Chapter Two: Discrete - Time Signals and System

Introduction:

- Signals: can be defined as
  1. A flow of information.
  2. a function of independent variable such as time (e.g: speech signal), position (e.g: image), etc.

- Examples of signals:-
  - Speech: 1-Dimension signal as a function of time, s(t).
  - Grey-scale image: 2-Dimension signal as a function of space i(x,y)
  - Video: 3x3-Dimension signal as a function of space and time {r(x,y,t), g(x,y,t), b(x,y,t)}

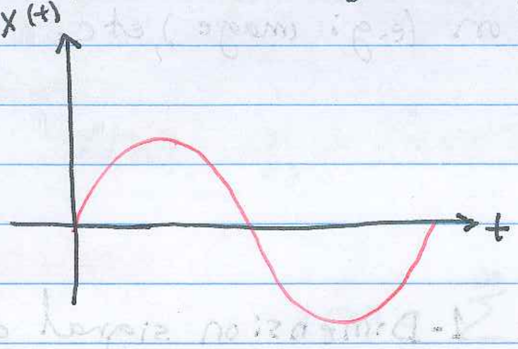
- Types of signals:
  - The independent variable may be either continuous or discret
    - * Continuous - time signal
    - * Discrete-time signals: are defined at discrete times and represented as a sequences of numbers.

• The signal amplitude may be either continuous or discrete

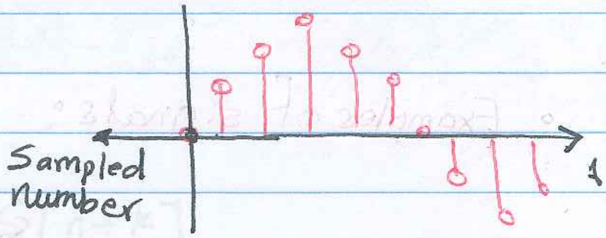
* Analog Signals: both time and amplitude are continuous.

* Analog Signals: both time and amplitude are continuous.

* Digital Signals: both are discrete.

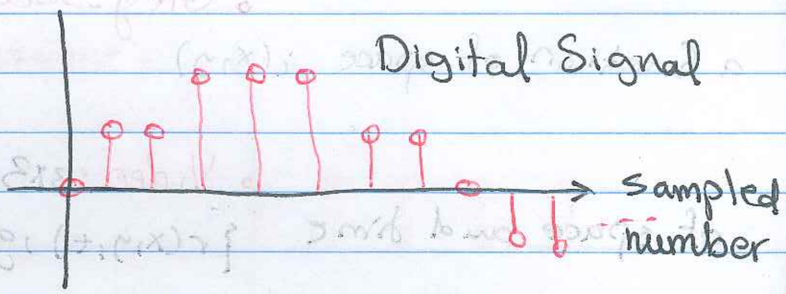


Analog Signal



Discret-time signal

Digital Signal



• Digital Signal Processing:

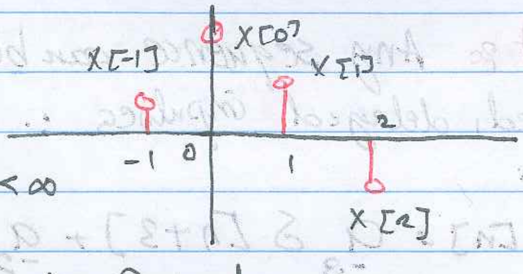
• Modifying and analyzing information with computers - So being measured as sequence of numbers.

• Representation, transformation and manipulation of signals and information they contain.

• Typical DSP System Components

Analog to Digital Converter (ADC) | Digital to Analog Converter (DAC)

$x[n] \equiv x(nT)$



• Discrete Signals:

$x[n] = x(nT) \qquad -\infty < n < \infty$

where n is an integer

T is called the sampling Period.

• Sequence Operations:

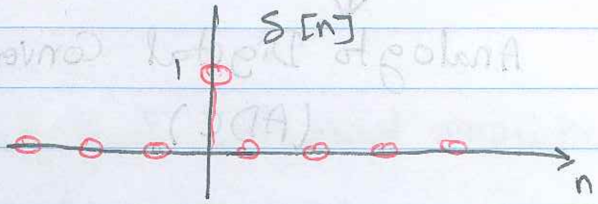
- The product and sum of two sequences $x[n]$ and $y[n]$: Sampled-by-Sampled production and sum, respectively.
- Multiplication of a sequence $x[n]$ by a number multiplication of each sample value.
- Delay or shift of a sequence $x[n]$

$y[n] = x[n - n_0]$ where n is an integer.

- Basic Sequences :

* Unit Sample sequence (Discrete-time impulse)

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



Note that : Any sequence can be represented as a sum of scaled, delayed impulses :

For example:

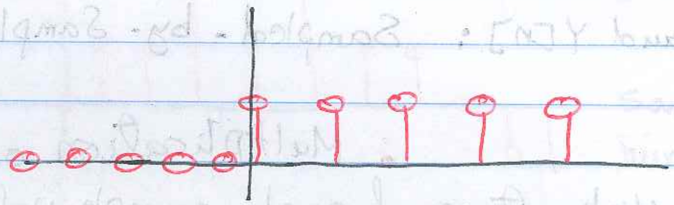
$$x[n] = a_{-3}\,\delta[n+3] + a_{-2}\,\delta[n+3] + \cdots + a_5\,\delta[n-5]$$

More genearlly :

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

* Unit Step Sequence :

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Also, it can be written in terms of impulse sequences,

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$$

or

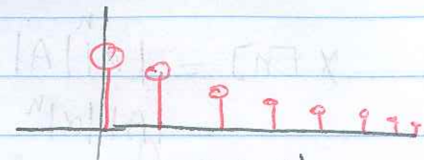
$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\,\delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$$

Conversely,

$$\delta[n] = u[n] - u[n-1]$$

- **Exponential Sequences:**

$$x[n] = A\alpha^n$$



$0 < \alpha < 1$

It can be noted :-

* If A and α are real numbers, the sequence is real.

