n] e^{-jw_o n}\right]$$

Linearity

$$= \mathcal{F}\left[\frac{1}{2} x[n] e^{jw_o n}\right] + \mathcal{F}\left[\frac{1}{2} x[n] e^{-jw_o n}\right]$$

$$= \frac{1}{2} X\left(e^{j(w-w_o)}\right) + \frac{1}{2} X\left(e^{j(w+w_o)}\right)$$

frequency shifting

using modulation theorem

$$\mathcal{F}\left[\frac{1}{2} x[n] e^{jw_o n}\right] = \frac{1}{2} \mathcal{F}\left[x[n] e^{jw_o n}\right]$$

constant

$$= \frac{1}{2} F[x[n]] * F(e^{jw_o n})$$

given that $F(1) = \delta(w)$

$$\therefore \mathcal{F}(1 e^{jw_o n}) = \delta(w - w_o)$$

$$\therefore \mathcal{F}\left[\frac{1}{2} x[n] e^{jw_o n}\right] = \frac{1}{2} F[x[n]] * \delta(w - w_o)$$

$$= \frac{1}{2} X\left(e^{j(w-w_o)}\right)$$

Example: Consider the following system

$$x[n] \longrightarrow \boxed{LTI} \longrightarrow y[n]$$

If $x[n] = \delta[n-n_d]$, find $Y(e^{j\omega})$

$$y[n] = x[n] * h[n]$$

$$= \delta[n-n_d] * h[n]$$

$$= \underbrace{h[n-n_d]}_{\text{time shifting}}$$

$$Y(e^{j\omega}) = \mathcal{F}(h[n-n_d]) = H(e^{j\omega})\, e^{-j\omega n_d}$$

Example: Evaluate the frequency response of the following LTI system with input/output relation given by the following difference equation

$$y[n] - \frac{1}{2} y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

$$x[n] \longrightarrow \overset{LTI}{\boxed{h[n]}} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$\mathcal{F}(y[n]) = \mathcal{F}[x[n] * h[n]]$$

$$Y(e^{jw}) = X(e^{jw})\,H(e^{jw})$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$\mathcal{F}\left(y[n] - \tfrac{1}{2}y[n-1] = x[n] + 2\cancel{x}[n-1] + x[n-2]\right)$$

$$Y(e^{jw}) - \tfrac{1}{2}Y(e^{jw})e^{-jw} = X(e^{jw}) + 2X(e^{jw})e^{-jw} + X(e^{jw})e^{-j2w}$$

$$Y(e^{jw})\left[1 - \tfrac{1}{2}e^{-jw}\right] = X[e^{jw}]\left[1 + 2e^{jw} + e^{-j2w}\right]$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1 + 2e^{jw} + e^{-j2w}}{1 - \tfrac{1}{2}e^{-jw}}$$

Example :- consider the following system $y[n] - y[n-1] = x[n]$
evaluate $H(e^{jw})$

$$y[n] - y[n-1] = x[n]$$

$$\mathcal{F}\left(y[n] - y[n-1] = x[n]\right)$$

$$= Y(e^{jw}) - Y(e^{jw})e^{-jw} = X(e^{jw})$$

$$Y(e^{jw})\left[1 - e^{-jw}\right] = X[e^{jw}]$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1 - e^{-jw}}$$

Example: Determine the fourier transform of the sequence

$$x[n] = a^n u[n-5].$$

to evaluate $X(e^{jw})$ the following steps should be considered

$$x[n] = a^n u[n-5]$$

$$x[n] = a^n u[n-5] \frac{a^{-5}}{a^{-5}}$$

$$= \frac{1}{a^{-5}} e^{n-5} u[n-5]$$

$$= a^{5} e^{n-5} u[n-5]$$

scaling time shifting

$$x[n] = a^n u[n]$$

$$X(e^{jw}) = \frac{1}{1-ae^{-jw}}$$

$$X(e^{jw}) = a^5 \left(\frac{1}{1-ae^{-jw}} \right) e^{-j5w}$$

Example:- Determine the impulse response of the high pass filter with delay

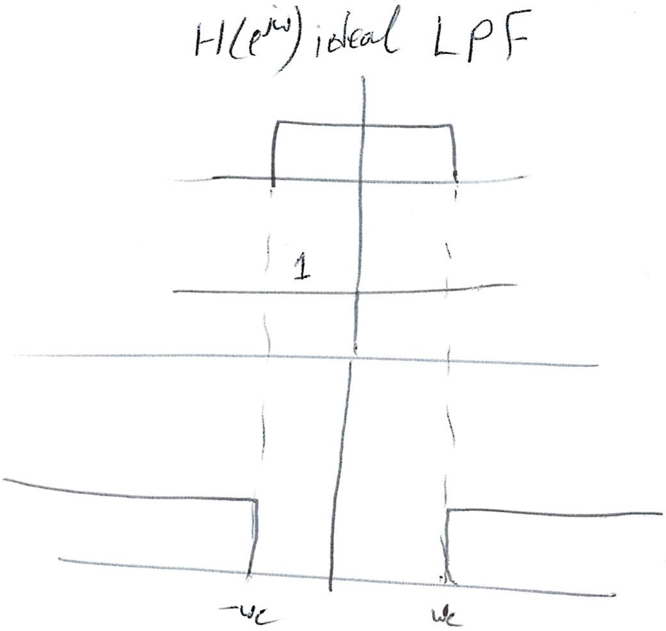
$$H_{Hp}(e^{jw}) = \begin{cases} e^{-jwnd} & , w_c < |w| < \pi \\ 0 & , |w| < w_c \end{cases}$$

for ideal High pass filter

$$H_{Hp}(e^{jw}) = 1 - H_{Lp}(e^{jw})$$

for non ideal

$$H_{Hp}(e^{jw}) = e^{-jwnd}\left(1 - H_{Lp}(e^{jw})\right)$$



$$h_{Hp}[n] = \delta[n-nd] - h_{Lp}(n-nd)$$

$$= \delta(n-nd) - \frac{\sin(w_c(n-nd))}{\pi(n-nd)}$$

Inverse Fourier transform

Example :- Suppose that

$$X(e^{jw}) = \frac{1}{(1-ae^{-jw})(1-be^{-jw})}$$

Evaluate $x[n]$

$$\frac{1}{(1-ae^{-jw})(1-be^{-jw})} = \frac{A}{1-ae^{-jw}} + \frac{B}{1-be^{-jw}}$$

$$1 = A(1-be^{-jw}) + B(1-ae^{-jw})$$

$$1-be^{-jw} = 0$$

$$e^{-jw} = \frac{1}{b}$$

$$1 = A(0) + B(1-a(\tfrac{1}{b}))$$

$$1 = B(1-\frac{a}{b})$$

$$1 = B(\frac{b-a}{b}) \Rightarrow B = \frac{b}{b-a}$$

$$1 - a e^{-jw} = 0$$

$$e^{-jw} = \frac{1}{a}$$

$$1 = A\left(1 - b\left(\frac{1}{a}\right)\right) + B(0)$$

$$1 = A\left(1 - \frac{b}{a}\right)$$

$$1 = A\left(\frac{a-b}{a}\right) \Rightarrow A = \frac{a}{a-b}$$

$$X[e^{jw}] = \frac{a}{a-b} \cdot \frac{1}{1 - a e^{-jw}} + \frac{b}{b-a} \cdot \frac{1}{1 - b e^{-jw}}$$

$$= \frac{a}{a-b} a^n u[n] + \frac{b}{b-a} b^n u[n]$$