* If $0<\alpha<1$ and A is positive, the sequence values are positive and decrease with increasing n.

* If $-1<\alpha<0$, the sequence values alternate in sign, but again decrease in magnitude with increasing n.

* If $|\alpha|>1$, the sequence values increase with increasing n.

* An exponential sequence that is zero for $n<0$ can be expressed as:

$$x[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n<0 \end{cases}$$

or

$$x[n] = A\alpha^n u[n]$$

- **Sinusoidal Sequences**

$$x[n] = A\cos(\omega_0 n + \phi), \text{ for all } n \quad \text{-----} \quad (*)$$

with A and ϕ are real constants.

This result can be obtained as follow

If we assume $\alpha = |\alpha| e^{j\omega_0}$ and $A = |A| e^{j\phi}$

$$x[n] = |\alpha|^n |A| e^{j\omega_0 n + j\phi}$$
$$= |A||\alpha|^n e^{j(\omega_0 n + \phi)}$$

By using Euler's Equation

$$x[n] = |A| |\alpha|^n [\cos(\omega_0 n + \phi) + j \sin(\omega_0 n + \phi)]$$

By taking real part then

$$x'[n] = \mathrm{Re}\{x[n]\}$$
$$= |A| |\alpha|^\alpha \cos(\omega_0 n + \phi)$$

and if we assume $|\alpha| = 1$ then we obtain

the same result expressed in (*)

$$x'[n] = |A| \cos(\omega_0 n + \phi)$$

- Periodic & a Periodic Signals :

In the discrete-time case, a periodic sequence is defined as :

$$x[n] = x[n+N], \text{ for all } n$$

where the period N is necessary to be an integer.

For sinusoid,

$$A \cos(\omega_0 n + \phi) = A \cos(\omega_0 n + \omega_0 N + \phi)$$

which requires that $\omega_0 N = 2\pi k$ or $N = 2\pi k / \omega_0$

where k is an integer.

Note that :-

- The signal should be priodic if

$$\frac{N}{k} = \frac{2\pi}{\omega_0} \Rightarrow \text{rational number.}$$

- otherwise

$$\frac{N}{k} : \text{irrational number}$$

$\Rightarrow$ The signal will be a periodic signal

Take care: N and k should be integer numbers.

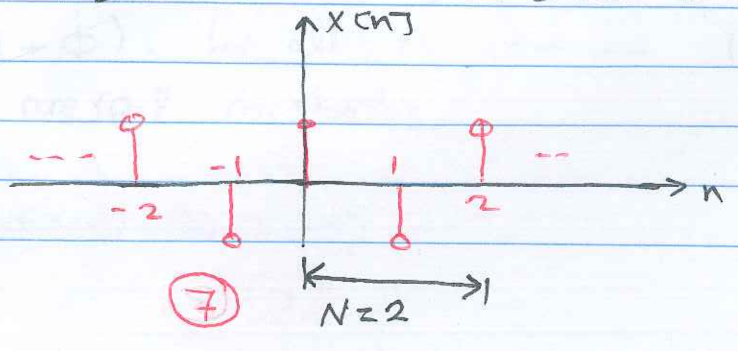
Example:- Consider the following signal

$$x[n] = \sum_{K=\infty}^{\infty} (-1)^k \delta(n-k)$$, check if $x[n]$ is periodic or not?

Ans:

$$x[n] = \sum_{K=-\infty}^{\infty} (-1)^k \delta(n-k)$$

$$= \cdots + \delta(n+2) - \delta(n+1) + \delta(0) - \delta(n-1) + \delta(n+2) + \cdots$$



Examples: Consider the following signal

$$x[n] = A \sin(\omega_0 n + \Phi)$$

check if $x[n]$ is periodic or not ?

Ans: If $x[n]$ is periodic with fundamental period N then

$$\omega_0 N = 2\pi k \quad , \; k = 1, 2, 3, \cdots$$

$$\frac{N}{k} = \frac{2\pi}{\omega_0} \quad \text{should be rational number}$$

If $\frac{N}{k}$ is irrational number then $x[n]$ is a periodic.

- Rational Number ($\frac{N}{k}$, where $q \neq 0$)

Rational Number $\left(\frac{N}{k}\right)$ could be
- → Finite fractional part e.g, $\frac{1}{2} = 0.5$
- → Infinite fractional part which is cyclic e.g, $\frac{1}{3} = 0.333$

OR : ratio of two integer is always rational.

Examples on irrational number:

$$\pi, \; \sqrt{2}, \; \frac{1}{\sqrt{2}}$$

Example: Check the periodicity of each signal, In case of periodic signal, specify the values of k, and N.

1. $x[n] = 3\cos(0.2\pi n)$

Ans:

$$x[n] = 3\cos(0.2\pi n)$$

where $\omega_0 = 0.2\pi$

$\Rightarrow \frac{N}{K} = \frac{2\pi}{\omega_0} = \frac{2\pi}{0.2\pi} = \frac{10}{1} \Rightarrow$ rational number

where $N = 10$ and $K = 1 \Rightarrow$ periodic @ $N = 10$ and $K = 1$

2. $x[n] = 5\sin(0.3\pi n)$

Ans:

$$x[n] = 5\sin(0.3\pi n)$$

where $\omega_0 = 0.3\pi$

$\Rightarrow \frac{N}{K} = \frac{2\pi}{0.3\pi} = \frac{20}{3} \Rightarrow$ rational number

periodic @ $N = 20$ and $k = 3$

Note:

- If we assume $k = 1 \Rightarrow N = \frac{20}{30}$ (Not integer number) ✗
- If we assume $k = 2$, $\Rightarrow N = \frac{40}{3}$ (Not integer number) ✗
- If we assume $K = 3$, $\Rightarrow N = 20$ (K and N are integer) ✓

Remember: For periodic signal K and N should be integer Numbers.

3. $X[n] = 4\cos(0.5n)$

Ans:

$$X[n] = 4\cos(0.5n)$$

where $\omega_0 = 0.5$

$\Rightarrow \frac{N}{K} = \frac{2\pi}{\omega_0} = \frac{2\pi}{0.5} = 4\pi \Rightarrow$ irrational number

$\Rightarrow X[n]$ is a periodic Signal.

4. $X[n] = 8\sin\left(\frac{\pi}{\sqrt{2}}n\right)$

Ans:

w/ $X[n] = 8\sin\left(\frac{\pi}{\sqrt{2}}n\right)$

where $\omega_0 = \frac{\pi}{\sqrt{2}}$

$\Rightarrow \frac{N}{K} = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2} \Rightarrow$ irrational number

$\Rightarrow X[n]$ is a periodic signal.

5. $X[n] = 10e^{-j1.1\pi n}$

Ans:

$$X[n] = 10e^{-j1.1\pi n}$$

where $x[n]$ can be written in form

$$x[n] = A e^{-j\omega_0 n}$$

$\Rightarrow \quad \omega_0 = 1.1\pi$

$$\frac{N}{K} = \frac{2\pi}{\omega_0} = \frac{2\pi}{1.1\pi} = \frac{20}{11}$$

The signal $x[n]$ will be periodic signal when $k = 11$ and $N = 20$.

6. $x[n] = 10 \, e^{-j1.1\pi(n-1)}$

Ans:

$$x[n] = 10 \, e^{-j1.1\pi(n-1)}$$
$$= 10 \, e^{-j1.1\pi n} \, e^{j1.1\pi}$$

where

$$e^{j1.1\pi} = \cos(1.1\pi) + j\sin(1.1\pi)$$

$\omega_0 = 1.1\pi =$

$\Rightarrow$

$$\frac{N}{K} = \frac{2\pi}{1.1\pi} = \frac{20}{11} \Rightarrow \text{rational number}$$

$x[n]$ is periodic signal @ $N = 20$ and $k = 11$

7. $x[n] = 10\, e^{-j1.1\pi(n-1)}\, u(n-1)$

Ans:

$$x[n] = 10\, e^{-j1.1\pi(n-1)}\, u(n-1)$$

$$= \begin{cases} 10\, e^{-j1.1\pi(n-1)} & n \geq 1 \\ 0 & \text{o.w} \end{cases}$$

$\Rightarrow$

$x[n]$ is a periodic signal.

- Periodicity of Composite Signal:

$$x[n] = x_1[n] + x_2[n] + x_3[n] + \cdots + x_K[n]$$

where the fundamental period for each signal is

$N_1, N_2, N_3, \cdots, N_K$, respectively.

$\Rightarrow$

Example: Check the periodicity of each signal. In case of periodic signal, specify the values of k, and N

1. $x[n] = 4\cos(0.2\pi n) - 3\sin(0.3\pi n) + 5\cos(0.4\pi n)$

Ans: $x[n+N] \stackrel{?}{=} x[n]$

$0.2\pi N_1 = 2\pi k_1$

$\frac{N_1}{k_1} = \frac{2}{0.2} = \frac{20}{2} = \frac{10}{1} \Rightarrow N_1 = 10$ and $k_1 = 1$

$0.3\pi N_2 = 2\pi k_2$

$\frac{N_2}{k_2} = \frac{2}{0.3} = \frac{20}{3} \Rightarrow N_2 = 20$ and $k_2 = 3$

and

$0.4\pi N_3 = 2\pi k_3$

$\frac{N_3}{k_3} = \frac{2}{0.4} = \frac{20}{4} = \frac{5}{1} \Rightarrow N_3 = 5$ and $k_3 = 1$

$N = LCM(10, 20, 5) = 20$

2.

$x[n] = 3\sin\left(\frac{\pi}{4}n\right) + 5\cos\left(\frac{\pi}{3}n\right) - 7\sin\left(\frac{\pi}{2}n\right)$

$\frac{\pi}{4}N_1 = 2\pi k_1 \Rightarrow \frac{N_1}{k_1} = \frac{8}{1} \Rightarrow N_1 = 8$ and $k_1 = 1$

$\frac{\pi}{3}N_2 = 2\pi k_2 \Rightarrow \frac{N_2}{k_2} = \frac{6}{1} \Rightarrow N_2 = 6$ and $k_2 = 1$

$\frac{\pi}{2} N_3 = 2\pi k_3$

$\frac{N_3}{k_3} = 4 \Longrightarrow N_3 = 4$ and $k_3 = 1$

$N = \text{LCM}(8, 6, 4) = 24$

3. $x[n] = 2\cos(\sqrt{2}\pi n) + 5\sin 2\sqrt{2}\pi n$

$\sqrt{2}\pi N_1 = 2\pi k_1$

$\frac{N_1}{k_1} = \frac{\sqrt{2}}{1}$ a periodic "irrational number"

$2\sqrt{2}\pi N_2 = 2\pi k_2$

$\frac{N_2}{k_2} = \frac{1}{\sqrt{2}}$

irrational number

aperiodic signal

$\Rightarrow$ A periodic Signal

4. $x[n] = 10\cos 0.1\pi n - 6\cos 0.9\pi n + 5\sin 0.7n$

$0.1\pi N_1 = 2\pi k_1$

$\frac{N_1}{k_1} = \frac{2}{0.1} = \frac{20}{1}$

periodic

$0.9\pi N_2 = 2\pi k_2$

$\frac{N_2}{k_2} = \frac{2}{0.9} = \frac{20}{9}$

periodic

$0.7 N_3 = 2\pi k_3$

$\frac{N_3}{k_3} = \frac{2\pi}{0.7}$

"irrational number"

a periodic

$\Longrightarrow$ A periodic Signal

Discrete - time Systems

A transformation or operator that maps input into output can be expressed as

$$y[n] = T\{x[n]\}$$

$$x[n] \longrightarrow \boxed{T\{.\}} \longrightarrow y[n]$$

- Examples on Discrete-time Systems

  - The ideal delay system

$$y[n] = x[n-n_d] \qquad -\infty < n < \infty$$

  - A memoryless system

$$y[n] = (x[n])^2 \qquad -\infty < n < \infty$$

- Properties of Discrete-time Systems

1. Linear and Non-Linear System

A system is linear if and only if the following properties are achieved :

ⓐ additivity property

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$

(b) Scaling Property

$$T\{a\, x[n]\} = a\, T\{x[n]\} = a\, y[n]$$

where a is an arbitrary constant

In other words, scaling and additivity should be achieved as

$$T\{a x_1[n] + b x_2[n]\} = \underbrace{a\, T\{x_1[n]\} + b\, T\{x_2[n]\}}_{c\,T\{x_3[n]\}} = \underbrace{a y_1[n] + b y_2[n]}_{c\, y_3[n]}$$

Example: Consider the following accumulator system which is defined by the following input-output equation

$$y[n] = \sum_{k=\infty}^{n} x[k]$$

check the linearity of the System

Ans:

$$\alpha_1 y_1[n] = \sum_{k=\infty}^{n} \alpha_1 x_1[k]$$

$$\alpha_2 y_2[n] = \sum_{k=\infty}^{n} \alpha_2 x_2[k]$$

$$\underbrace{\alpha_1 y_1[n] + \alpha_2 y_2[n]}_{\alpha_3 y_3[n]} = \sum_{k=-\infty}^{n} [\underbrace{\alpha_1 x_1[k] + \alpha_2 x_2[k]}_{\alpha_3 x_3[k]}]$$

$\Rightarrow$ The system is linear

Example: Consider the system defined by

$$w[n] = \log_{10}(|x[n]|)$$

check the linearity of the system.

Ans:

$$\alpha_1 w_1[n] = \log_{10}(|\alpha_1 x_1[n]|)$$
$$\alpha_2 w_2[n] = \log_{10}(|\alpha_2 x_2[n]|)$$

$$\underbrace{\alpha_1 w_1[n] + \alpha_2 w_2[n]}_{} = \underbrace{\log_{10}(|\alpha_1 x_1[n]|) + \log_{10}(|\alpha_2 x_2[n]|)}_{}$$

$$= \alpha_3 w_3[n] \qquad \neq \log_{10}(|\alpha_3 x_3[n]|)$$
$$= \log_{10}(|\alpha_1 x_1[n] + \alpha_2 x_2[n]|)$$

The system is Non-linear.

2. Time-invariant Systems

- For which a time shift or delay of the input sequence causes a corresponding shift in the output sequence

Example: Consider the compressor system which is defined by the relation

$$y[n] = x[Mn] \qquad -\infty < n < \infty$$

check if the system is time-invariant or time-variant

Ans:

$$y_1[n-n_0] = x_1[M[n-n_0]] \quad \text{D} \quad \text{“time shift”}$$
$$y_2[n-n_0] = x_2[Mn-n_0] \quad \text{“Delay of the input sequence”}$$

Since $y_1[n-n_0] \neq y_2[n-n_0]$
$\Rightarrow$ The system is time-variant

Example: Consider accumulator system which is defined by

$$y[n] = \sum_{k=\infty}^{n} x[k]$$

check if the system is time-invariant or time-variant

Ans:
$$y_1[n-n_0] = \sum_{k=\infty}^{n-n_0} x_1[k] \qquad \text{"Time shift"}$$

$$y_2[n-n_0] = \sum_{k=\infty}^{n-n_0} x_2[k] \qquad \text{"Delay of the input sequence"}$$

$\Rightarrow y_1[n-n_0] = y_2[n-n_0] \Rightarrow$ The system is time invariant.

3. Causal and Non-Causal System

- The system will be causal if the output sequence value at the index $n=n_0$ depends only ~~the~~ on the input sequence values for $n <= n_0$

Example: The following system $y[n] = x[n-n_d]$; $-\infty < n < \infty$
will be :
- Causal for $n_d >= 0$
- Non-causal for $n_d < 0$

Example: Consider the forward difference system defined by the relationship

$$y[n] = x[n+1] - x[n]$$

check if the system is causal or non-causal

Ans: To check if the system is causal or non-causal.

Assume $n = 1$

$$\Rightarrow \underbrace{y[1]}_{\text{output value}} = \underbrace{x[2]}_{\text{Future}} - \underbrace{x[1]}_{\text{present}}$$

$\Rightarrow$ The system is non-causal

Example: Consider the backward difference system, defined as

$$y[n] = x[n] - x[n-1]$$

check if the system is causal or non-causal

Ans: To check if the system is causal or non-causal.

Assume $n=1$

$$\Rightarrow \underbrace{y[1]}_{\text{output value}} = \underbrace{x[1]}_{\text{present}} - \underbrace{x[0]}_{\text{past}}$$

$\Rightarrow$ causal system

4. Stability

- A system is stable in the bounded-input, bounded output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence.

• The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that

$$|x[n]| \leq B_x < \infty \qquad \text{for all } n$$

• stability requires that, for every bounded input, there exist a fixed positive finite value B_y such as

$$|y[n]| \leq B_y < \infty \qquad \text{for all } n$$

Example: Check the stability of the following systems

1. $y[n] = (x[n])^2$

Ans: $|y[n]| = |x[n]|^2 \leq B_x^2 < \infty$

$\Rightarrow$ BIBO $\Rightarrow$ The system is stable

2. $y[n] = \log(x[n])$

Ans:

The system is unstable since $y[n] = -\infty$ when $x[n] = 0$.

3. $y[n] = \sum_{k=-\infty}^{n} u[k]$

$$= \begin{cases} 0 & n < 0 \\ (n+1) & n \geq 0 \end{cases}$$

There is no finite choice for B_y such that $(n+1) \leq B_y < \infty$ for all n;

thus the system is unstable.

Example: For each of the systems, determine whether the system is (1) stable, (2) causal, (3) linear, (4) time invariant, and (5) memoryless

a. $T(x[n]) = g[n]\, x[n]$ with $g[n]$ is given.

1. The system will be stable if $|g[n]| < \infty$ since $|x[n]| \le M < \infty$

2. The system is causal since the system depends on the present value of n.

3. The system is linear, because:

$$\alpha_1 y_1[n] = \alpha_1 g[n]\, x_1[n]$$
$$\alpha_2 y_2[n] = \alpha_2\, g[n]\, x_2[n]$$
$$\underbrace{(\alpha_1 y_1[n] + \alpha_2 y_2[n])} = g[n]\, \underbrace{(\alpha_1 x_1[n] + \alpha_2 x_2[n])}$$
$$\alpha_3 y_3[n] = g[n]\, \alpha_3 x_3[n]$$

$\Rightarrow$ The system is linear

4. The system is time variant, because:

$$T[x(n-n_0)] = g[n]\, x[n-n_0]$$
$$\neq y[n-n_0] = g[n-n_0]\, x[n-n_0]$$

5. The ystem is Memoryles because

$y[n] = T(x[n])$ depends only on the n^{th} value of x.

b. $T(x[n]) = \sum_{k=n_0}^{n} x[k]$

1. The system will be unstable when

$$\left|T(x[n])\right| \leq \sum_{k=n_0}^{n} |x[k]| \leq |(n-n_0)| M \longrightarrow \infty$$

when $n \longrightarrow \infty$

2. The system will be non causal in case of $n > n_0$

3. The system is linear since

$$\alpha_1 y_1[n] = \sum_{k=n_0}^{n} \alpha_1 x_1[k]$$

$$\alpha_1 T[x_2[n]] = \sum_{k=n_0}^{n} \alpha_2 x_2[k]$$

$$\underbrace{\alpha_1 T[x_1[n]] + \alpha_2 T[x_2[n]]}_{\alpha_3 T[x_3[n]]} = \sum_{k=n_0}^{n} (\underbrace{\alpha_1 x_1[n] + \alpha_2 x_2[k]}_{\alpha_3 x_3[n]})$$

4. The system is time-variant since

$$T[x[n-n_0]] = \sum_{k=n_0}^{n} x[k-n_0] = \sum_{k=0}^{n-n_0} x[k]$$

$$\neq y[n-n_0] = \sum_{k=n_0}^{n-n_0} x[k]$$

5. The system is Memory since it depends on the past value $n > n_0$.

C. $T[x[n]] = \sum_{k=n-n_0}^{n+n_0} x[k]$

1. The system is stable since $|T[x[n]]| \leq |2n_0+1| M < \infty$ BIBO.

2. The system is Non-causal since it depends on the future values of $x[n]$

3. The system is linear since

$$\alpha_1 T(x_1[n]) = \sum_{k=n-n_0}^{n+n_0} \alpha_1 x_1[k]$$

$$T(\alpha_2 x_2[n]) = \sum_{k=n-n_0}^{n+n_0} \alpha_2 x_2[k]$$

$$T(\alpha_1 x_1[n] + \alpha_2 x_2[n]) = \sum_{k=n-n_0}^{n+n_0} (\alpha_1 x_1[n] + \alpha_2 x_2[n])$$

$$T(\alpha_3 x_3[n]) = \sum_{k=n-n_0}^{n+n_0} \alpha_3 x_3[n]$$

4. The system is Time-invariant since

$$T(x[n-n_0]) = \sum_{k=n-n_0}^{n+n_0} x[k-n_0]$$

$$= \sum_{k=n-2n_0}^{n} x[k] = y[n-n_0]$$

5. The system is memory since it depends on different values of $x[n]$.

D. $T(x[n]) = x[n-n_0]$

1. The system is stable since $|T(x[n])| \leq |x[n-n_0]| \leq M < \infty$

2. The system will be causal if $n_0 \geq 0$, otherwis it is non-causal

3. The system is linear since

$$T(\alpha_1 x_1[n]) = \alpha_1 x_1[n-n_0]$$
$$T(\alpha_2 x_2[n]) = \alpha_2 x_2[n-n_0]$$

$$T(\underbrace{\alpha_1 x_1[n] + \alpha_2 x_2[n]}) = \underbrace{\alpha_1 x_1[n-n_0] + \alpha_2 x_2[n-n_0]}$$

$$T(\alpha_3 x_3[n]) = \alpha_3 x_3[n-n_0]$$

4. The system is Time-invariant since

$$T(x[n-n_d]) = \cancel{x[n-2n_0}\ x[n-n_d-n_0] = y[n-n_d]$$

5. The system will be memoryless only if $n_0 = 0$, otherwise the system is memory.

E. $T(x[n]) = e^{x[n]}$

1. The system is stable since $|T(x[n])| = |e^{x[n]}| \leq e^{M} < \infty$

2. The system is causal.

3. The system is non-linear since

$$T(\alpha_1 x_1[n]) = e^{\alpha_1 x_1[n]} \quad \text{and} \quad T(\alpha_2 x_2[n]) = e^{\alpha_2 x_2[n]}$$

$$\Rightarrow T(\alpha_1 x_1[n] + \alpha_2 x_2[n]) = e^{\alpha_1 x_1[n]} + e^{\alpha_2 x_2[n]}$$

$$\neq$$

$$T(\alpha_3 x_3[n]) = e^{\alpha_3 x_3[n]} = e^{\alpha_1 x_1[n] + \alpha_2 x_2[n]}$$

4. The system is Time-invariant since

$$T(x[n-n_0]) = e^{x[n-n_0]} = y[n-n_0]$$

5. The system is memoryless since it depend only on the present value of $x[n]$.

F. $T(x[n]) = a\,x[n] + b$

1. The system is stable since $|T(x[n])| = |a\,x[n] + b| \leq a|x[n]| + b < \infty$ where a, and b are finite values.

2. The system is causal.

3. The system is non-linear

4. The system is time-invariant

5. The system is memoryless.

G. $T(x[n]) = x[-n]$

1. The system is stable since $|T(x[n])| = |x[-n]| \leq M < \infty$

2. The system will be non-causal if $n < 0$, otherwise the system is causal

3. The system is linear

4. The system is time-variant since

$$T(x[n-n_0]) = x[-n-n_0] \neq y[n-n_0] = x[-n+n_0]$$

5. The system is Memory for all values ex, of n except $n=0$.

H. $T(x[n]) = x[n] + 3u[n+1]$

1. The system is stable since

$$|T(x[n])| = |x[n]| + |3u[n+1]| \leq M+3 \quad \text{for } n \geq -1$$

and $|T(x[n])| \leq M$ for $n < -1$

2. The system is causal

3. The system is non linear

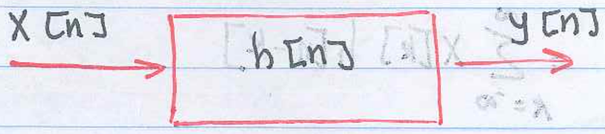
4. The system is Time-variant since

$$T(x[n-n_0]) = x[n-n_0] + 3u[n+1]$$
$$\neq y[n-n_0] = x[n-n_0] + 3u[n-n_0+1]$$

5. The system is Memoryless.

• Linear Time-Invariant Systems

For LTI system shown below



The output signal y[n] can be expressed as :-

$$y[n] = x[n] * h[n]$$

where

$*$ represents convolution operation

$\Rightarrow$

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

or

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

Example :- Consider LTI system with impulse response

$$h[n] = u[n] - u[n-N]$$

$$= \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & o.w \end{cases}$$

and the input signal is given by

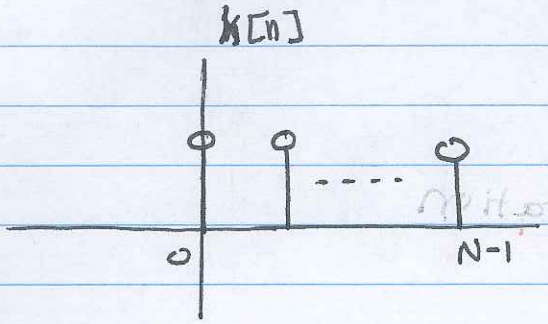
$$x[n] = a^n u[n] \; ; \quad 0 < a < 1$$

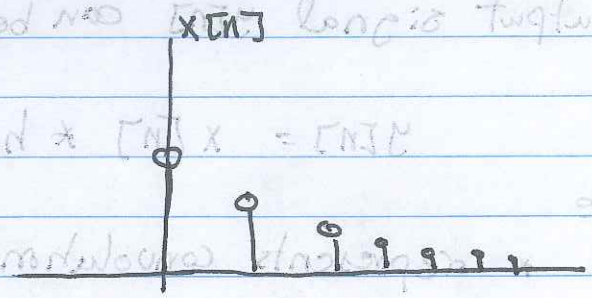
Ans:- For LTI system

$$y[n] = x[n] * h[n]$$

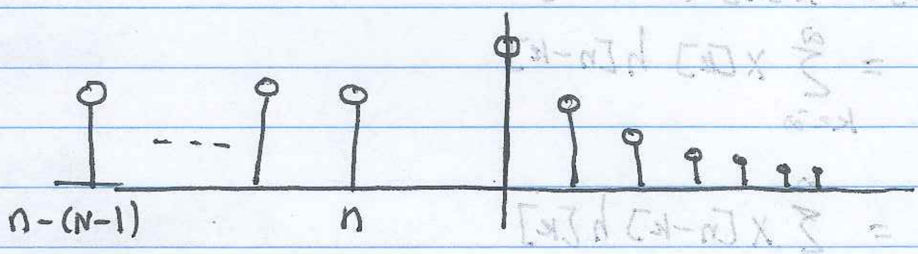
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

where





$\Rightarrow$

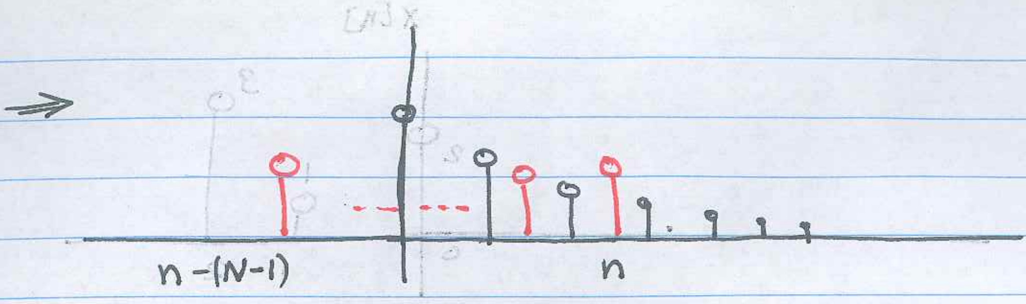


when $n < 0$

$$y[n] = 0$$

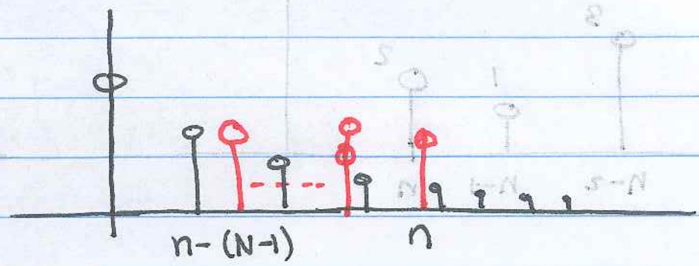
when $0 \leq n \leq N-1$, and by using the general formula of the closed form expression of the sum, where,

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \qquad N_2 \geq N_1$$



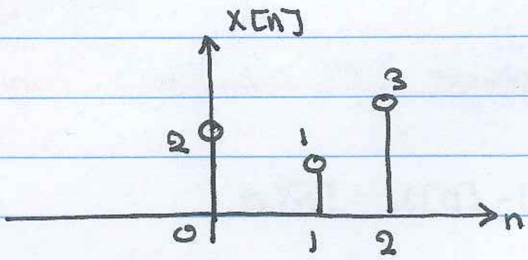
$$y[n] = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \qquad 0 \le n \le N-1$$

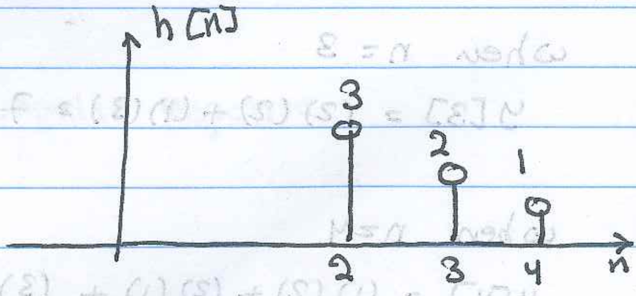
when $\quad n > N-1$



$$y[n] = \sum_{k=n-(N-1)}^{n} a^k = \frac{a^{n-N+1} - a^{n+1}}{1-a} \qquad n > N-1$$

Example: Consider LTI system in which $x[n]$ and $h[n]$ shown below



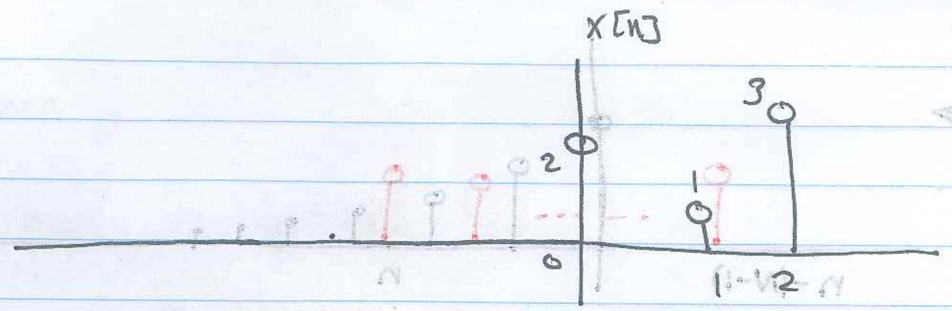


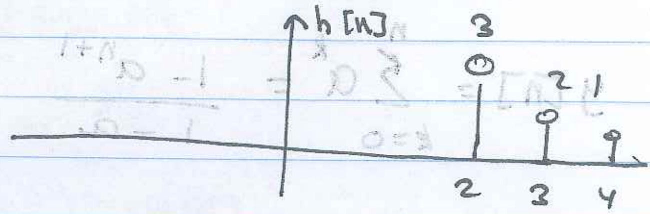
Find $y[n]$.

when

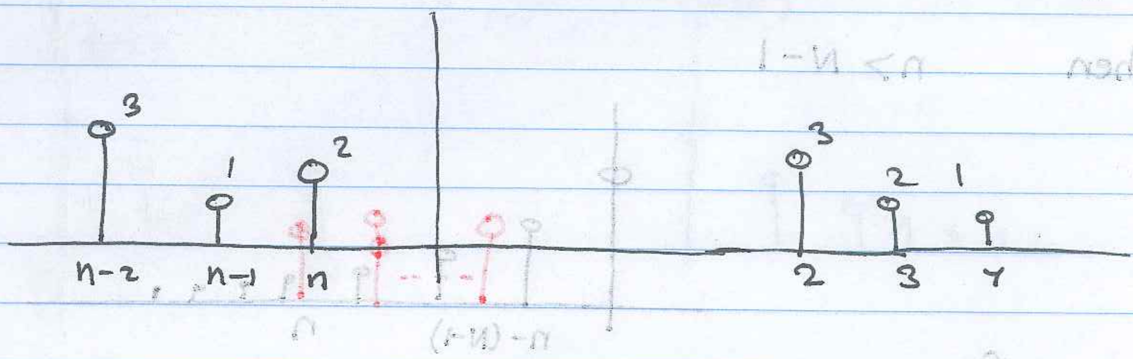
Ans:

Method 1 :-





$\Rightarrow$



when $n < 2$

$y[n] = 0$

when $n = 2$

$y[2] = (2)(3) = 6$

when $n = 3$

$y[3] = (2)(2) + (1)(3) = 7$

when $n = 4$

$y[4] = (1)(2) + (2)(1) + (3)(3) =$
$= 13$

when $n = 5$

$y[5] = (1)(1) + (3)(2) = 7$

when $n = 6 \Rightarrow y[6] = (3)(1) = 3$

when $n = 7$

$y[7] = 0$

Method 2:

n	0	1	2	3	4		
$x[n]$	$x[0]$	$x[1]$	$x[2]$	0	0		
$h[n]$	0	0	$h[2]$	$h[3]$	$h[4]$		
	0	0	$h[2]x[0]$	$h[2]x[1]$	$h[2]x[2]$		
	0	0	0	$h[3]x[0]$	$h[3]x[1]$	$h[3]x[2]$	
	0	0	0	0	$h[4]x[0]$	$h[4]x[1]$	$h[4]x[2]$
Σ	0	0	6	7	13	7	3

Exercise: An LTI system has the impulse response $h[n] = a^n u[n]$ with $|a| < 1$. The input to the system is $x[n] = B^n (u[n] - u[n-5])$ with no restriction on the value of B.

a. Find the general closed-form equation for the system output $y[n]$.

b. Evaluate $y[n]$ at $n = 0, 2,$ and 10 for $a = 0.6$ and $B = 0.8$.

c. Create stem plots of $x[n]$, $h[n]$, and $y[n]$ over the time range $0 \leq n \leq 10$ for $a = 0.6$ and $B = 0.8$.

d. Repeat Part (c) for $a = 0.6$ and $B = -0.8$

Examples: An LTI system has the impulse response $h[n] = \{1, \underline{2}, 0, -3\}$; the underline locates the $n=0$ value. For each input sequence below, find the output sequence $y[n] = x[n] * h[n]$ expressed both as a list (underline the $n=0$ value) and as a stem plot.

a. $x_1[n] = \delta[n]$

b. $x_2[n] = \delta[n+1] + \delta[n-2]$

c. $x_3[n] = \{\underline{1}, 1, 1\}$

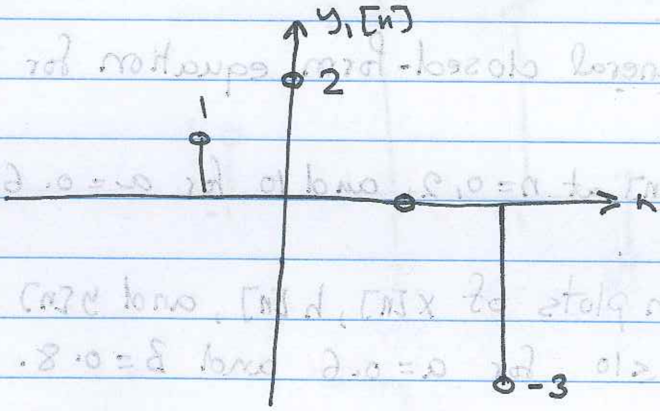
d. $x_4[n] = \{2, 1, \underline{-1}, -2, -3\}$

Ans:

a. $y[n] = x[n] * h[n]$

$= \delta[n] * h[n]$

$= h[n] = \{1, \underline{2}, 0, -3\}$

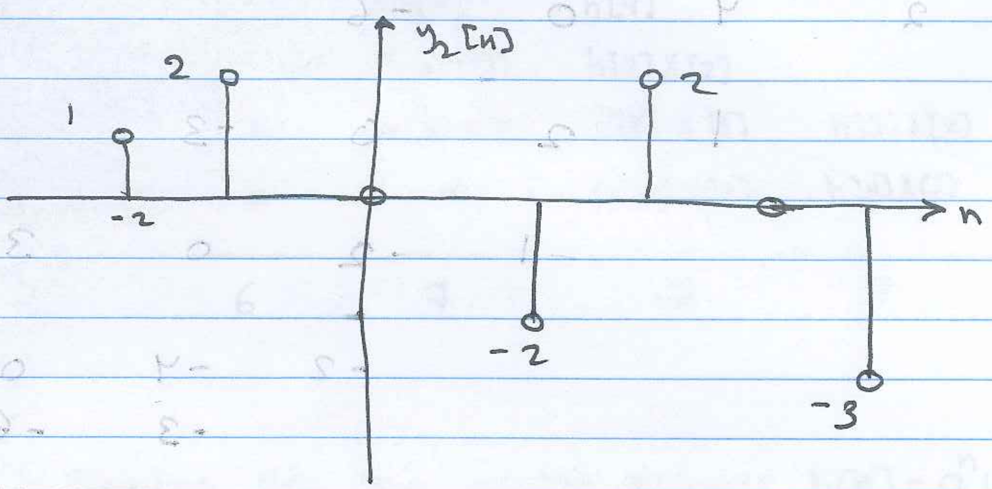


b. $y_2[n] = x_2[n] * h[n]$

$= (\delta[n+1] + \delta[n-2]) * h[n]$

$= h[n+1] + h[n-2]$

$h[n]$		1	$\underline{2}$	0	-3		
$h[n+1]$	1	2	$\underline{0}$	-3	0	0	0
$h[n-2]$	0	0	$\underline{0}$	1	2	0	-3
$h[n+1]+h[n-2]$	1	2	$\underline{0}$	-2	2	0	-3



c. $y_3[n] = x_3[n] * h[n]$

$= h[n] + h[n-1] + h[n-2]$

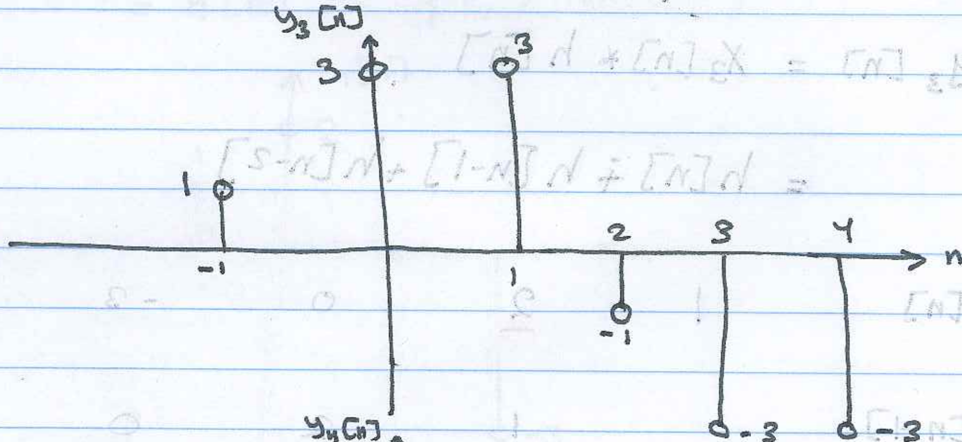
$h[n]$	1	$\underline{2}$	0	-3		
$h[n-1]$		1	2	0	-3	
$h[n-2]$			1	2	0	-3
	1	$\underline{3}$	3	-1	-3	-3

d. $y_4[n] = x_4[n] * h[n]$

$$= 2h[n+2] + h[n+1] - h[n] - 2h[n-1] - 3h[n-2]$$

n	-3	-2	-1	0	1	2	3	4
	2	4	0	-6				
		1	2	0	-3			
			-1	-2	0	3		
				-2	-4	0	6	
					-3	-6	0	9
Σ	2	5	1	-10	-10	-3	6	9

c.



d.